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Research Paper

ABSOLUTE-(p, r)-$*$-PARANORMALITY AND BLOCK MATRIX OPERATORS

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Abstract. In this paper, we introduce a new model of a block matrix operator $\mathcal{M}(\zeta, \eta)$ induced by two sequences ζ and η and characterize its absolute-(p, r)-$*$-paranormality. Next, we give examples of these operators to show that absolute-(p, r)-$*$-paranormal classes are distinct.

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1. Introduction and preliminaries

Let $\mathcal{H}$ be the infinite dimensional complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Let $T = U|T|$ be the canonical polar decomposition for $T \in \mathcal{L}(\mathcal{H})$. An operator T is said to be paranormal if $\|Tx\|^2 \leq \|T^2x\|$, for any unit vector $x \in \mathcal{H}$. Further, T is said to be $*$-paranormal if $\|T^*x\|^2 \leq \|T^2x\|$, for any unit vector $x \in \mathcal{H}$. An operator T is $\mathcal{A}(k^*)$ class operator if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \leq |T^*|^2$, for every $k > 0$. In the paper [5], there were introduced absolute-k-$*$-paranormal class of operators as follows: $\||T|^kTx\| \geq \|T^*x\|^{k+1}$, for $x \in \mathcal{H}, \|x\| = 1$ and for any $k > 0$. The $\mathcal{A}(k^*)$ class operators is included in the absolute-k-$*$-paranormal operators for any $k > 0$ (see Theorem 2.4 in [9]). An operator T is said to be p-$*$-paranormal if $\||T|^pU|T|^px\| \geq \||T|^pU^*x\|^2$, for all unit vectors $x \in \mathcal{H}$ and $p > 0$. Braha and Hoxha [1] introduced the absolute-(p, r)-$*$-paranormality which is a further generalization of both absolute-k-$*$-paranormality and p-$*$-paranormality. For each $p > 0$, $r \geq 0$, an operator T is absolute-(p, r)-$*$-paranormal if

$$\||T|^pU|T|^rx\|^r \geq \||T|^rU^*x\|^{p+r},$$

for any unit vector $x \in \mathcal{H}$. Also, they introduced (p, r, q)-$*$-paranormal operators. For each $p > 0$, $r \geq 0$ and $q > 0$, an operator T is (p, r, q)-$*$-paranormal if $\||T|^pU|T|^rx\|^{\frac{1}{q}}\|x\|^p \geq \||T|^{\frac{p+r}{q}}U^*x\|$, for all unit vectors $x \in \mathcal{H}$.

Let (X, Σ, μ) be a complete σ-finite measure space and let $\mathcal{A}$ be a sub-σ-finite algebra of Σ. We use the notation $L^2(\mathcal{A})$ for $L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$ and henceforth we write μ in place of $\mu_{|\mathcal{A}}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. The support of a measurable function f is defined as $S(f) = \{x \in X; f(x) \neq 0\}$. We denote the vector space of all equivalence classes of almost everywhere finite valued measurable

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functions on X by $L^0(\Sigma)$. Let $\varphi : X \to X$ be a transformation such that $\varphi^{-1}(\Sigma) \subseteq \Sigma$ and $\mu \circ \varphi^{-1} \ll \mu$. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ \varphi^{-1}/d\mu$ is finite-valued or equivalently $(X, \varphi^{-1}(\Sigma), \mu)$ is σ-finite. The composition operator C_φ acting on $L^2(\Sigma) := L^2(X, \Sigma, \mu)$ is defined by $C_\varphi = f \circ \varphi$. The condition $h \in L^\infty(\Sigma)$ assures that C_φ is bounded. For any non-negative Σ-measurable function f as well as for any $f \in L^p(\Sigma)$ by the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E(f)$ such that

$$\int_A Ef d\mu = \int_A f d\mu, \text{ for all } A \in \mathcal{A}.$$

Hence we obtain a operator E from $L^p(\Sigma)$ to $L^p(\mathcal{A})$ which is conditional expectation operator associated with the σ-algebra $\mathcal{A}$. This operator will play major role in our work. For further information on conditional type operators, see [6, 10]. Composition operators as an extension of shift operators are a good tool for separating weak paranormal classes. Classic seminormal composition operators have been extensively studied by Harrington and Whitley [5], Lambert [8], Singh [11], Campbell [3] and many other mathematicians.

In this paper, we will restrict ourselves to the Hilbert space $\ell^2(\mathbb{N})$ of complex-valued functions on the natural numbers. The space of $\ell^2(\mathbb{N})$ can also be denoted by $L^2(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $2^{\mathbb{N}}$ is the power set of natural numbers and μ is a measure on $2^{\mathbb{N}}$ define by $\mu(\{n\}) = m_n$ where $\{m_n\}_{n=1}^\infty$ be a sequance of positive real numbers.
Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\ell^2(\mathbb{N})$, and $f = \sum_{n\in\mathbb{N}} f_n e_n$ be in $\ell^2(\mathbb{N})$. Then some direct computations show that for each $k \in \mathbb{N}$:

$$h(k) = \frac{1}{m_k} \sum_{j\in\varphi^{-1}(k)} m_j, \qquad E(f)(k) = \frac{\sum_{j\in\varphi^{-1}(\varphi(k))} f_j m_j}{\sum_{j\in\varphi^{-1}(\varphi(k))} m_j}; \tag{1.1}$$

In [4], Exner, Jung and Lee introduced a model of block matrix operator and by using composition operators, characterize its p-hyponormality. In this paper we define a new block matrix on $\ell^2(\mathbb{N})$. Next, we obtain the (p, r, q)-$*$-paranormality and the absolute-(p, r)-$*$ paranormality criteria of these type block matrices. Finally, some examples presented which show that block matrix operators can distinguish between these classes.

2. Characterizations

Let $\zeta := \{\zeta_i^n\}_{\substack{1\le i\le t \\ 0\le n<\infty}}$ and $\eta := \{\eta_j^n\}_{\substack{1\le j\le s \\ 0\le n<\infty}}$ be bounded sequences of positive real numbers. Let $\mathcal{M}(\zeta, \eta) := [E_{ij}]_{0\le i,j<\infty}$ be a block matrix operator whose blocks are $(t + s) \times (t + 1)$ matrices such that $E_{ij} = 0, i \ne j$, and

$$E_n := E_{nn} = \begin{bmatrix} \zeta_1^{(n)} & & & O \\ & \ddots & & \\ & & \zeta_t^{(n)} & \\ & & & \eta_1^{(n)} \\ & O & & \vdots \\ & & & \eta_s^{(n)} \end{bmatrix}. \tag{2.1}$$

where other entries are 0 except ζ_*^n and η_*^n in (2.1). It is clear that block matrix $\mathcal{M}$ is bounded.

Definition 2.1. For two bounded sequences $\zeta := \{\zeta_i^n\}_{\substack{1\le i\le t\\ 0\le n<\infty}}$ and $\eta := \{\eta_j^n\}_{\substack{1\le j\le s\\ 0\le n<\infty}}$, the block matrix operator $\mathcal{M} := \mathcal{M}(\zeta,\eta)$ satisfying in (2.1) is called a block matrix operator with weight sequence (ζ,η).

Let $\mathcal{M}$ be a block matrix operator with weight sequence (ζ,η) and let $\mathcal{W}_(\zeta,\eta)$ be its corresponding operator on ℓ^2 relative to some orthonormal basis. Then $\mathcal{W}_(\zeta,\eta)$ may provide a repetitive form; for example $t=2$, $s=4$ and $\zeta_i^{(n)} = \eta_j^{(n)} = 1$ for all $i,j,n\in\mathbb{N}$, then the block matrix operator with (ζ,η) is unitarily equivalent to the following operator $\mathcal{W}_{\zeta,\eta}$ on ℓ^2 defined by

$$\mathcal{W}_{\zeta,\eta}(x_1,x_2,x_3,x_4,...) = (x_1,x_2,\underbrace{x_3,x_3,x_3,x_3}_{(4)},x_4,x_5,\underbrace{x_6,x_6,x_6,x_6}_{(4)},x_7,x_8,...).$$

We put $X=\mathbb{N}_0$ and the power set $\mathcal{P}(X)$ of X for the σ-algebra Σ. Define a non-singular measurable transformation φ on $\mathbb{N}_0$ such that

$$\begin{aligned} &\varphi^{-1}(k(t+1)+t) = \{k(t+s)+i-1+t : 1\le i\le s\}, \quad k=0,1,2,...,\\ &\varphi^{-1}(k(t+1)+i-1) = k(t+s)+i-1, \quad 1\le i\le t, \quad k=0,1,2,... \end{aligned} \tag{2.2}$$

If we choose s and t in such a way that their sum is always an even number, then we have

$$\varphi^2(n) = \begin{cases} k(t+1)+t & n=k(t+s)+i+t-1, 1\le i\le s \quad k\in\mathbb{N}_0;\\ k(t+1)+t & n=k(t+s)+i-1, 1\le i\le t, k\in\mathbb{N}_0,\ k\,\text{is odd};\\ k(t+1)+i-1 & n=k(t+s)+i-1, 1\le i\le t, k\in\mathbb{N}_0,\ k\,\text{is even}. \end{cases} \tag{2.3}$$

Throughout this paper, we assume that $t+s$ is even. We write $m(\{i\}) := m_i, i\in\mathbb{N}_0$, for the underlying point mass measure on X, and we suppose that each m_i is strictly positive.

Proposition 2.2. *The composition operator C_φ on ℓ^2 defined by $C_\varphi f = f\circ\varphi$ is unitarily equivalent to the block matrix operator $\mathcal{M}(\zeta,\eta)$, where $\zeta := \{\zeta_i^n\}_{\substack{1\le i\le t\\ 0\le n<\infty}}$ and $\eta := \{\eta_j^n\}_{\substack{1\le j\le s\\ 0\le n<\infty}}$ and for each $n\in\mathbb{N}_0$*

$$\zeta_i^{(n)} = \sqrt{\frac{m_{n(t+s)+i-1}}{m_{n(t+1)+i-1}}} \quad (1\le i\le t),$$

$$\eta_j^{(n)} = \sqrt{\frac{m_{n(t+s)+j+t-1}}{m_{n(t+1)+t}}} \quad (1\le j\le s).$$

Proof. Let $e_i = \frac{1}{\sqrt{m_i}}\chi_i \quad (i\in\mathbb{N}_0)$. Then $\{e_i\}_{i\in\mathbb{N}_0}$ is an orthonormal basis for ℓ^2. We have

$$C_\varphi e_j = e_j\circ\varphi = \frac{1}{\sqrt{m_j}}\chi_{\varphi^{-1}\{j\}} = \frac{1}{\sqrt{m_j}}\sum_{i\in\varphi^{-1}(j)} e_i\sqrt{m_i}.$$

Then, for each $k \in \mathbb{N}_0$, we have

$$C_\varphi e_j = \begin{cases} \sum_{1\le i\le s} \sqrt{\frac{m_{k(t+s)+i-1+t}}{m_{k(t+1)+t}}} e_{k(t+s)+i-1+t} & j = k(t+1)+t; \\ \sqrt{\frac{m_{k(t+s)+i-1}}{m_{k(t+1)+i-1}}} e_{k(t+s)+i-1} & j = k(t+1)+i-1, 1 \le i \le t. \end{cases}$$

Now, we set weight sequences $\zeta := \{\zeta_i^n\}_{\substack{1\le i\le t \\ 0\le n<\infty}}$ and $\eta := \{\eta_j^n\}_{\substack{1\le j\le s \\ 0\le n<\infty}}$, where

$$\zeta_i^{(n)} = \sqrt{\frac{m_{n(t+s)+i-1}}{m_{n(t+1)+i-1}}}, \quad 1 \le i \le t, 0 \le n < \infty,$$

and

$$\eta_j^{(n)} = \sqrt{\frac{m_{n(t+s)+j+t-1}}{m_{n(t+1)+t}}} \quad 1 \le j \le s, 0 \le n < \infty.$$

Therefore, it is easy to check that C_φ is unitarily equivalent to the block matrix operator $\mathcal{M}(\zeta, \eta)$ with weight sequence (ζ, η). □

Remark 2.3. In Proposition 2.2, put $t = 2$ and $s = 4$. Then, $C_\varphi e_0 = \sqrt{\frac{m_0}{m_0}} e_0$, $C_\varphi e_1 = \sqrt{\frac{m_1}{m_1}} e_1$, $C_\varphi e_2 = \sqrt{\frac{m_2}{m_2}} e_2 + \sqrt{\frac{m_3}{m_2}} e_3 + \sqrt{\frac{m_4}{m_2}} e_4 + \sqrt{\frac{m_5}{m_2}} e_5$ and $\cdots$. Therefore, in this case, C_φ is equivalent to the block matrix

$$\begin{bmatrix} E_1 & & & \\ & E_2 & & \\ & & E_3 & \\ & & & E_4 \\ & & \ddots & \end{bmatrix} \text{ where } E_1 := E_{11} = \begin{bmatrix} \sqrt{\frac{m_0}{m_0}} & 0 & 0 \\ 0 & \sqrt{\frac{m_1}{m_1}} & 0 \\ 0 & 0 & \sqrt{\frac{m_2}{m_2}} \\ 0 & 0 & \sqrt{\frac{m_3}{m_2}} \\ 0 & 0 & \sqrt{\frac{m_4}{m_2}} \\ 0 & 0 & \sqrt{\frac{m_5}{m_2}} \end{bmatrix}. \tag{2.4}$$

Proposition 2.4. [4] *Let $\mathcal{M}(\zeta, \eta)$ be a block matrix operator with weight sequence (ζ, η), where $\zeta := \{\zeta_i^n\}_{\substack{1\le i\le r \\ 0\le n<\infty}}$ and $\eta := \{\eta_j^n\}_{\substack{1\le j\le s \\ 0\le n<\infty}}$. Then there exists a measurable transformation φ on a σ-finite measure space $(\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0), m)$ such that $\mathcal{M}(\zeta, \eta)$ is unitarily equivalent to the composition operator C_φ on ℓ^2.*

Proposition 2.5. [7] *The folllowing are equivalent:*

(i) *C_φ is absolute-(p, r)-$*$-paranormal.*

(ii) *$(h^r \circ \varphi)E(h^p) \ge h^{p+r} \circ \varphi^2$ on $S(h)$.*

(iii) *C_φ is (p, r, q)-$*$-paranormal.*

Theorem 2.6. *Let φ be a non-singular measurable transformation on ℓ^2 as in (2.2) and let $p>0$, $r\geq 0$ and $q>0$. Then the following assertions are equivalent*

(*i*) *C_φ is absolute-(p,r)-$*$-paranormal on ℓ^2;*

(*ii*) *C_φ is (p,r,q)-$*$-paranormal.*

(*iii*) *the block matrix operator $\mathcal{M}(\zeta,\eta)$ as in Proposition 2.2 is absolute-(p,r)-$*$-paranorma and (p,r,q)-$*$-paranormal.*

(*iv*) *$(h^r\circ\varphi)(n)E(h^p)(n)\geq h^{p+r}\circ\varphi^2(n)$ on $S(h)$.*

(*v*) *the following inequality for $n\in\mathbb{N}_0$, holds*

$$\left(\frac{m(\varphi^{-1}(\varphi(n)))}{m_{\varphi(n)}}\right)^r\frac{1}{m(\varphi^{-1}(\varphi(n)))}\sum_{l\in\varphi^{-1}(\varphi(n))}\frac{m(\varphi^{-1}(j))^p}{m_j^p}m_j \geq\left(\frac{m(\varphi^{-1}(\varphi^2(n)))}{m_{\varphi^2(n)}}\right)^{p+r}, \tag{2.5}$$

Proof. Because of Propositions 2.4 and 2.5 we have (i), (ii), (iii) and (iv) are equivalent. Also, by a similar argument as in the proof of [Theorem 2.1, [4]], it is easy to see that (iv) and (v) are equivalent. □

The conditions above simplify considerably if we specialize to the case of a repeated block Let $\mathcal{M}(\zeta,\eta)$ be a block matrix operator where $\zeta:=\{\zeta_i^n\}_{\substack{1\leq i\leq t\\0\leq n<\infty}}$ and $\eta:=\{\eta_j^n\}_{\substack{1\leq j\leq s\\0\leq n<\infty}}$ as follows:

$$\begin{aligned}\mathcal{M}(\zeta,\eta):E\equiv E_1\equiv E_1=E_2=\cdots\\ \zeta:\zeta_i^{(n)}=\zeta_i,\ n\in\mathbb{N}_0,\ 1\leq i\leq t;\\ \eta:\eta_j^{(n)}=\eta_j,\ n\in\mathbb{N}_0, 1\leq j\leq s.\end{aligned}\tag{2.6}$$

For any $n\in\mathbb{N}_0$, let i_n denote the solution to the conditions $1\leq i_n\leq t$ and $n=k_1(t+1)+i_n-1$ for some $k_1\in\mathbb{N}_0$. Similarly, let v_n satisfy $1\leq v_n\leq s$ and $n=k_2(t+s)+v_n-1+t$ for some $k_2\in\mathbb{N}_0$.

Theorem 2.7. *Let $\mathcal{M}(\zeta,\eta)$ be as in (2.6). Then the block matrix operator $\mathcal{M}(\zeta,\eta)$ is absolute-(p,r)-$*$-paranormal if and only if the following three conditions hold:*
(*i*) *if $n=k(t+s)+i-1+t$ for $1\leq i\leq s$, then for all $1\leq i_j\leq t$ and $1\leq v_j\leq s$ we have*

$$\left(\sum_{1\leq i\leq s}\eta_i^2\right)^r\sum_{\substack{j\in\varphi^{-1}(\varphi(n))\\ j\equiv t\ mod(t+1)}}\left(\sum_{1\leq i\leq s}\eta_i^2\right)^p\left(\frac{\eta_{v_j}^2}{\sum_{1\leq i\leq s}\eta_i^2}\right) + \sum_{\substack{j\in\varphi^{-1}(\varphi(n))\\ j\not\equiv t\ mod(t+1)}}(\zeta_{i_j})^{2p}\left(\frac{\eta_{v_j}^2}{\sum_{1\leq i\leq s}\eta_i^2}\right)\geq\left(\sum_{1\leq i\leq s}\eta_i^2\right)^{p+r}\tag{2.7}$$

(ii) if $n = k(t+s)+q-1$ and k is even, for $1 \le q \le t$, we have

$$(ii-a) \quad \zeta_q^{2r}(\sum_{1\le i\le s}\eta_i^2)^p \ge (\zeta_q^2)^{p+r} \quad n \equiv t \quad mod(t+1)$$

$$(ii-b) \quad \zeta_q^{2r}\zeta_{i_n}^{2p} \ge \zeta_q^{2(p+r)} \qquad n \equiv i_n - 1 \quad mod(t+1)\, and\, 1 \le i_n \le t.$$

(iii) if $n = k(r+s)+q-1$ and k is odd, then for $1 \le q \le t$ we have

$$(ii-a) \quad \zeta_q^{2r}(\sum_{1\le i\le s}\eta_i^2)^p \ge (\sum_{1\le v_n\le s}\eta_{v_n}^2)^{p+r} \qquad n \equiv t \quad mod(t+1)$$

$$(ii-b) \quad \zeta_q^{2r}\zeta_{i_n}^{2p} \ge (\sum_{1\le v_n\le s}\eta_{v_n}^2)^{p+r} \qquad n \equiv i_n - 1 \quad mod(t+1)\, with\, 1 \le i_n \le t.$$

Proof. First, we proof (i): since $n = k(t+s)+i-1+t$ for $1 \le i \le s$. Thus $\varphi(n) = k(t+1)+t$ and $\varphi^{-1}(\varphi(n)) = \{k(t+s)+i-1+t : 1 \le i \le s\}$. By using Proposition 2.2, we have

$$m(\varphi^{-1}(\varphi(n))) = \sum_{1\le i\le s} m_{k(t+s)+i-1+t} = \sum_{1\le i\le s}(\eta_i^{(k)})^2 m_{k(t+1)+t},$$

since for any $k \in \mathbb{N}_0$, $\eta_i^{(k)} = \eta_i$. So $m(\varphi^{-1}(\varphi(n))) = \sum_{1\le i\le s}\eta_i^2 m_{k(t+1)+t}$. Also, since in this case $\varphi^2(n) = \varphi(n)$, therefore we have

$$\left(\frac{m(\varphi^{-1}(\varphi(n)))}{m_{\varphi(n)}}\right) = \left(\frac{m(\varphi^{-1}(\varphi^2(n)))}{m_{\varphi^2(n)}}\right) = \left(\frac{\sum_{1\le i\le s}\eta_i^2 m_{\varphi(n)}}{m_{\varphi(n)}}\right) = \sum_{1\le i\le s}\eta_i^2.$$

Now, we will calculate $\frac{1}{m(\varphi^{-1}(\varphi(n)))}\sum_{j\in\varphi^{-1}(\varphi(n))}\frac{m(\varphi^{-1}(j))^p}{m_j^p}m_j$. By using Proposition 2.2, we deduce that

$$\frac{m_j}{m(\varphi^{-1}(\varphi(n)))} = \frac{\eta_{v_j}^2 m_{k(t+1)+t}}{\sum_{1\le i\le s}\eta_i^2 m_{k(t+1)+t}} = \frac{\eta_{v_j}^2}{\sum_{1\le i\le s}\eta_i^2}, \quad 1 \le v_j \le s.$$

In sequel, we compute $\left(\frac{m(\varphi^{-1}(j))}{m_j}\right)^p$ for $j \in \varphi^{-1}(\varphi(n))$. To do so we consider two subcases.

Case1a: $j = k_1(t+1)+t, k_1 \in \mathbb{N}_0$, then we have $\varphi^{-1}(j) = \{k_1(t+s)+i-1+t : 1 \le i \le s\}$. By Proposition 2.2, we have

$$\left(\frac{m(\varphi^{-1}(j))}{m_j}\right)^p = \left(\frac{\sum_{1\le i\le s}\eta_i^2 m_{k_1(t+1)+t}}{m_{k_1(t+1)+t}}\right)^p = \left(\sum_{1\le i\le s}\eta_i^2\right)^p.$$

Case1b: $j = k_1(t+1)+i_j-1$ for $k_1 \in \mathbb{N}_0$ and $1 \le i_j \le t$. In this case we get that $\varphi^{-1}(j) = k_1(t+s)+i_j-1 : 1 \le i_j \le t$, so Proposition 2.2 implies that

$$\left(\frac{m(\varphi^{-1}(j))}{m_j}\right)^p = \left(\zeta_{i_j}^2\right)^p.$$

Therefore, for $n = k(t+s)+i-1+t$ and $1 \le i \le t$, we conclude that (2.5) is equivalent to (2.7).

Now, we proof (ii): In this case $n = k(t+s)+q-1$ for $1 \le q \le t$ and k is even. By (2.2) and (2.3), it is easy to see that $\varphi(n) = \varphi^2(n) = k(t+1)+q-1$ and $\varphi^{-1}(\varphi(n)) = \varphi^{-1}(\varphi^2(n)) = n$, by using Proposition 2.2, we get that

$$\frac{m(\varphi^{-1}(\varphi(n)))}{m(\varphi(n))} = \frac{m(\varphi^{-1}(\varphi^2(n)))}{m(\varphi^2(n))} = \frac{m_{k(t+s)+q-1}}{m_{k(t+1)+q-1}} = \frac{\zeta_q^2 m_{k(t+1)+q-1}}{m_{k(t+1)+q-1}} = \zeta_q^2,$$

Since $\varphi^{-1}(\varphi(n)) = n$ for $n = k(t+s)+q-1$, obviously $\frac{m(\varphi^{-1}(\varphi(n)))}{m_j} = 1$ for $j \in \varphi^{-1}(\varphi(n))$. Now we consider two subcases for computations of $\left(\frac{m(\varphi^{-1}(j))}{m_j}\right)^p$, $j \in \varphi^{-1}(\varphi(n))$.

Case2a: $j(=n) = k_2(t+1)+t$ for some $k_2 \in \mathbb{N}_0$. Hence, we have $\varphi^{-1}(j) = \{k_2(t+s)+i-1+t : 1 \le i \le s\}$. Hence

$$\frac{m(\varphi^{-1}(j))}{m_j} = \frac{\sum_{1\le i\le s} \eta_i^2 m_{k_2(t+1)+t}}{m_{k_2(t+1)+t}} = \sum_{1\le i\le s} \eta_i^2.$$

Case2b: $j(=n) = k_2(t+1)+i_n-1$ for some $k_2 \in \mathbb{N}_0$, with $1 \le i_n \le t$. Obviously $\varphi^{-1}(j) = \{k_2(t+s)+i_n-1 : 1 \le i_n \le t\}$, consequently

$$\frac{m(\varphi^{-1}(j))}{m_j} = \frac{\zeta_{i_n}^2 m_{k_2(t+1)+v_n-1}}{m_{k_2(t+1)+v_n-1}} = \zeta_{i_n}^2.$$

Thus we get that in this case (2.5) is equivalent to

$$\begin{cases} \zeta_q^{2r}\left(\sum_{1\le i\le s}\eta_i^2\right)^p \ge \zeta_q^{2(p+r)} & n \equiv t, \quad \mathrm{mod}(t+1), \\ \zeta_q^{2r}\zeta_{i_n}^{2p} \ge \zeta_q^{2(p+r)} & n \equiv i_n - 1, \quad \mathrm{mod}(t+1). \end{cases}$$

Finally, we proof (iii): $n = k(t+s)+q-1$ for $1 \le q \le t$ and k is odd. By (2.2) and (2.3), we have $\varphi(n) = k(t+1)+q-1$, $\varphi^{-1}(\varphi(n)) = n$, $\varphi^2(n) = k(t+1)+t$ and $\varphi^{-1}(\varphi^2(n)) = \{k(t+s)+v_n-1+t : 1 \le v_n \le s\}$ by using Proposition 2.2, we get that

$$\frac{m(\varphi^{-1}(\varphi(n)))}{m(\varphi(n))} = \zeta_q^2, \ \frac{m(\varphi^{-1}(\varphi^2(n)))}{m(\varphi^2(n))} = \sum_{1\le v_n\le s} \eta_{v_n}^2$$

Also, by a similar argument as in the proof of (ii), we have

$$\frac{m(\varphi^{-1}(j))}{m_j} = \begin{cases} \sum_{1\le i\le t}\eta_i^2, & n \equiv t, \quad \mathrm{mod}(t+1), \\ \zeta_{i_n}^2 & n \equiv i_n - 1, \quad \mathrm{mod}(t+1) \end{cases}$$

Consequently, for $n = k(t+s)+q-1$ where k is odd and $1 \le q \le t$, we get that (2.5) is equivalent to

$$\begin{cases} \zeta_q^{2r}\left(\sum_{1\le i\le t}\eta_i^2\right)^p \ge \left(\sum_{1\le v_n\le s}\eta_{v_n}^2\right)^{p+r} & n \equiv t, \quad \mathrm{mod}(t+1), \\ \zeta_q^{2r}\zeta_{i_n}^{2p} \ge \left(\sum_{1\le v_n\le s}\eta_{v_n}^2\right)^{p+r} & n \equiv i_n - 1, \quad \mathrm{mod}(t+1). \end{cases}$$

□

Example 2.8. Let

$$E := \begin{bmatrix} c & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{M} := \begin{bmatrix} E & & \\ & E & \\ & & \ddots \end{bmatrix}.$$

Note that c is a fixed positive real number. Then some direct computations show that the conditions for $\mathcal{M}$ to be absolute (p, r)-$*$-paranormal in Theorem 2.7 is equivalent to the following condition:

$$c^{2p} \geq 3^p \quad \text{and} \quad c^{2(p+r)} \geq 3^{p+r} \tag{2.8}$$

Then by using (2.8) we can find c such that $\mathcal{M}$ is absolute-$(2, 3)$-$*$-paranormal but it is not absolute-$(2, 4)$-$*$-paranormal. Namely, put $c = 1.8$

Example 2.9. Let

$$F := \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{M} := \begin{bmatrix} F & & \\ & F & \\ & & \ddots \end{bmatrix}.$$

where a, b, c are fixed positive real number. Hence, by using Theorem 2.7, it is easy to see that $\mathcal{M}$ is absolute-(p, r)-$*$-paranormal if and only if the following conditions hold:

$$\begin{aligned} &16^p + 2a^2 + b^2 + 9c^2 \geq 16^{p+1}; \\ &a^{2(p+r)} \geq 16^{p+r}; \\ &b^{2(p+r)} \geq 16^{p+r}; \\ &c^{2(p+r)} \geq 16^{p+r}. \end{aligned} \tag{2.9}$$

Therefore by using (2.9), we can find a, b and c such that $\mathcal{M}$ is absolute-$(3, 4)$-$*$-paranormal, but it is not absolute-$(1, 3)$-$*$-paranormal. Put $a = 5$, $b = 6$ and $c = 4$, so this yields that the classes of absolute-(p, r)-$*$-paranormal operators are distinct for $p > 0$ and $r \geq 0$. Also, by Theorem 2.6 we deduce that this block matrix operator can separate the classes of (p, r, q)-$*$-paranormal operators for $p > 0$, $r \geq 0$ and $q > 0$.

References

[1] N. Braha and I. Hoxha, (p, r, q)-$*$-paranormal operators and Absolute-(p, r)-$*$-paranormal operators, J. Math. Anal, **3**(2013), 14-22.
[2] C. Burnap, I. Jung and A. Lambert, Separating partial normality classes with composition operators, J. Operator Theory, **53** (2005), 381-397.
[3] J. Campbell and W. Hornor, Seminormal composition operators, J. Operator Theory **29** (1993), 323-343.
[4] G. Exner, I. Jung and M. Lee, Block matrix operators and weak hyponormalities, J. Integr. equ. oper. theory **65** (2009), 345-362.
[5] D. Harrington and R. Whitley, Seminormal composition operators, J. Operator Theory, **11** (1984), 125-135.
[6] J. Herron, Weighted Conditional Expectation Operators, Oper. Matrices, **5**(2011), 107-118.

[7] H. Emamalipour, M.R. Jabbarzadeh and M. Sohrabi Chegeni, Some weak p-Hyponormality classes of weighted composition operators, Filomat, **9**(2017), 2643-2656.
[8] A. Lambert, Hyponormal composition operators, Bull London Math Soc **18** (1986), 395-400.
[9] S. Panayappan and A. Radharamani, A note on p-$*$-paranormal operators and absolute k-$*$-paranormal operators, Int. J. Math. Anal. **2**(2008), no.25-28, 1257-1261.
[10] M. M. Rao, Conditional measure and applications, Marcel Dekker, New York, 1993.
[11] R. K. Singh and J. S. Manhas, Composition Operators on Function Spaces, North Holland Math. Studies 179, Amsterdam, 1993.

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