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Research Paper

NEW MODIFIED IMPLICIT ITERATIVE ALGORITHM FOR FINITE FAMILIES OF TWO TOTAL ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS

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Abstract. In this article, we propose a modified implicit iterative algorithm for approximation of common fixed points of finite families of two uniformly L-Lipschitzian total asymptotically pseudocontractive mappings in Banach spaces. Our new iterative algorithm contains some well known iterative algorithm which has been used by several authors for approximating fixed points of different classes of mappings. We prove some convergence theorems of our new iterative method and validate our main result with an example. Our result is an improvement and generalization of the results of many well-known authors in the existing literature.

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1. Introduction

Let X be a real Banach space with dual E^* and C a nonempty closed convex subset of X. We denote by J the *normalized duality* mapping from X into 2^{X^*} defined by

$$J(x) = \{f^* \in X^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\},\ \forall\, x \in X, \tag{1.1}$$

where $\langle ., .\rangle$ denotes the generalized duality pairing. Let j denotes the single-valued-normalized duality mapping, $\Re^+$ the set of positive real numbers, $\mathbb{N}$ the set of natural numbers and $F(T)$ denotes the set of fixed points of mapping $T : X \to X$, i.e., $F(T) = \{x \in X : Tx = x\}$.

Definition 1.1. A mapping $T : C \to C$ is said to be:

1. *uniformly Lipschitzian,* if there exists a constant $L \geq 0$ such that for any $x, y \in C$,
$$\|T^n x - T^n y\| \leq L\|x - y\|,\ \ \forall\, n \geq 1; \tag{1.2}$$
2. *pseudocontractive,* if for any $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that
$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2; \tag{1.3}$$
3. *strictly pseudocontractive,* if there exists a constant $\lambda \in (0, 1)$ and for any given $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that
$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \lambda\|(I - T)x - (I - T)y\|^2; \tag{1.4}$$

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4. *asymptotically pseudocontractive,* if there exists a sequence $\{h_n\} \subset [1, \infty)$ with $h_n \to 1$ as $n \to \infty$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \le h_n \|x-y\|^2,\ \forall\, n \ge 1,\ \text{and}\ \ x, y \in C. \tag{1.5}$$

The class of asymptotically pseudocontractive mappings was introduced by Schu [15] (see also [16]). In [19], Rhoades gave an example to show that the class of asymptotically pseudocontractive mappings contains properly the class of asymptotically nonexpansive mappings, see [19] for more details.

5. *asymptotically strictly pseudocontractive,* if there exists a constant $\lambda \in (0,1)$ and a sequence $h_n \subset [1, \infty)$ with $h_n \to 1$ as $n \to \infty$ for any given $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\begin{aligned} \langle T^n x - T^n y, j(x-y)\rangle \ &\le\ h_n\|x-y\|^2 \\ &\quad -\lambda\|(I-T^n)x-(I-T^n)y\|^2,\ \ \forall\, n \ge 1; \end{aligned} \tag{1.6}$$

6. *asymptotically pseudocontractive in the intermediate sense,* if there exists a sequence $h_n \subset [1, \infty)$ with $h_n \to 1$ as $n \to \infty$ and $j(x-y) \in J(x-y)$ such that

$$\limsup_{n\to\infty} \sup_{(x,y)\in C} (\langle T^n x - T^n y, j(x-y)\rangle - h_n\|x-y\|^2) \le 0. \tag{1.7}$$

Set

$$\tau_n = \max\left\{0, \sup_{x,y\in C}(\langle T^n x - T^n y, j(x-y)\rangle - h_n\|x-y\|^2)\right\}.$$

It follows that $\tau_n \ge 0$, $\tau_n \to 0$ as $n \to \infty$. Hence, (1.7) yields the following inequality:

$$\langle T^n x - T^n y, j(x-y)\rangle \le h_n\|x-y\|^2 + \tau_n,\ \forall\ n \ge 1,\ x, y \in C. \tag{1.8}$$

This class of mappings was introduced by Qin et al. [12].

7. *Total asymptotically pseudocontractive,* if there exists sequences $\{\mu_n\} \subset [0, \infty)$ and $\xi_n \subset [0, \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \le \|x-y\|^2 + \mu_n\phi(\|x-y\|) + \xi_n, \tag{1.9}$$

$\forall n \ge 1$ and $x, y \in C$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$. This class of mapping was introduced by Qin et al. [13].

Remark 1.2. If $\phi(t) = t^2$, then (1.9) reduces to the class of total asymptotically pseudocontractive mappings in the intermediate sense as follows:

$$\langle T^n x - T^n y, j(x-y)\rangle \le (1+\mu_n)\|x-y\|^2 + \xi_n \tag{1.10}$$

for all $n \ge 1,\ x, y \in C$. Put

$$\tau_n = \max\left\{0, \sup_{x,y\in C}(\langle T^n x - T^n y, j(x-y)\rangle - (1+\mu_n)\|x-y\|^2)\right\}.$$

Then, the class of total asymptotically pseudocontractive mappings in the intermediate sense is a proper subclass of the class of total asymptotically pseudocontractive mappings.

Remark 1.3. If $\tau_n = 0, \forall n \geq 1$, then the class of asymptotically pseudocontractive mappings in the intermediate sense reduces to the class of asymptotically pseudocontractive mappings. Again, if $h_n = 1, \forall\, n \geq 1$, then the class of asymptotically pseudocontractive mappings reduces to the class of pseudocontractive mappings.

From the implications in Remarks 1.2 and 1.3, it is clear that the class of total asymptotically pseudocontractive mappings properly contains all other classes of pseudocontractive mappings mentioned above.

Many iterative methods for approximating fixed points of total asymptotically pseudocontractive mappings have been studied by some authors. In 2011, Qin et. al. [5] proved a weak convergence theorem for a total asymptotically pseudocontractive mappings by the modified Ishikawa iterative process which was introduced by Schu [1]. In 2012, Ding and Quan [4] introduced a modified Mann iterative algorithm for a total asymptotically pseudocontractive mapping. Very recently, Chima and Osilike [3] studied the split common fixed point problem (SCFP) for a class of total asymptotically pseudocontractive mappings.

In recent years, the implicit iteration scheme for approximating fixed point of nonlinear mappings has been introduced and studied by various authors (see, e.g., [1, 5, 6, 7, 8, 9, 17, 22]). In 2001, Xu and Ori [22] introduced the following implicit iteration process for a finite family of nonexpansive self-mappings in Hilbert spaces:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n,\ n \geq 1, \tag{1.11}$$

where $\{\alpha_n\}$ is a sequence in [0,1] and $T_n = T_{n\,mod\,N}$. They proved that the sequence (1.11) converges weakly to a common fixed point of T_n, $n = 1, 2, 3, ..., N$. Later, Osilike and Akuchu [10], and Chen et al. [11] extended the iteration process (1.11) to a finite family of asymptotically pseudo-contractive mapping and a finite family of continuous pseudo-contractive self-mapping, respectively. Zhou and Chang [23] studied the convergence of a modified implicit iteration process to the common fixed point of a finite family of asymptotically nonexpansive mappings.

In 2003, Sun [17] modified the implicit iteration process of Xu and Ori [22] and applied the modified averaging iteration process for the approximation of fixed points of asymptotically quasi-nonexpansive mappings. Sun introduced the following implicit iteration process for common fixed points of a finite family of asymptotically quasi-nonexpansive mappings $\{T_i\}_{i=1}^N$ in Banach spaces:

$$\begin{cases} x_0 \in C, \\ x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} x_n, \end{cases} \qquad \forall n \geq 1, \tag{1.12}$$

where $\{\alpha_n\}$ is a sequence in [0,1], $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

Recently, Thakur et al. [18] considered the following implicit iterative process for approximating the common fixed point of a finite family $\{T_i\}_{i=1}^N$ of asymptotically pseudocontractive mappings and established convergence results in Banach spaces:

(1.13)

$$\begin{aligned} x_n &= (1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n, \\ y_n &= (1-\beta_n-\gamma_n-\delta_n)x_n + \beta_n x_{n-1} + \gamma_n T_{i(n)}^{k(n)} x_n + \delta_n T_{i(n)}^{k(n)} x_{n-1} \end{aligned}$$

$\forall n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in [0,1] satisfying $\beta_n+\gamma_n+\delta_n \leq 1$, $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

Motivated and inspired by the above facts, the purpose of this paper is to modify (1.13) for finite families of two uniformly L-Lipschitzian total asymptotically pseudocontractive mappings. By using a different approach, we prove that our new iterative algorithm (1.14) converges strongly to the common fixed points of finite families of two uniformly L-Lipschitzian total asymptotically pseudocontractive mappings in Banach spaces. We also give examples of mappings and prototype of the sequences which satisfies all the conditions in our main result. The result of this paper improves the corresponding result in [5, 11, 10, 18] and several others in the literature.

(1.14)

$$\begin{aligned} x_n &= (1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n, \\ y_n &= (1-\beta_n-\gamma_n-\delta_n)x_n + \beta_n x_{n-1} + \gamma_n S_{i(n)}^{k(n)} x_n + \delta_n T_{i(n)}^{k(n)} x_{n-1} \end{aligned}$$

$\forall n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in [0,1] satisfying $\beta_n+\gamma_n+\delta_n \leq 1$, $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

Remark 1.4. Clearly, our new iterative scheme (1.14), contains some well known iterative schemes in the literature. This is illustrated as follows:

(a) (1.14) reduces to (1.13) when $S_i = T_i$.
(b) (1.14) reduces to (1.12) when $\beta_n = \gamma_n = \delta_n = 0$.
(c) (1.14) reduces to (1.11) when $\beta_n = \gamma_n = \delta_n = 0$, $T^n = T$.
(d) When $\beta_n = 1$, $\gamma_n = \delta_n = 0$, $T^n = T$, $N = 1$ then (1.14) reduces to:

$$\left\{ \begin{array}{l} x_0 \in C, \\ x_n = (1-\alpha_n)x_{n-1} + \alpha_n T x_{n-1}, \end{array} \right. \qquad \forall n \geq 1, \tag{1.15}$$

where $\{\alpha_n\}$ is a sequence in [0,1]. The iterative process (1.15) is the well known Mann iterative process.

(e) If we take $\beta_n = 1$, $\gamma_n = \delta_n = 0$ in (1.14) then we obtain

$$\left\{ \begin{array}{l} x_0 \in C, \\ x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} x_{n-1}, \end{array} \right. \qquad \forall n \geq 1, \tag{1.16}$$

where $\{\alpha_n\}$ is a sequence in [0,1], $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

The above modified averaging iteration process (1.16) was considered in 2014 by Saluja [14] for the approximation of common fixed point of a finite family of strictly asymptotically pseudocontractive mappings in the intermediate sense in Hilbert spaces.

Now, we show that (1.14) can be employed to approximate the fixed points of total asymptotically pseudocontractive mappings which is assumed to be continuous. Let T_i be a L_t^i–Lipschitz total asymptotically pseudocontractive mapping with sequences $\lambda_n^i \in [0, \infty)$ and $\nu_n^i \in [0, \infty)$ with $\nu_n^i \to 0$ and $\lambda_n^i \to 0$ as $n \to \infty$ for each $1 \leq i \leq N$. Let S_i be a L_s^i–Lipschitz total asymptotically pseudocontractive mapping with sequences $\eta_n^i \in [0, \infty)$ and $l_n^i \in [0, \infty)$ with $\eta_n^i \to 0$ and $l_n^i \to 0$ as $n \to \infty$ for each $1 \leq i \leq N$.

Define a mapping $W_n : C \to C$ by

$$\begin{aligned} W_n(x) &= (1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)}[(1-\beta_n-\gamma_n-\delta_n)x + \beta_n x_{n-1} \\ &\quad + \gamma_n S_{i(n)}^{k(n)} x + \delta_n T_{i(n)}^{k(n)} x_{n-1}], \ \forall n \geq 1. \end{aligned} \tag{1.17}$$

From (1.17), we have

$$\begin{aligned} \|W_n(x) - W_n(y)\| &= \alpha_n \|T_{i(n)}^{k(n)}[(1-\beta_n-\gamma_n-\delta_n)x + \beta_n x_{n-1} + \gamma_n S_{i(n)}^{k(n)} x \\ &\quad + \delta_n T_{i(n)}^{k(n)} x_{n-1}] - T_{i(n)}^{k(n)}[(1-\beta_n-\gamma_n-\delta_n)y \\ &\quad + \beta_n x_{n-1} + \gamma_n S_{i(n)}^{k(n)} y + \delta_n T_{i(n)}^{k(n)} x_{n-1}]\| \\ &\leq \alpha_n L[(1-\beta_n-\gamma_n-\delta_n)\|x-y\| \\ &\quad + \gamma_n \|S_{i(n)}^{k(n)} x - S_{i(n)}^{k(n)} y\|] \\ &\leq \alpha_n L[(1-\beta_n-\gamma_n-\delta_n)\|x-y\| + \gamma_n L\|x-y\|] \\ &= \alpha_n L[(1-\beta_n+\gamma_n(L-1)-\delta_n)]\|x-y\|, \end{aligned} \tag{1.18}$$

for all $x, y \in C$, where $L = \max\{L_t^1, ..., L_t^N, L_s^1, ..., L_s^N\}$.

If $\alpha_n L[(1-\beta_n+\gamma_n(L-1)-\delta_n)] < 1$ for all $n \geq 1$, then from (1.18), it follows that W_n is a contraction mapping. By Banach contraction principle, we see that there exists a unique point $x_n \in C$ such that

$$\begin{aligned} x_n = W(x_n) &= (1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)}[(1-\beta_n-\gamma_n-\delta_n)x + \beta_n x_{n-1} \\ &\quad + \gamma_n S_{i(n)}^{k(n)} x + \delta_n T_{i(n)}^{k(n)} x_{n-1}], \ \forall n \geq 1. \end{aligned} \tag{1.19}$$

That is, the implicit iteration process (1.14) is well defined. Hence, the iterative sequence (1.14) can be employed for the approximation of common fixed points for a finite family of uniformly L–Lipschitzian total asymptotically pseudo-contractive mappings.

2. Preliminaries

The following definition, lemmas and proposition will be useful in proving our main results.

Definition 2.1 (see [2])**.** A family $\{T_i\}_{i=1}^N : C \to C$ with $\Im = \cap_{i=1}^N F(T_i) \neq \emptyset$ is said to satisfy *condition (B)* on C if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in C$

$$\max_{1 \leq i \leq N} \{\|x - T_i x\|\} \geq f(d(x, \Im)). \tag{2.1}$$

Lemma 2.2 (see [21])**.** *Let $J : X \to 2^{X^*}$ be the normalized duality mapping. Then for any $x, y \in X$, one has*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \ \forall j(x+y) \in J(x+y). \tag{2.2}$$

Lemma 2.3 (see [20])**.** *Let $\{\vartheta_n\}$, $\{\Lambda_n\}$ and $\{\Omega_n\}$ be sequences of nonnegative real numbers satisfying the following inequality:*

$$\vartheta_n \leq (1+\Lambda_n)\vartheta_n + \Omega_n, \ n \geq 1. \tag{2.3}$$

If $\sum_{n=1}^\infty \Lambda_n < \infty$ and $\sum_{n=1}^\infty \Omega_n < \infty$ then $\lim\limits_{n\to\infty} \vartheta_n$ exists, additionally, if $\{\vartheta_n\}$ has a subsequence $\{\vartheta_{n_i}\}$ such that $\vartheta_{n_i} \to 0$, then $\lim\limits_{n\to\infty} \vartheta_n = 0$.

3. MAIN RESULTS

Lemma 3.1. *Let C be a nonempty close convex subset of a real Banach space X, let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $T_i : C \to C$ be a finite family of uniformly L_t^i–Lipschitzian total asymptotically pseudocontractive mapping with sequences $\{\nu_n^i\} \subset [0, \infty)$ and $\{\lambda_n^i\} \subset [0, \infty)$, where $\nu_n^i \to 0$ and $\lambda_n^i \to 0$ as $n \to \infty$ and $S_i : K \to K$ be a finite uniformly L_s^i–Lipscitzian total asymptotically pseudocontractive mappings with sequences $\{\eta_n^i\} \subset [0, \infty)$ and $\{l_n^i\} \subset [0, \infty)$, where $\eta_n^i \to 0$ and $l_n^i \to 0$ as $n \to \infty$, for each $i \in I$. Let $\mu_n = \max\{\nu_n, \eta_n\}$, where $\nu_n = \max\{\nu_n^i : i \in I\}$ and $\eta_n = \max\{\eta_n^i : i \in I\}$. Let $\xi_n = \max\{\lambda_n, l_n\}$, where $\lambda_n = \max\{\lambda_n^i : i \in I\}$ and $l_n = \max\{l_n^i : i \in I\}$. Assume that $\Im = (\bigcap_{i=1}^N F(T_i)) \bigcap (\bigcap_{i=1}^N F(S_i)) \neq \emptyset$. Let $\phi(\wp) = \max\{\phi_i(\wp) : i \in I\}$, for each $\wp \geq 0$. Suppose that there exist $M, M^* > 0$ such that $\phi(k) \leq M^* k^2$ for all $k \geq M$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in [0,1] such that $\beta_n + \gamma_n + \delta_n \leq 1$ for each $n \geq 1$. Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;

(iii) $\sum_{n=1}^{\infty} \alpha_n \mu_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n \xi_n < \infty$;

(iv) $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n \gamma_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n \delta_n < \infty$;

(v) $\alpha_n L[(1 - \beta_n + \gamma_n(L - 1) - \delta_n)] < 1$, $\forall n \geq 1$, *where* $L = \max\{L_t^1, ..., L_t^N, L_s^1, ..., L_s^N\}$.

Then for arbitrary $x_0 \in C$, let $\{x_n\}$ be a sequence generated in (1.14). Then, $\lim_{n \to \infty} \|x_n - p\|$ exists for all $p \in \Im$.

Proof. For any $p \in \Im$, from (1.14) we have

$$
\begin{aligned}
\|y_n - p\| &= \|(1 - \beta_n - \gamma_n - \delta_n)x_n + \beta_n x_{n-1} + \gamma_n S_{i(n)}^{k(n)} x_n + \delta_n T_{i(n)}^{k(n)} x_{n-1} - p\| \\
&= \|(1 - \beta_n - \gamma_n - \delta_n)(x_n - p) + \beta_n(x_{n-1} - p) \\
&\quad + \gamma_n(S_{i(n)}^{k(n)} x_n - p) + \delta_n(T_{i(n)}^{k(n)} x_{n-1} - p)\| \\
&\leq (1 - \beta_n - \gamma_n - \delta_n)\|x_n - p\| + \beta_n\|x_{n-1} - p\| \\
&\quad + \gamma_n\|S_{i(n)}^{k(n)} x_n - p\| + \delta_n\|T_{i(n)}^{k(n)} x_{n-1} - p\| \\
&\leq \|x_n - p\| + \beta_n\|x_{n-1} - p\| \\
&\quad + \gamma_n\|S_{i(n)}^{k(n)} x_n - p\| + \delta_n\|T_{i(n)}^{k(n)} x_{n-1} - p\|.
\end{aligned} \tag{3.1}
$$

Now, from (1.14) and Lemma 2.2, we obtain

$$
\begin{aligned}
\|x_n - p\|^2 &= \|(1 - \alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n - p\|^2 \\
&= \|(1 - \alpha_n)(x_{n-1} - p) + \alpha_n(T_{i(n)}^{k(n)} y_n - p)\|^2 \\
&\leq (1 - \alpha_n)^2\|x_{n-1} - p\|^2 + 2\alpha_n\langle T_{i(n)}^{k(n)} y_n - p, j(x_n - p)\rangle \\
&= (1 - \alpha_n)^2\|x_{n-1} - p\|^2 + 2\alpha_n\langle T_{i(n)}^{k(n)} y_n - T_{i(n)}^{k(n)} x_n + T_{i(n)}^{k(n)} x_n - p, j(x_n - p)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\alpha_n)^2\|x_{n-1}-p\|^2 + 2\alpha_n\langle T_{i(n)}^{k(n)}y_n - T_{i(n)}^{k(n)}x_n, j(x_n-p)\rangle \\
&\quad +2\alpha_n\langle T_{i(n)}^{k(n)}x_n - p, j(x_n-p)\rangle \\
&\leq (1-\alpha_n)^2\|x_{n-1}-p\|^2 + 2\alpha_n\|T_{i(n)}^{k(n)}y_n - T_{i(n)}^{k(n)}x_n\|\|x_n-p\| \\
&\quad +2\alpha_n\langle T_{i(n)}^{k(n)}x_n - p, j(x_n-p)\rangle \\
&\leq (1-\alpha_n)^2\|x_{n-1}-p\|^2 + 2\alpha_n L\|y_n - x_n\|\|x_n-p\| \\
&\quad +2\alpha_n\langle T_{i(n)}^{k(n)}x_n - p, j(x_n-p)\rangle. \qquad (3.2)
\end{aligned}
$$

Using (1.14), we have

$$
\begin{aligned}
\|y_n - x_n\| &= \|[(1-\beta_n-\gamma_n-\delta_n)x_n + \beta_n x_{n-1} + \gamma_n S_{i(n)}^{k(n)}x_n + \delta_n T_{i(n)}^{k(n)}x_{n-1}] - x_n\| \\
&= \|\beta_n(x_{n-1}-x_n) + \gamma_n(S_{i(n)}^{k(n)}x_n - x_n) + \delta_n(T_{i(n)}^{k(n)}x_{n-1}-x_n)\| \\
&= \|\beta_n(x_{n-1}-p+p-x_n) + \gamma_n(S_{i(n)}^{k(n)}x_n - p + p - x_n) \\
&\quad +\delta_n(T_{i(n)}^{k(n)}x_{n-1}-p+p-x_n)\| \\
&\leq \beta_n\|x_{n-1}-p\| + \beta_n\|x_n-p\| + \gamma_n\|S_{i(n)}^{k(n)}x_n - p\| + \gamma_n\|x_n-p\| \\
&\quad +\delta_n\|T_{i(n)}^{k(n)}x_{n-1}-p\| + \delta_n\|x_n-p\| \\
&\leq \beta_n\|x_{n-1}-p\| + \beta_n\|x_n-p\| + \gamma_n L\|x_n-p\| + \gamma_n\|x_n-p\| \\
&\quad +\delta_n L\|x_{n-1}-p\| + \delta_n\|x_n-p\| \\
&= (\beta_n+\delta_n L)\|x_{n-1}-p\| + (\beta_n+\gamma_n L+\gamma_n+\delta_n)\|x_n-p\| \\
&= (\beta_n+\delta_n L)\|x_{n-1}-p\| \\
&\quad +(\beta_n+\gamma_n(L+1)+\delta_n)\|x_n-p\|. \qquad (3.3)
\end{aligned}
$$

Putting (3.3) into (3.2), we have

$$
\begin{aligned}
\|x_n-p\|^2 &\leq (1-\alpha_n)^2\|x_{n-1}-p\|^2 + 2\alpha_n L[(\beta_n+\delta_n L)\|x_{n-1}-p\| \\
&\quad +(\beta_n+\gamma_n(L+1)+\delta_n)\|x_n-p\|]\|x_n-p\| \\
&\quad +2\alpha_n\langle T_{i(n)}^{k(n)}x_n - p, j(x_n-p)\rangle \\
&= (1-\alpha_n)^2\|x_{n-1}-p\|^2 + 2\alpha_n L(\beta_n+\delta_n L)\|x_{n-1}-p\|\|x_n-p\| \\
&\quad +2\alpha_n L(\beta_n+\gamma_n(L+1)+\delta_n)\|x_n-p\|^2 \\
&\quad +2\alpha_n\langle T_{i(n)}^{k(n)}x_n - p, j(x_n-p)\rangle. \qquad (3.4)
\end{aligned}
$$

From classical analysis, it is well known that

$$
\|x_{n-1}-p\|\|x_n-p\| \leq \frac{1}{2}(\|x_{n-1}-p\|^2 + \|x_n-p\|^2). \qquad (3.5)
$$

Using (3.4) and (3.5), we have

$$\begin{aligned}
\|x_n - p\|^2 &\leq (1-\alpha_n)^2\|x_{n-1}-p\|^2 + 2\alpha_n L(\beta_n + \delta_n L) \\
&\quad \times \frac{1}{2}(\|x_{n-1}-p\|^2 + \|x_n - p\|^2) \\
&\quad + 2\alpha_n L(\beta_n + \gamma_n(L+1) + \delta_n)\|x_n - p\|^2 \\
&\quad + 2\alpha_n \langle T_{i(n)}^{k(n)} x_n - p, j(x_n - p)\rangle \\
&= [(1-\alpha_n)^2 + \alpha_n L(\beta_n + \delta_n L)]\|x_{n-1}-p\|^2 \\
&\quad + [\alpha_n L(\beta_n + \delta_n L) + 2\alpha_n L(\beta_n + \gamma_n(L+1) + \delta_n)]\|x_n - p\|^2 \\
&\quad + 2\alpha_n \langle T_{i(n)}^{k(n)} x_n - p, j(x_n - p)\rangle. \qquad (3.6)
\end{aligned}$$

Since each T_i $(i = 1, 2..., N)$ is a total asymptotically pseudocontractive mapping, from (3.6) we have

$$\begin{aligned}
\|x_n - p\|^2 &\leq [(1-\alpha_n)^2 + \alpha_n L(\beta_n + \delta_n L)]\|x_{n-1}-p\|^2 \\
&\quad + [\alpha_n L(\beta_n + \delta_n L) + 2\alpha_n L(\beta_n + \gamma_n(L+1) + \delta_n)]\|x_n - p\|^2 \\
&\quad + 2\alpha_n(\|x_n - p\|^2 + \mu_n \phi(\|x_n - p\|) + \xi_n) \\
&= [(1-\alpha_n)^2 + \alpha_n L(\beta_n + \delta_n L)]\|x_{n-1}-p\|^2 \\
&\quad + [\alpha_n L(\beta_n + \delta_n L) + 2\alpha_n L(\beta_n + \gamma_n(L+1) + \delta_n) + 2\alpha_n]\|x_n - p\|^2 \\
&\quad + 2\alpha_n \mu_n \phi(\|x_n - p\|) + 2\alpha_n \xi_n. \qquad (3.7)
\end{aligned}$$

Since ϕ is a strictly increasing function, it follows that $\phi(k) \leq \phi(M)$, if $k \leq M$; $\phi(k) \leq M^* k^2$, if $k \geq M$. In either case, we can obtain

$$\phi(k) \leq \phi(M) + M^* k^2. \qquad (3.8)$$

Using (3.7) and (3.8), we obtain

$$\begin{aligned}
\|x_n - p\|^2 &\leq [(1-\alpha_n)^2 + \alpha_n L(\beta_n + \delta_n L)]\|x_{n-1}-p\|^2 \\
&\quad + [\alpha_n L(\beta_n + \delta_n L) + 2\alpha_n L(\beta_n + \gamma_n(L+1) + \delta_n) + 2\alpha_n]\|x_n - p\|^2 \\
&\quad + 2\alpha_n \mu_n \phi(M) + 2\alpha_n M^* \mu_n \|x_n - p\|^2 + 2\alpha_n \xi_n \\
&= [(1-\alpha_n)^2 + \alpha_n L(\beta_n + \delta_n L)]\|x_{n-1}-p\|^2 \\
&\quad + [\alpha_n L(\beta_n + \delta_n L) + 2\alpha_n L(\beta_n + \gamma_n(L+1) + \delta_n) \\
&\quad + 2\alpha_n + 2\alpha_n M^* \mu_n]\|x_n - p\|^2 + 2\alpha_n \mu_n \phi(M) + 2\alpha_n \xi_n \\
&= \psi_n \|x_{n-1}-p\|^2 + \Psi_n \|x_n - p\|^2 + 2\alpha_n \mu_n \phi(M) + 2\alpha_n \xi_n, \qquad (3.9)
\end{aligned}$$

where

$$\begin{aligned}
\psi_n &= (1-\alpha_n)^2 + \alpha_n L(\beta_n + \delta_n L), \\
\Psi_n &= \alpha_n L(\beta_n + \delta_n L) + 2\alpha_n L(\beta_n + \gamma_n(L+1) + \delta_n) \\
&\quad + 2\alpha_n + 2\alpha_n M^* \mu_n.
\end{aligned}$$

By transposing and simplifying (3.9), we obtain

$$
\begin{aligned}
\|x_n - p\|^2 &\le \frac{\psi_n}{1-\Psi_n}\|x_{n-1}-p\|^2 + \frac{2\alpha_n\mu_n\phi(M)}{1-\Psi_n} + \frac{2\alpha_n\xi_n}{1-\Psi_n} \\
&= \left(1+\frac{\psi_n+\Psi_n-1}{1-\Psi_n}\right)\|x_{n-1}-p\|^2 + \frac{2\alpha_n\mu_n\phi(M)}{1-\Psi_n} \\
&\quad + \frac{2\alpha_n\xi_n}{1-\Psi_n}. \qquad (3.10)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\psi_n + \Psi_n - 1 &= \alpha_n^2 + 2\alpha_n L(\beta_n+\delta_n L) + 2\alpha_n L(\beta_n + \gamma_n(L+1)+\delta_n) \\
&\quad + 2\alpha_n M^*\mu_n.
\end{aligned}
$$

Now, set

$$
\varpi_n = \psi_n + \Psi_n - 1. \qquad (3.11)
$$

Since $\lim_{n\to\infty}\alpha_n = 0$, then from conditions (iii) and (iv), we obtain

$$
\begin{aligned}
\Psi_n &= \alpha_n L(\beta_n+\delta_n L) + 2\alpha_n L(\beta_n+\gamma_n(L+1)+\delta_n) \\
&\quad + 2\alpha_n + 2\alpha_n M^*\mu_n \to 0 \text{ as } n\to\infty,
\end{aligned}
$$

therefore, there exists a positive integer n_0 such that

$$
\frac{1}{2} < 1 - \Psi_n \le 1, \ \forall\ n \ge n_0.
$$

Thus, from (3.10) we have

$$
\begin{aligned}
\|x_n - p\|^2 &\le (1+2\varpi_n)\|x_{n-1}-p\|^2 + 4\alpha_n\mu_n\phi(M) + 4\alpha_n\xi_n \\
&= (1+\zeta_n)\|x_{n-1}-p\|^2 + \sigma_n,, \ \forall\ n\ge n_0, \qquad (3.12)
\end{aligned}
$$

where

$$
\begin{aligned}
\zeta_n &= 2\varpi_n, \\
\sigma_n &= 4\alpha_n\mu_n\phi(M) + 4\alpha_n\xi_n.
\end{aligned}
$$

From conditions (ii)–(iv), it is easy to see that $\sum_{n=1}^{\infty}\zeta_n < \infty$ and $\sum_{n=1}^{\infty}\sigma_n < \infty$. Clearly, from (3.12), we see that all the conditions of Lemma 2.3 are satisfied. Thus, $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in \Im$. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Banach space X. Let $N \ge 1$ be a positive integer and $I = \{1,2,3,...,N\}$. Let $T_i : C \to C$ be a finite family of uniformly L_t^i–Lipschitzian total asymptotically pseudocontractive mapping with sequences $\{\nu_n^i\} \subset [0,\infty)$ and $\{\lambda_n^i\} \subset [0,\infty)$, where $\nu_n^i \to 0$ and $\lambda_n^i \to 0$ as $n\to\infty$ and $S_i : C \to C$ be a finite uniformly L_s^i–Lipscitzian total asymptotically pseudocontractive mappings with sequences $\{\eta_n^i\} \subset [0,\infty)$ and $\{l_n^i\} \subset [0,\infty)$, where $\eta_n^i \to 0$ and $l_n^i \to 0$ as $n\to\infty$, for each $i \in I$. Let $\mu_n = \max\{\nu_n, \eta_n\}$, where $\nu_n = \max\{\nu_n^i : i \in I\}$ and $\eta_n = \max\{\eta_n^i : i\in I\}$. Let $\xi_n = \max\{\lambda_n, l_n\}$, where $\lambda_n = \max\{\lambda_n^i : i \in I\}$ and $l_n = \max\{l_n^i : i \in I\}$. Assume that $\Im = (\bigcap_{i=1}^N F(T_i))\bigcap(\bigcap_{i=1}^N F(S_i)) \ne \emptyset$. Let $\phi(\wp) = \max\{\phi_i(\wp) : i \in I\}$, for each $\wp \ge 0$. Suppose that there exist $M, M^* > 0$ such that $\phi(k) \le M^*k^2$ for all $k \ge M$. Let $\{\alpha_n\}$, $\{\beta_n\}$,*

$\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in [0,1] such that $\beta_n+\gamma_n+\delta_n \leq 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (1.14). Assume that the following conditions are satisfied:

(i) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n^2 < \infty$;

(iii) $\sum\limits_{n=1}^{\infty} \alpha_n\mu_n < \infty,\ \sum\limits_{n=1}^{\infty} \alpha_n\xi_n < \infty$;

(iv) $\sum\limits_{n=1}^{\infty} \alpha_n\beta_n < \infty,\ \sum\limits_{n=1}^{\infty} \alpha_n\gamma_n < \infty,\ \sum\limits_{n=1}^{\infty} \alpha_n\delta_n < \infty$;

(v) $\alpha_n L[(1-\beta_n+\gamma_n(L-1)-\delta_n)] < 1,\ \forall n \geq 1,\ L=\max\{L_t^1,...,L_t^N,L_s^1,...,L_s^N\}$.

Hence the sequence $\{x_n\}$ converges strongly to a common fixed point in $\Im$ if and only if

$$\liminf_{n\to\infty} d(x_n,\Im)=0, \tag{3.13}$$

where $d(x,\Im)$ denotes the distance of x to set $\Im$, i.e., $d(x,\Im)=\inf_{y\in\Im} d(x,y)$.

Proof. The necessity of condition (3.13) is obvious.

Next, we prove the sufficiency of Theorem 3.2. For any given $p \in F$, from (3.12) in Lemma 3.1 we have that

$$[d(x_n,\Im)]^2 \leq (1+\zeta_n)[d(x_{n-1},\Im)]^2+\sigma_n,\ \forall\, n \geq n_0. \tag{3.14}$$

Clearly, from conditions (ii), (iii) and (iv), it is easy to see that $\sum\limits_{n=1}^{\infty} \zeta_n < \infty$ and $\sum\limits_{n=1}^{\infty} \sigma_n < \infty$. It follows from (3.14) and Lemma 2.3 that $\lim\limits_{n\to\infty}[d(x_n,\Im)]^2$ exists, further, $\lim\limits_{n\to\infty} d(x_n,\Im)$ exists. By condition (3.13), we get

$$\lim_{n\to\infty} d(x_n,\Im)=0. \tag{3.15}$$

Next we prove that the sequence $\{x_n\}$ is a Cauchy sequence in C. Clearly, since $\sum_{n=1}^{\infty}\sigma_n < \infty$, then $1+t \leq e^t$ for all $t>0$ and from (3.12) we therefore have

$$\|x_n-p\|^2 \leq e^{\zeta_n}\|x_{n-1}-p\|^2+\sigma_n,\ \geq n_0. \tag{3.16}$$

Hence, for any positive integers $n,m \geq n_0$, from (3.16) we have

$$\begin{aligned}
\|x_{n+m}-p\|^2 &\leq e^{\zeta_{n+m}}\|x_{n+m-1}-p\|^2+\sigma_{n+m}\\
&\leq e^{\zeta_{n+m}}[e^{\zeta_{n+m-1}}\|x_{n+m-2}-p\|^2+\sigma_{n+m-1}]+\sigma_{n+m}\\
&\leq e^{\zeta_{n+m}+\zeta_{n+m-1}}\|x_{n+m-2}-p\|^2+\sigma_{n+m-1}]+\sigma_{n+m}\\
&\leq \cdots\\
&\leq e^{\sum_{i=n+1}^{n+m}\zeta_i}\|x_n-p\|^2+e^{\sum_{i=n+2}^{n+m}\zeta_i}\sum_{i=n+1}^{n+m}\sigma_i\\
&\leq \vartheta\|x_n-p\|^2+\vartheta\sum_{i=n+1}^{\infty}\sigma_i, \qquad (3.17)
\end{aligned}$$

where $\vartheta=e^{\sum_{n=1}^{\infty}\zeta_n}<\infty$.

Since $\lim\limits_{n\to\infty} d(x_n, \Im) = 0$ and $\lim\limits_{n\to\infty} \sigma_n < \infty$, for any given $\epsilon > 0$, there exists a positive integer $n_1 \geq n_0$ such that

$$[d(x_n, \Im)]^2 < \frac{\epsilon^2}{8(\vartheta+1)}, \quad \sum_{i=n+1}^{\infty} \sigma_i < \frac{\epsilon^2}{4\vartheta}, \ \forall\, n \geq n_1. \tag{3.18}$$

Therefore there exists $p_1 \in \Im$ such that

$$\|x_n - p_1\|^2 < \frac{\epsilon^2}{8(\vartheta+1)}, \ \forall\, n \geq n_1. \tag{3.19}$$

Consequently, for any $n \geq n_1$ and for all $m \geq 1$ we have

$$\begin{aligned} \|x_{n+m} - x_n\|^2 &\leq 2(\|x_{n+m} - p_1\|^2 + \|x_n - p_1\|^2) \\ &\leq 2(1+\vartheta)\|x_n - p_1\|^2 + 2\vartheta \sum_{i=n+1}^{\infty} \sigma_i \\ &< 2 \cdot \frac{\epsilon^2}{4(\vartheta+1)}(1+\vartheta) + 2\vartheta \cdot \frac{\epsilon^2}{4\vartheta} \\ &= \epsilon^2, \end{aligned}$$

i.e.,

$$\|x_{n+m} - x_n\| < \epsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence in C. By the completeness of C, we can assume that $x_n \to x^* \in C$.

Now, we have to prove that $p^* \in \Im$. By contradiction, we assume that p^* is not in $\Im = (\bigcap_{i=1}^N F(T_i)) \bigcap (\bigcap_{i=1}^N F(S_i)) \neq \emptyset$. Since $\Im$ is a closed subset of E, we have that $d(p^*, \Im) > 0$. Thus, for all $p^* \in \Im$, we have

$$\|p^* - p\| \leq \|p^* - x_n\| + \|x_n - p\|, \tag{3.20}$$

which implies that

$$d(p^*, \Im) \leq \|x_n - p^*\| + d(x_n, \Im), \tag{3.21}$$

so that, we obtain $d(p^*, \Im) = 0$ as $n \to \infty$, which contradicts $d(p^*, \Im) > 0$. Hence, $p^* \in \Im$. This completes the proof. □

We obtain the following results immediately from Theorem 3.2

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Banach space X. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $T_i : C \to C$ be a finite family of uniformly L_t^i–Lipschitzian total asymptotically pseudocontractive mapping with sequences $\{\nu_n^i\} \subset [0, \infty)$ and $\{\lambda_n^i\} \subset [0, \infty)$, where $\nu_n^i \to 0$ and $\lambda_n^i \to 0$ as $n \to \infty$, for each $i \in I$. Let $\mu_n = \max\{\nu_n^i : i \in I\}$ and $\xi_n = \max\{\lambda_n^i : i \in I\}$. Assume that $\Im = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\phi(\wp) = \max\{\phi_i(\wp) : i \in I\}$, for each $\wp \geq 0$. Suppose that there exist $M, M^* > 0$ such that $\phi(k) \leq M^* k^2$ for all $k \geq M$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in [0,1] such that $\beta_n + \gamma_n + \delta_n \leq 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (1.13). Assume that the following conditions are satisfied:*

(i) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n^2 < \infty$;

(iii) $\sum\limits_{n=1}^{\infty} \alpha_n\mu_n < \infty$, $\sum\limits_{n=1}^{\infty} \alpha_n\xi_n < \infty$;

(iv) $\sum\limits_{n=1}^{\infty} \alpha_n\beta_n < \infty$, $\sum\limits_{n=1}^{\infty} \alpha_n\gamma_n < \infty$, $\sum\limits_{n=1}^{\infty} \alpha_n\delta_n < \infty$;

(v) $\alpha_n L[(1-\beta_n+\gamma_n(L-1)-\delta_n)] < 1$, $\forall n \geq 1$, $L = \max\{L_t^1, ..., L_t^N, L_s^1, ..., L_s^N\}$.

Then the sequence $\{x_n\}$ *converges strongly to a common fixed point in* $\Im$ *if and only if*

$$\liminf_{n\to\infty} d(x_n, \Im) = 0, \tag{3.22}$$

where $d(x, \Im)$ *denotes the distance of* x *to set* $\Im$*, i.e.,* $d(x, \Im) = \inf_{y\in\Im} d(x, y)$.

Proof. Set $S_i = T_i$ in Theorem 3.2, then we obtain the required result. □

Corollary 3.4. *Let* C *be a nonempty closed convex subset of a real Banach space* X*. Let* $N \geq 1$ *be a positive integer and* $I = \{1, 2, 3, ..., N\}$*. Let* $T_i : C \to C$ *be a finite family of uniformly* L_t^i*–Lipschitzian total asymptotically pseudocontractive mapping with sequences* $\{\nu_n^i\} \subset [0, \infty)$ *and* $\{\lambda_n^i\} \subset [0, \infty)$ *, where* $\nu_n^i \to 0$ *and* $\lambda_n^i \to 0$ *as* $n \to \infty$*, for each* $i \in I$*. Let* $\mu_n = \max\{\nu_n^i : i \in I\}$ *and* $\xi_n = \max\{\lambda_n^i : i \in I\}$*. Assume that* $\Im = \bigcap_{i=1}^N F(T_i) \neq \emptyset$*. Let* $\phi(\wp) = \max\{\phi_i(\wp) : i \in I\}$*, for each* $\wp \geq 0$*. Suppose that there exist* $M, M^* > 0$ *such that* $\phi(k) \leq M^*k^2$ *for all* $k \geq M$*. Let* $\{\alpha_n\}$ *be a sequence in [0,1]. Let* $\{x_n\}$ *be a sequence generated in* (1.12)*. Assume that the following conditions are satisfied:*

(i) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n^2 < \infty$;

(iii) $\sum\limits_{n=1}^{\infty} \alpha_n\mu_n < \infty$, $\sum\limits_{n=1}^{\infty} \alpha_n\xi_n < \infty$;

(v) $\alpha_n L < 1$, $\forall n \geq 1$, $L = \max\{L_t^1, ..., L_t^N, L_s^1, ..., L_s^N\}$.

Then the sequence $\{x_n\}$ *converges strongly to a common fixed point in* $\Im$ *if and only if*

$$\liminf_{n\to\infty} d(x_n, \Im) = 0, \tag{3.23}$$

where $d(x, \Im)$ *denotes the distance of* x *to set* $\Im$*, i.e.,* $d(x, \Im) = \inf_{y\in\Im} d(x, y)$.

Proof. Put $\beta_n = \gamma_n = \delta_n = 0$ in Corollary 3.3, then we obtain the required result. □

These are just but a few of the numerous results that can be obtained from Theorem 3.2

Example 3.5. Let X be the real line with the usual metric $|.|$ and let $C = (-1, 1)$. Now for $N = 1$, let $Tx = \sin x$ and $Sx = \sin(-x)$ for all $x \in C$. Let ϕ be a strictly increasing continuous function such that $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$. Let $\{\mu_n\}_{n\geq1}$ and $\{\xi_n\}_{n\geq1}$ in $\mathbb{R}^+$ be two sequences defined by $\mu_n = \xi_n = \frac{1}{n+1}$, for all $n \geq 1$, then $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$. Clearly, the mappings T and S are total asymptotically nonexpansive mappings with $F(T) = F(S) = \{0\}$. This implies that the mappings T and S are total asymptotically pseudocontractive mappings. Obviously, $\Im = F(T) \bigcap F(S) = \{0\} \neq \emptyset$. Put

$$\alpha_n = \frac{1}{n}, \beta_n = \gamma_n = \frac{1}{2n+1} \text{ and } \delta_n = \frac{1}{n+1}$$

for all $n \geq 1$.

From Example 3.5, we see that all the conditions in Theorem 3.2 are satisfied. Hence, our result is applicable.

4. Conclusion

In this paper, we have seen that the classes of nonexpansive mappings and classes pseudocontractive mappings are proper subclasses of the class of total asymptotically pseudocontractive mappings. Also, we have demonstrated that our new iterative algorithm properly contains several well known iterative schemes which have been considered by Osilike and Akuchu [10], Qin et al. [12], Thakur [18], Saluja [14], Chen [11], Xu and Ori [21] and several others in the existing literature. Hence, our results generalize, improve and extend the corresponding results of Osilike and Akuchu [10], Qin et al. [12], Thakur [18], Saluja [14], Chen [11], Xu and Ori [21] and several others in the existing literature.

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References

[1] G. L. Acedo and H.K. Xu, Iterative methods for strict pseudo-contractions in Hilbert spaces, Nonlinear Anal., **67**, 2258–2271, 2007.

[2] C. E. Chidume and N. Shahzad, Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings, Nonlinear Anal., TMA, **62**(6), 1149–1156, 2005.

[3] E. E. Chima and M. O. Osilike, Split Common Fixed Point Problem for a Class of Total Asymptotic Pseudocontractions, Journal of Applied Mathematics Volume 2016, Article ID 3435078, 10 pages.

[4] C. Ding and J. Quan, A Strong Convergence Theorem for Total Asymptotically Pseudocontractive Mappings in Hilbert Spaces, Abstract and Applied Analysis, Vol. 2012, Article ID 127851, 8 pages, 2012.

[5] G. Marino and H. K. Xu, Weak and strong convergence theorems for strict pseudocontractions in Hilbert spaces, J. Math. Anal. Appl., **329**, 336–346, 2007.

[6] A. E. Ofem, *Strong convergence of modified implicit hybrid S-iteration scheme for finite family of nonexpansive and asymptotically generalized Φ-hemicontractive mappings*, Malaya Journal of Matematik, **8**(4), 1643–1649, 2020

[7] A. E. Ofem, *Strong convergence of a multi-step implicit iterative scheme with errors for common fixed points of uniformly L-Lipschitzian total asymptotically strict pseudocontractive mappings*, Results in Nonlinear Analysis, **3**(2), 100–116, 2020.

[8] A. E. Ofem, D. I. Igbokwe, X. A. Udo-utun, Implicit iteration process for Lipschitzian α-hemicontraction semigroup, MathLAB Journal, **7**, 43–52, 2020.

[9] M.O. Osilike, Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps, J. Math. Anal. Appl., **294**(1), 73–81, 2004.

[10] M.O. Osilike, B.G Akuchu, Common fixed points of finite family of asymptotically pseudocontractive maps, Fixed Point Theory Appl., 2004, 81–88, 2004.

[11] R.D. Chen, Y.S. Song, H. Zhou, Convergence theorems for implicit iteration process for a finite family of continuous pseudocontractive mappings, J. Math. Anal. Appl., 314, 701–706, 2006.

[12] X. Qin, S. Y. Cho, and J. K. Kim, Convergence theorems on asymptotically pseudocontractive mappings in the intermediate sense, Fixed Point Theory and Applications, vol. 2010, Article ID 186874, 14 pages, 2010.

[13] X. Qin, S. Y. Cho, and S. M. Kang, Weak Convergence Theorem for Total Asymptotically Pseudocontractive Mappings in Hilbert Spaces, Fixed Point Theory and Applications, Vol. 2011, Article ID 859795, 11 pages, 2011.

[14] G. S. Saluja, Convergence of the explicit iteration method for strictly asymptotically pseudocontractive mappings in the intermediate sense, Novi Sad J. Math., **44**(1), 75-90, 2014.

[15] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, Journal of Mathematical Analysis and Applications, **158**(2), 407–413, 1991.

[16] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bulletin of the Australian Mathematical Society, vol. **43**(1), 153–159, 1991.

[17] Z. H. Sun, Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings, J. Math. Anal. Appl., **286**, 351–358, 2003.

[18] B. S. Thakur, R. Dewangan and M. Postolache, general composite implicit iteration process for a finite family of asymptotically pseudo-contractive mappings, Thakur et al. Fixed Point Theory and Applications 2014, 2014:90.

[19] B. E. Rhoades, Comments on two fixed point iteration methods, Journal of Mathematical Analysis and Applications, **56**(3), 741–750, 1976.

[20] Y. Wang and C. Wang, Convergence of a new modified Ishikawa type iteration for common fixed Points of total asymptotically strict pseudocontractive semigroups, Volume 2013, Article ID 319241, 7 pages.

[21] H. K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal., **16**, 1127–1138, 1991.

[22] H.K. Xu and R.G. Ori, An implicit iteration process for nonexpansive mappings, Numer. Funct. Anal. Optim., **22**, 767–773, 2001.

[23] Y. Zhou and S. S. Chang, Convergence of implicit iteration process for a finite family of asymptotically nonexpansive mappings in Banach spaces, Numer. Funct. Anal. Optim., **23**, 911–921, 2002.

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