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Research Paper

A NEW APPROACH TO THE CHROMATIC POLYNOMIAL STRUCTURE ON FINSLER MANIFOLDS

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Abstract. In this paper, the chromatic polynomial structure on Riemannian manifolds and the almost golden structure on the tangent bundle of a Finsler manifold have been studied. A class of g-natural metrics on the tangent bundle of a Finsler manifold have been considered and some conditions under which the golden structure is compatible with the above-mentioned metric are proposed. The Levi-Civita connection associated with the mentioned metric is calculated and the results of it are presented. Finally, the integrability of the golden structure and its compatibility with the covariant derivative is studied.

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1. Introduction and preliminary concepts

The golden structure has been used in many different areas, in architecture, music, and arts. Research on the properties of the golden structure on manifolds is an interesting topic in mathematics. Hretcanu and Crasmareno in [3] investigated the geometry of the golden structure on manifolds and in [7] presented some applications of the golden ratio in differential geometry. The integrability of this structure has been investigated in [5]. Later, authors in [4] discussed the compatible Riemannian metrics and adapted covariant derivatives to a golden structure. The past thirty years have seen increasingly rapid advances in the field of polynomial structures on the tangent bundle of a Riemannian manifold [9]. In [12], Özkan studied the complete and horizontal lifts of golden structures in the tangent bundle.

Analogously to the geometry of the polynomial structures of a Riemannian manifold, the geometry of the polynomial structures of a Finsler manifold has not been studied. Only a few authors have studied these structures on the Finsler manifolds. Bing Ye Wu with consideration of the Sasaki metric studied the complex structures in the tangent bundle of a Finsler manifold.

Recently, some researchers have focused on Finsler manifolds and obtained many geometrical results in such spaces. For example, in [1] M. T. K. Abbassi and G. Calvaruso investigated geometric properties of "g-natural" metrics on the tangent bundle TM. Peyghan and Tayebi in [13] showed the almost complex structure is not compatible with the Miron metric, but they provided another definition of the complex structure that is compatible with the Miron metric.

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Our study will firstly focus on the metric $\hat{g}$ on TM^0. We consider the (M, F) as a finsler metric with the Chern connection and obtain the Levi-Civita connection of the metric $\hat{g}$ that is defined in section 2. We proved that the map F has constant relatively isotropic Landsberg $\mathbf{L} = -\frac{1}{2}\mathbf{C}$ if and only if the vertical distribution $\mathcal{V}T\hat{M}^0$ is totally geodesic in $TT\hat{M}^0$. We next prove that F is a weakly Berward metric if $di\hat{v}(X^v) = 0$.

In recent years, several authors have studied the polynomial structures such as golden structure, metallic structure, product, and complex structures of manifolds [8]. In 2008, Mirecea Grasmareanu and Cristina Elena examined the golden structure on differentiable manifolds (see [3]). Kazan et al later proved that the metallic structure $\tilde{J}$ on the tangent bundle TM with Levi-Civita connection ∇ and Riemannian curvature tensor R is integrable if and only if Riemannian manifold M is flat, i.e. $R \equiv 0$.

In section 3, we give the almost golden structure $\hat{\phi}$ on the slice tangent bundle $TT\hat{M}^0$. We show that $\hat{\phi}$ is a golden structure on $TT\hat{M}^0$ if and only if the metric F is of constant flag curvature [10]. Later, we examine the almost golden structure compatibility with the Levi-Civita connection on TM^0. We show that with metric $\hat{g}$ the almost golden structure $\hat{\phi}$ is compatible with covariant derivative $\hat{\nabla}$ if and only if M is a flat Riemannian manifold. Throughout the paper, all manifolds, connections,... are assumed to be differentiable of class C^∞. All data generated or analyzed during this study are included in this published article. In this section, we give a brief presentation of some definitions and propositions that use in the rest of this paper. For general background in the subject, the reader can consult [2]. Suppose that M is an $n-$dimensional C^∞ manifold and let T_xM be the tangent space of M at x. The tangent bundle TM is the union of all tangent space to M at all points $x \in M$, that is

$$TM := \bigcup_{x\in M} T_xM = \{(x, y)\,|x \in M,\ y \in T_xM\}.$$

Let $TM^0 = TM\backslash 0$, the natural bundle projection $\pi : TM \longrightarrow M$ is given by $\pi(x, y) = x$. A Finsler manifold is a manifold M and a function $F : TM \longrightarrow [0, \infty)$ such that,

i. the function F is smooth on TM^0 (on the entire tangent bundle TM^0),
ii. the function F is positive $1-$homogeneous $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$,
iii. the Hessian matrix $(g_{ij}) = \left[\frac{1}{2}F_2(x, y)\right]_{y^iy^j}$ is positive- definite at every point of TM^0.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0} = \frac{1}{4}\frac{\partial^3}{\partial r\partial s\partial t}\Big[F^2(y + ru + sv + tw)\Big]_{r=s=t=0},$$

where $u, v, w \in T_xM$. By definition, $\mathbf{C}_y$ is a symmetric trilinear form on T_xM. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by $\mathbf{I}_y(u) = \sum_{i=1}^n g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j)$, where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. Thus, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

Define the Landsberg curvature $\mathbf{L}_y : T_xM\times T_xM\times T_xM \to \mathbb{R}$ by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$ where $L_{ijk} := C_{ijk|s}y^s$. F is called a Landsberg metric if $\mathbf{L} = 0$. A Finsler metric F on a manifold M is called of relatively isotropic Landsberg curvature if $\mathbf{L} + cF\mathbf{C} = 0$, where $c = c(x)$ is a scalar function on M.

The horizontal covariant derivatives of the mean Cartan torsion $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y : T_xM \to \mathbb{R}$ which are defined by $\mathbf{J}_y(u) := J_i(y)u^i$, where $J_i := I_{i|s}y^s$. The family $\mathbf{J} := \{\mathbf{J}_y\}_{y\in TM_0}$ is called the mean Landsberg curvature. A Finsler metric F on a manifold M is called of relatively isotropic mean Landsberg curvature if $\mathbf{J} + cF\mathbf{I} = 0$, where $c = c(x)$ is a scalar function on M.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4}g^{il}\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\}, \quad y \in T_xM.$$

The $\mathbf{G}$ is called the spray associated with (M, F). In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For $y \in T_xM_0$, define the Berwald curvature $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ and the mean Berwald curvature $\mathbf{E}_y : T_xM \times T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y)u^jv^k$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2}B^m{}_{jkm},$$

$u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. Finsler metrics satisfying $\mathbf{B} = 0$ or $\mathbf{E} = 0$ are called Berwald metrics and weakly Berwald metrics, respectively.

Then, for a non-zero vector $y \in T_xM_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_xM \to T_xM$ with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2\mathbf{R}_y$, $\forall\lambda > 0$ which is defined by $\mathbf{R}_y(u) := R^i_k(y)u^k\frac{\partial}{\partial x^i}$, where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{1.1}$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y\in TM_0}$ is called the Riemann curvature. Let us put

$$R^i{}_{kl} := \frac{1}{3}\Big\{\frac{\partial R^i_k}{\partial y^l} - \frac{\partial R^i_l}{\partial y^k}\Big\}, \quad R^i_{j\ kl} := \frac{1}{3}\Big\{\frac{\partial^2 R^i_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i_l}{\partial y^j \partial y^k}\Big\}. \tag{1.2}$$

Then

$$R^i_k = R^i_{j\ kl}y^jy^l, \quad R^i{}_{kl} = R^i_{j\ kl}y^j, \quad R^i_{j\ kl} + R^i_{j\ lk} = 0, \tag{1.3}$$

$$R^h{}_{ijk} + R^h{}_{jki} + R^h{}_{kij} = 0. \tag{1.4}$$

Lemma 1.1. The Riemann curvature $\mathbf{R}_y$ is well-defined linear transformation satisfying following

$$\mathbf{R}_y(y) = 0, \qquad \mathbf{g}_y\big(\mathbf{R}_y(u), v\big) = \mathbf{g}_y\big(u, \mathbf{R}_y(v)\big).$$

Let π be natural projection map of TM to M. We obtain the tangent mapping of the projection π follows:

$$\pi_* : T(TM) \longrightarrow TM$$

Let us put

$$\mathcal{V}TM := \bigcup_{v\in TM} Ker\pi^v_*.$$

The set $\mathcal{V}TM$ is an $n-$dimensional sub-bundle of $T(TM^0)$ that is called the vertical tangent bundle of TM^0. A non-linear connection on TM is an extension distribution $\mathcal{H}TM$ for $\mathcal{V}TM$ on $T(TM)$. In the other words:

$$T(TM) = \mathcal{V}TM \oplus \mathcal{H}TM, \tag{1.5}$$

where $\mathcal{H}TM$ is called horizontal vector bundle that is an $n-$dimensional sub-bundle of $T(TM^0)$. It is known that $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right\}$ is a basis for $T(TM)$. Let us introduce a basis for $T(TM^0)$ that is proportional to the above decomposition. If π_*^{υ} is the natural projection map from $T_{\upsilon(TM)}$ to TM, in this case;

$$\pi_*^{\upsilon}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i}, \quad \pi_*^{\upsilon}\left(\frac{\partial}{\partial y^i}\right) = 0.$$

According to the definition of vertical vector bundle, the set of vectors $\left\{\frac{\partial}{\partial y^i}\right\}$ is the basis for $\mathcal{V}TM$, based on the above decomposition for $T(TM)$. The basis $\left\{\frac{\partial}{\partial y^i}\right\}$ can be expanded to basis $\left\{S_i, \frac{\partial}{\partial y^i}\right\}$ for $T(TM)$. Since $\left\{\frac{\partial}{\partial x^i}\right\}$ is a member of $T(TM)$. The set $\frac{\partial}{\partial x^i} = A_i^j(x,y)S_j + N_i^j(x,y)\frac{\partial}{\partial y^j}$, where the above A_i^j and N_i^j are differential functions such that, they are defined locally on TM. So $\left\{A_i^j(x,y)S_j\right\}$ is the locally basis for $\mathcal{H}TM$. Then we have $A_i^j(x,y)S_j = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j}$.

We put $A_i^j(x,y)S_j = \frac{\delta}{\delta x^i}$. Therefore the set $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right\}$ is a basis for $T_{\upsilon}(TM)$ proportional to the above decomposition. Then N_i^j are called the coefficients for a non-linear connection. These coefficients on TM^0 satisfying transformation rules $\frac{\partial \tilde{x}^j}{\partial x^i}\tilde{N}_j^h = \frac{\partial \tilde{x}^h}{\partial x^j}N_i^j - \frac{\partial^2 \tilde{x}^h}{\partial x^i \partial x^j}y^j$.

2. Some results on g-natural metrics

Let (M,F) be a Finsler manifold. For the horizontal-vertical decomposition of the Sasaki metric $\tilde{g}$ on the slit tangent bundle, TM^0 is defined by

$$\tilde{g}_{ij}(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j. \tag{2.1}$$

It easily shows that $\tilde{g}$ is a Riemannian metric on TM^0. A general metrics that can be defined on TM^0 is a family of Riemannian metrics that we show it $\hat{g}$. The Sasaki metric is particular case of this metric. It is defined by

$$\hat{g}(x,y) = c_1 g_{ij}dx^i \otimes dx^j + (c_2F^2 g_{ij} + c_3F^2 y_i y_j)\delta y^i \otimes \delta y^j + c_4 g_{ij}dx^i \otimes \delta y^j \tag{2.2}$$

for all $(x,y) \in TM^0$, with real functions $c_1, c_2, c_3, c_4 : [0,\infty] \longrightarrow [0,\infty]$ such that $c_1, c_2, c_3 > 0$. The Sasaki metric is obtained for $c_1 = c_2 = 1$ and $c_3 = c_4 = 0$. Therefore $\hat{g}$ is a Riemannian metric on TM^0. The Levi-Civita connection $\hat{\nabla}$ on TM^0 with respect to $\hat{g}$ is given by the Koszul formula

$$\begin{aligned} 2\hat{g}\left(\hat{\nabla}_{\hat{X}}^{\hat{Y}}, \hat{Z}\right) &= \hat{X}\hat{g}\left(\hat{Y}, \hat{Z}\right) + \hat{Y}\hat{g}\left(\hat{Z}, \hat{X}\right) - \hat{Z}\hat{g}\left(\hat{X}, \hat{Y}\right) \\ &+ \hat{g}\left(\left[\hat{X}, \hat{Y}\right], \hat{Z}\right) - \hat{g}\left(\left[\hat{Y}, \hat{Z}\right], \hat{X}\right) + \hat{g}\left(\left[\hat{Z}, \hat{X}\right], \hat{Y}\right), \end{aligned}$$

and the Lie bracket of a tangent bundle with the Chern connection by the following:

Proposition 2.1. Let (M, F) be a Finsler manifold with the vertical basis $\frac{\partial}{\partial y^i}\mid_y \in \mathcal{V}_yTM$ and the horizontal basis $\frac{\delta}{\delta x^i}\mid_y \in \mathcal{H}_yTM$ with the Chern connection. Then the Lie bracket of the tangent bundle TM of M satisfies:

i. $\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -R^k{}_{ij}\frac{\partial}{\partial y^k}$,

ii. $\left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0$,

ii. $\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = (\Gamma^k_{ij} + L^k_{ij})\frac{\partial}{\partial y^k}$

where $R^k{}_{ij}$ define a skew-symmetric Finsler tensor field of type (1.2) and $(\Gamma^k_{ij} + L^k_{ij})$ are the local coefficients of the Berward connection. Some other Finsler tensor fields defined by R^k_{ij} will be useful in study of Finsler manifolds of constant flag curvature:

$$R_{hij} = g_{hk}R^k_{ij},\ R_{hj} = R_{hij}y^i,\ R^k_j = g^{kh}R_{hj}. \tag{2.3}$$

Consequently, we have

$$y^hR_{hij} = 0,\ y^hR_{hj} = 0,\ R_{ij} = R_{ji}, \tag{2.4}$$

and

$$R^k_{ij} = \frac{1}{3}\left\{\frac{\partial R^k_j}{\partial y^i} - \frac{\partial R^k_i}{\partial y^j}\right\}. \tag{2.5}$$

Also the Cartan tensor field is given by its local components:

$$C^k_{ij} = \frac{1}{2}g^{kh}\frac{\partial g_{ij}}{\partial y^h},\ C_{ijk} = \frac{1}{2}\frac{\partial g_{ik}}{\partial y^j}. \tag{2.6}$$

It is easy to see that C_{ijk} is symmetric with respect i, j, k. Furthermore, we deduce that (M, F) becomes a Riemannian manifold, that is g_{ij} depend on (x^k) alone if and only if we have $C^k_{ij} = 0$ for all $i, j, k \in \{1, ..., n\}$. By the homogeneity condition for F, we obtain $y^iC^k_{ij} = 0$.

Lemma 2.2. Let (M, F) be a Finsler manifold, then the Levi-Civita connection $\hat{\nabla}$ on the Riemannian manifold $(TM^0, \hat{g})$ is locally expressed as follows:

$$\hat{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = \frac{2}{3}\Big(C^k_{ij} + 2L^k_{ij}\Big)\frac{\delta}{\delta x^k} + \frac{2}{3}\Big(C^k_{ij} - L^k_{ij}\Big)\frac{\partial}{\partial y^k}, \tag{2.7}$$

$$\hat{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} = \frac{2}{3}\Big(2C^k_{ij} + y^lR_{lij}^{\ k} + L^k_{ij}\Big)\frac{\delta}{\delta x^k} - \frac{1}{3}\Big(2C^k_{ij} + y^lR^k_{lij} + 4L^k_{ij}\Big)\frac{\partial}{\partial y^k}, \tag{2.8}$$

$$\hat{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = \frac{2}{3}\Big(2C^k_{ij} - y^lR^k_{lij} + L^k_{ij}\Big)\frac{\delta}{\delta x^k} + \frac{1}{3}\Big(3\Gamma^k_{ij} - 2C^k_{ij} + y^lR^k_{lij} - L^k_{ij}\Big)\frac{\partial}{\partial y^k}, \tag{2.9}$$

$$\hat{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = \frac{1}{3}\Big(3\Gamma^k_{ij} + 2C^k_{ij} + L^k_{ij}\Big)\frac{\delta}{\delta x^k} - \frac{1}{3}\Big(4C^k_{ij} + y^lR^k_{lij} + 2L^k_{ij}\Big)\frac{\partial}{\partial y^k}. \tag{2.10}$$

where

$$\Gamma^k_{ij} = \frac{1}{2}g^{kh}\Big\{\frac{\delta g_{ki}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h}\Big\}. \tag{2.11}$$

The vertical distribution $\mathcal{V}T\hat{M}^0$ is totally geodesic (resp. minimal) in $TT\hat{M}^0$ if $\mathcal{H}\hat{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = 0$ (resp. $g^{ij}\mathcal{H}\hat{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = 0$), where $\mathcal{H}$ denotes the horizontal projection. Similarly, if we denote by $\mathcal{V}$ the vertical projection, then we say that the horizontal distribution $\mathcal{H}T\hat{M}^0$ is totally geodesic (resp. minimal) in $TT\hat{M}^0$ if $\mathcal{V}\hat{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = 0$ (resp. $g^{ij}\mathcal{V}\hat{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = 0$. So by a simple calculation, we can get the following:

Proposition 2.3. Let (M, F) be a Finsler manifold, then the following hold:

i. the function F has constant relatively isotropic Landsberg $\mathbf{L} = -\frac{1}{2}\mathbf{C}$ if and only if the vertical distribution $\mathcal{V}T\hat{M}^0$ is totally geodesic in $TT\hat{M}^0$.
ii. the function F has constant relatively isotropic mean Landsberg $\mathbf{J} = -\frac{1}{2}\mathbf{I}$ if and only if the vertical distribution $\mathcal{V}T\hat{M}^0$ is minimal in $TT\hat{M}^0$.
iii. the function F is flat metric if and only if it has constant relatively isotropic Landsberg curvature $\mathbf{L} = -2\mathbf{C}$.

Proof. **Part [i]**: Assume that $\mathcal{H}\hat{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = 0$. By (2.7) we have $C^k_{ij} + 2L^k_{ij} = 0$. This means that F is a constant relatively isotropic Landsberg metric.

Part [ii]: Let $g^{ij}\mathcal{H}\hat{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = 0$. Then we have $g^{ij}(C^k_{ij} + 2L^k_{ij}) = 0$ and so $I_j + 2J_j = 0$ which means that F is a constant relatively isotropic mean Landsberg metric.

Part [iii]: Let $\mathcal{H}T\hat{M}^0$ be totally geodesic. Then we get $16C^k_{ij} + 8L^k_{ij} + 3y^l R^k_{lij} = 0$. It is clear that $R = 0$ if and only if $L = -2C$. □

Corollary 2.4. Let (M, F) be a Finsler manifold. If $di\hat{v}(X^v) = 0$. Then F is a weakly Berward metric.

Proof. Let $di\hat{v}(X^v) = 0$. So we have $\Gamma^k_{ij} - 2I_j - J_j = 0$. Contracting it with y^i imply that $N^m_m = 0$. It follows that $E_{ij} := \frac{1}{2}\frac{\partial^2 N^m_m}{\partial y^i \partial y^j} = 0$, which means that F is a weakly Berward metric. □

Lemma 2.5. Let (M, F) be a compact Finsler manifold. Then the following relations are equivalent.

i. The vertical distribution $\mathcal{V}T\hat{M}^0$ is totally geodesic in $TT\hat{M}^0$.
ii. The vertical distribution $\mathcal{V}T\hat{M}^0$ is minimal in $TT\hat{M}^0$.
iii. $\mathbf{R} = 0$.

In any case, the function F reduces to Riemannian metric.

Proof. According to Part [iii] of Proposition 2.3, $R = 0$ if and only if $\mathbf{L} = -2\mathbf{C}$. By the same argument in ([11]), we have the proof. □

3. The Chromatic polynomial structure on Finsler manifold

This is a reminder of the chromatic polynomial structure notions we will use in the following. More details can be found in [3, 6, 8, 15]. The Golden mean canon is found in the linear proportions of masterpieces of architecture, human, animal, and plant bodies. The Golden mean canon arose from the division of a unit segment line AB into two parts x and $1x$, such that $\frac{x}{1-x} = \frac{1}{x}$. On the other hand, one can say that Golden mean canon follows from a square equation $x^2 - px - q = 0$, where $p = 1, q = 1$, which solution is $\phi = \frac{1+\sqrt{5}}{2}$.
As it is known, it is very easy to find the members of "the metallic means family" (Spinadel, 1999) as solutions of the above equation, for fix two various values of the positive integers p and q. In fact, if $p = q = 1$, we have the Golden mean canon. Analogously, for $p = 2$ and $q = 1$, we obtain the Silver mean; for $p = 3$ and $q = 1$, we get the Bronze mean. For $p = 4, q = 1$ we have the next metallic mean, etc. It should be mentioned that many authors wrote about the close relation of some the metallic means family to classical Fibonacci numbers, fractal geometry, dynamical systems, quasicrystals, etc.

Let $\mathbf{P}[x]$ be the algebra of all polynomials, and $P_n(x)$ be a polynomial of degree n, in $\mathbf{P}[x]$. We define $\mathbf{A}_P$, called the induced algebra with respect to $P_n(x)$, to be the set of all polynomials of degree less than n together with the addition and scalar product induced by $\mathbf{P}[x]$. The multiplication in $\mathbf{A}_P$ is defined in such away. Therefore $\mathbf{A}_P$ is isomorphic to the quotient algebra $\mathbf{P}[x]/ < P_n(x) >$.

Definition 3.1. Let M be a C^∞ Riemannian space. A C^∞ tensor field $\mathbb{W}$ of type (1.1) on M is said to define a chromatic polynomial structure of degree d on M if d is the smallest integer for which the power $1, \mathbb{W}, ..., \mathbb{W}^d$ are dependent, and $\mathbb{W}$ has constant rank on M.

If $\dim M = 2n$, an almost complex structure on M is a chromatic polynomial structure of degree 2. If $\dim M = 2n - 1$, an almost contact structure on M is a chromatic polynomial structure of degree 3. Quartic structures are chromatic polynomial structures of degree 4.

Example 3.2. A vector $(X^1, ..., X^p)$ from $T_{(x^1,..,x^p)}E^p$ is tangent to $\mathbb{S}^{p-1}(r)$ if and only if we have

$$\sum_{i=1}^{p} x^i X^i = 0. \tag{3.1}$$

Consider a vector (X, Y, Z) in $\mathbb{R}^3$ that tangent to $\mathbb{S}^2$. We define a tensor field $\mathbb{W}$ of type (1.1) by $\mathbb{W}(X, Y, Z) = (\phi X, (1 - \phi)Y, (2 - \phi)Z)$, where $\phi = \dfrac{1}{2}$. The tensor field $\mathbb{W}$ satisfying

$$\mathbb{W}^2 + \frac{4}{5}\mathbb{W} - \frac{20}{13} = 0. \tag{3.2}$$

Thus, $\mathbb{W}$ is a chromatic polynomial structure of degree 2 on the sphere $\mathbb{S}^2$.

Definition 3.3. Suppose (M, g) be a Riemannian space and $\mathbb{W}$ is a chromatic polynomial structure of degree d on M. We say that the metric g is $\mathbb{W}$-compatible if the equality

$$g(\mathbb{W}(X), Y) = g(X, \mathbb{W}(Y)) \tag{3.3}$$

is satisfied for every tangent vector fields $X, Y \in \chi(M)$.

Definition 3.4. A Riemannian space (M, g) endowed with a chromatic polynomial structure $\mathbb{W}$ of degree d so that the Riemannian metric g is $\mathbb{W}$-compatible is named a chromatic polynomial Riemannian space of degree d and $(g, \mathbb{W})$ is named a chromatic polynomial Riemannian structure of degree d on M.

Proposition 3.5. If $(M, g_1, \mathbb{W}_1)$ and $(N, g_2, \mathbb{W}_2)$ are two chromatic polynomial Riemannian spaces, then the product manifold $M \times N$ admits a chromatic polynomial Riemannian structure.

Proof. We define the Riemannian metric g and $(1, 1)$ tensor field $\mathbb{W}$ on $M \times N$ by

$$g((X_1, Y_1), (X_2, Y_2)) = g_1(X_1, X_2)g_2(Y_1, Y_2) \tag{3.4}$$

$$\mathbb{W} : T(M \times N) \longrightarrow T(M \times N) \tag{3.5}$$

$$\mathbb{W}(X, Y) = \mathbb{W}_1(X)\mathbb{W}_2(Y).$$

The metric g is $\mathbb{W}$-compatible. □

Definition 3.6. Let $(g, \mathbb{W})$ and $(g, \mathbb{W}')$ are tow systems of chromatic Polynomial Riemannian structure of degree d on Riemannian orbifold (M, g). We say that $(g, \mathbb{W})$ and $(g, \mathbb{W}')$ are equivalent or($(g, \mathbb{W}) \sim_P (g, \mathbb{W}')$) if and only if there exists a diffeomorphism $Q : TM \longrightarrow TM$ such that:
1) For all $m \in M$ $Q_m : T_mM \longrightarrow T_mM$ is a vector space isomorphism.
2) $\forall m \in M$, $\mathbb{W}_m = \mathbb{W}'_m o\ Q_m$.

Example 3.7. Let the Lie group $M = \mathbb{R}^2 \times \mathbb{R}$ where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ as $\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$. Therefore multiplication is given by $(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + e^{t_1}x_2, y_2 + e^{-t_1}y_2, t_1 + t_2)$ together with the left invariant Riemannian metric

$$ds^2 = e^{-2t}dx^2 + e^{2t}dy^2 + dt^2 \tag{3.6}$$

whenever $X_1 = e^t\dfrac{\partial}{\partial x},\ X_2 = -e^{-t}\dfrac{\partial}{\partial y},\ X_3 = \dfrac{\partial}{\partial t}$ then $(X_1, X_1) = e^{-2t}, (X_2, X_2) = e^{2t}, (X_3, X_3) = 1$.
A C^∞ tensor field $\mathbb{W}$ such $\mathbb{W}(X_1) = 3X_1, \mathbb{W}(X_2) = -2X_2, \mathbb{W}(X_3) = -X_3$ of type $(1, 1)$ on M define a polynomial Riemannian structure of degree 3 satisfying in $\mathbb{W}^3 - 7\mathbb{W} - 6I = 0$. Now consider another polynomial Riemannian structure of degree 3, $\mathbb{W}'(X_1) = -3X_1, \mathbb{W}'(X_1) = -4X_2, \mathbb{W}'(X_3) = -X_3$ that is zero of $\phi^3 + 8\phi^2 + 19\phi + 12I = 0$.
Tow above polynomial Riemannian structures are equivalent. Because if the diffeomorphism $Q : TM \longrightarrow TM$ is given by

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(or\ Q_m(X_1) = -X_1,\ Q_m(X_2) = 2X_2,\ Q_m(X_3) = X_3)$ for all $m \in M$, then $\mathbb{W}'_m = \mathbb{W}_m \circ Q_m$.

3.1. **Some results of chromatic polynomial (metallic and golden) structures on Riemannian spaces.** In 2009 Hretcanu and Crasmareano introduce the golden structure on Riemannian manifolds [7]. Let p and q be non zero integer and that (the discriminate) $R = p^2 - 4q$ is also not zero. The numbers α and β be the zeros of the companion sequences as follows: $A_n(p, q) = (\alpha^n - \beta^n)/(\alpha - \beta)$, and $B_n(p, q) = \alpha^n + \beta^n$.
Therefore $A_0 = 0, A_1 = 1, B_0 = 2$ and $B_1 = p$; and the sequences, follow the recurrence

relations given above. These sequences are both called Lucas sequences and the number in them are the generalized Lucas numbers.

Remark 3.8. If $p = 1$ and $q = -1$ or in other words a and b be the golden ratio, then $A_n(p, q)$ and $B_n(p, q)$ are the Fibonacci and Lucas sequences respectively, where the Fibonacci sequence is 011235... and the Lucas sequence is 21347... .

These sequences have many useful properties such as $A_{2n} = A_n B_n$ and if p is an odd prime, then p divides $A_{p-(R/p)}$ is (R/p) is the Legendre symbol.

Remark 3.9. If $p = 1$ and $q = -1$ then an $(1, 1)$ tensor field $\mathbb{W}$ that, is the zero of the polynomial structure $x^2 - px + q$ be a golden structure. Therefore, a golden structure is a polynomial structure of degree 2.

Definition 3.10. A chromatic polynomial structure on a Riemannian manifold M is called a metallic structure if it is determined by an $(1, 1)$ tensor field P which satisfies the equation

$$\mathbb{W}^2 = p\mathbb{W} + qI \tag{3.7}$$

where p, q are positive integers and I is the identity operator on the Lie algebra $\mathfrak{X}(M)$ of the vector fields on M.

Example 3.11. A vector $(X^1, ..., X^p)$ from $T_{(x^1,..,x^p)}E^p$ is tangent to $\mathbb{S}^{p-1}(r)$ if and only if we have

$$\sum_{i=1}^{p} x^i X^i = 0. \tag{3.8}$$

The tangent space at the point $m = (m_1, m_2, m_3) \in \mathbb{S}^2$ is given by

$$T_m S^2 \cong \{(\frac{-m_2}{m_1}v - \frac{m_3}{m_1}w, v, w) \mid v, w \in \mathbb{R}\}$$

where $\mathbb{S}^2$ is the unit sphere. Now suppose $X = (\frac{-m_2}{m_1}v - \frac{m_3}{m_1}w, v, w)$ and $Y = (\frac{-m_2}{m_1}v' - \frac{m_3}{m_1}w', v', w')$ be two independent vectors in $T_m\mathbb{S}^2$. We can define a polynomial structure of degree 2 on $\mathbb{S}^2$ by a $(1, 1)$ tensor field $\mathbb{W}(X) = \lambda X$ and $\mathbb{W}(Y) = (2 - \lambda)Y$ where λ is an arbitrary scaler. Its polynomial is $\mathbb{W}^2 - 2\mathbb{W} - (\lambda^2 - 2\lambda)I = 0$.
A $(1, 1)$ tensor field P that given by $\mathbb{W}(X) = -X - Y$ and $\mathbb{W}(Y) = X - Y$ satisfies in $\mathbb{W}^2 + 2\mathbb{W} + 2I = 0$. It is another example of polynomial structure on the sphere $\mathbb{S}^2$.

Consider two Riemannian spaces M and N. Suppose $(g_1, \mathbb{W})$ is a golden Riemannian structure on M. If $f : M \longrightarrow N$ is a diffeomorphism, then N admits a golden Riemannian structure induces by f. For its to hold, we define a Riemannian metric g_2 and the golden structure $\mathbb{W}_2$ on N by

$$g_2(X, Y) := g_1(f_*^{-1}X, f_*^{-1}Y) \tag{3.9}$$

$$\begin{aligned} \mathbb{W}_2 : TN &\longrightarrow TN \\ Y' &\longmapsto f_*(\mathbb{W}(f_*^{-1}Y')) \end{aligned} \tag{3.10}$$

the Riemannian metric g_2 is $\mathbb{W}_2$-compatible.

Definition 3.12. let $(g, \mathbb{W})$ and $(g, \mathbb{W}')$ are two systems of golden Riemannian structure on Riemannian space (M, g). We say that $(g, \mathbb{W})$ and $(g, \mathbb{W}')$ are equivalent or($(g, \mathbb{W}) \sim_G (g, \mathbb{W}')$) if and only if there exists a diffeomorphism $Q : TM \longrightarrow TM$ such that:
1) For all $m \in M$, the $Q_m : T_mM \longrightarrow T_mM$ is a vector space isomorphism.
2) $\forall m \in M$, we have $\mathbb{W}_m = \mathbb{W}'_m o\, Q_m$.

Proposition 3.13. Let $\mathbb{W}$ be a golden structure on a Riemannian space (M, g). If $Q = \lambda I$ and $\mathbb{W}' = \mathbb{W} \circ Q = -(1 + \lambda)I$, then $(g, \mathbb{W}) \sim_G (g, \mathbb{W}')$, (where λ is a scaler).

Proposition 3.14. Suppose $(g, \mathbb{W})$ is a golden Riemannian structure on Riemannian manifold space M and the diffeomorphism $Q : TM \longrightarrow TM$ satisfies in the following condition:
1) $(\mathbb{W}_m \circ Q_m) - (\mathbb{W}_m \circ Q_m)^{-1} = I$
2) $g(\mathbb{W}_m(X), Q_m(Y)) = g(Q_m(X), \mathbb{W}_m(Y)),\ for\ (X, Y) \in \chi(M)$
then $(g, \mathbb{W}) \sim_G (g, \mathbb{W} \circ Q)$.

Proof. Suppose that $\tilde{\mathbb{W}}_m$ and $\tilde{Q}_m$ are the matrix associate the linear transformation $\mathbb{W}_m$ and Q_m for all $m \in M$. If $\tilde{\mathbb{W}}_m\tilde{Q}_m - (\tilde{\mathbb{W}}_m\tilde{Q}_m)^{-1} = I$, then $\tilde{\mathbb{W}}_m\tilde{Q}_m\tilde{\mathbb{W}}_m = \tilde{\mathbb{W}}_m + I\tilde{Q}_m^{-1}$. Therefore $(\tilde{\mathbb{W}}_m\tilde{Q}_m)^2 = \tilde{\mathbb{W}}_m\tilde{Q}_m + I$. Since $g(X, \mathbb{W}_mQ_m(Y)) = g(\mathbb{W}_m(X), Q_m(Y))$ and $g(\mathbb{W}_mQ_m(X), Y) = g(Q_m(X), \mathbb{W}_m(Y))$. Therefore, g is $(\mathbb{W} \circ Q)$-compatible. □

Definition 3.15. The Fibonacci sequence's initial terms are $F_0 = 0$ and $F_1 = 1$, with $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$. The polynomial $\mathbb{G}_n(x) = x^n - F_nx - F_{n-1}$ (F_n is the Fibonacci sequence) is called a generalized golden polynomial of degree n.

Proposition 3.16. The generalized golden polynomial $\mathbb{G}_n(x)$ is decomposed as follows:

$$\mathbb{G}_n(x) = \mathbb{G}_2(x)(\sum_{i=0}^{n-2} F_ix^{n-i-2})$$

where $\mathbb{G}_2(x)$ is the golden polynomial.

Proof. The main basic idea of the proof is to take mathematical induction and the above definition. □

Definition 3.17. [14] A $\mathbb{P}$-structure on a differentiable Riemannian manifold space M of dimension n is a tensor field $\mathbb{P}$ of type (1.1) satisfying $\mathbb{P}^3 - \mathbb{P} = 0$ on M.

Proposition 3.18. If there exists a $\mathbb{P}$-structure of rank $(n - 1)$, then there exists a polynomial Riemannian orbifold structure of degree ≤ 3 on M.

Proof. It is sufficient to show that, if there exists a $\mathbb{P}$-structure of rank $(n - 1)$, then there exists a Riemannian metric g such that $g(\mathbb{P}X, Y) = g(X, \mathbb{P}Y)$.
Suppose there exists a $\mathbb{P}$-structure of rank $(n - 1)$. We define, the operators $\mathbb{L}$ and $\mathbb{T}$ respectively by

$$\mathbb{L} = -\mathbb{P}^2 \tag{3.11}$$

$$\mathbb{T} = \mathbb{P}^2 - I. \tag{3.12}$$

Thus $\mathbb{L} + \mathbb{T} = -I$. We denote two types of distributions corresponding to $\mathbb{L}$ and $\mathbb{T}$ by $D_{\mathbb{L}}$ and $D_{\mathbb{T}}$ corresponding to $\mathbb{L}$ and $\mathbb{T}$ respectively.
Consider the mutually orthogonal unit vectors in $D_{\mathbb{L}}$ and $D_{\mathbb{T}}$. Let e^i_X, for $X, Y, Z, ... = 1, 2, 3, ...r$ be the mutually orthogonal unit vectors in $D_{\mathbb{L}}$ and e^i_x for $x, y, z, ... = r + 1, r +$

$2, ..., n$ be the mutually orthogonal unit vectors in $D_{\mathbb{T}}$. Let the local co-ordinates of $\mathbb{P}$ and $\mathbb{T}$ respectively be $\mathbb{P}^i_j$ and $\mathbb{T}^i_j$. Since $\mathbb{P}\mathbb{T} = 0$ or in local co-ordinates

$$\mathbb{P}^i_j \mathbb{T}^j_k = 0 \tag{3.13}$$

multiplying by e^k_x we find

$$\mathbb{P}^i_j \mathbb{T}^j_k e^k_x = 0 \Rightarrow \mathbb{P}^i_j (\mathbb{T}^j_k e^k_x) = 0 \Rightarrow \mathbb{P}^i_j e^j_x = 0. \tag{3.14}$$

Thus we have

$$\mathbb{P}^i_j e^j_x = 0. \tag{3.15}$$

We denote by $\{\eta^X_h, \eta^x_h\}$ the matrix inverse to $\{e^i_X, e^i_x\}$, then η^X_h and η^x_h are both components of linearly independent covariant vectors. Similarly, as $\mathbb{T}\mathbb{P} = 0$, we have

$$\mathbb{P}^i_j \eta^x_i = 0. \tag{3.16}$$

Define a tensor field of type $(0, 2)$ by

$$h_{ji} = \eta^X_j \eta^X_i + \eta^x_j \eta^x_i \tag{3.17}$$

where the repeated index X or x do not represent the summation.
Then h_{ji} is well defined and it is a Riemannian metric on M such that

$$\eta^X_j = h_{ji} e^i_x \ and \ \eta^x_j = h_{ji} e^i_x. \tag{3.18}$$

Now define g by

$$g_{ji} = \frac{1}{2}[h_{ji} + P^t_j P^s_i h_{ts} + \eta^x_j \eta^x_i]. \tag{3.19}$$

Then g is Riemannian metric on M such that (3.3) is valid. $\square$

Corollary 3.19. If in a Riemannian space M of dimension n, there exists an almost Lorentzian paracontact structure $(\mathbb{P}, u, \omega)$ then there exist a Riemannian metric g such that g is $\mathbb{P}$-compatible.

Proof. Use [14] and the previous Proposition. $\square$

Corollary 3.20. If the Riemannian space M admits an almost product structure $\mathbb{P}$, then there exists a Riemannian metric g such that, $(g, \mathbb{P})$ is a polynomial Riemannian structure of degree ≤ 2.

Remark 3.21. The polynomial relation $\sim_{\mathbb{P}}$ is an equivalence relation.

Definition 3.22. [16] An f-structure on a differentiable manifold M of dimension n is a tensor field f of type (1.1) satisfying $f^3 + f = 0$ on M.

Proposition 3.23. Suppose (M, g) be a connected Riemannian manifold space and the (1.1) tensor field P is a f-structure on M such that g is $\mathbb{P}$-compatible. If the diffeomorphism $Q : TM \longrightarrow TM$ satisfies in the following conditions:
1) $(\mathbb{P} \circ Q)$ be an almost complex structure on M.
2) $g(\mathbb{P}_m(X), Q_m(Y)) = g(Q_m(X), \mathbb{P}_m(Y)) \quad \forall X, Y \in \chi(M)$,
then $(g, \mathbb{P}) \sim_P (g, \mathbb{P} \circ Q)$.

Proof. If $\mathbb{P}$ is a tensor field of type $(1, 1)$ on M satisfying $\mathbb{P}^3 + \mathbb{P} = 0$, then the function from M to the integers assigning to x the rank of $\mathbb{P}(x)$ is continuous. In particular, the rank of $\mathbb{P}$ is automatically constant on the components of M. $\square$

Being given any polynomial with real coefficients $\mathcal{P}(x) = a_m x^m + ... + a_2 x^2 + a_1 x$ with $a_1 \neq 0$, the set of f in $Hom(TM, TM)$ satisfying $\mathcal{P}(f) = 0$ behaves quite similarly.

3.2. **The golden structure on Finsler manifold space.** In [13] Peyghan and Tayebi define the new almost complex structure and found that it is a complex structure if and only if the Finsler metric F is of scalar flag curvature.

Following the research works of Tayebi et al., we are studying these structures on Finsler manifolds. We also will obtain a condition of the integrability of the golden structure on the Finsler manifold. Let us define the polynomial structures of degree r on a manifold M with Finsler metric F.

Definition 3.24. ([4]) Let M be a manifold with Finsler metric $F : TM \longrightarrow [0, \infty)$. The polynomial structures of degree r are the tensor fields $\hat{J}$ of type (1.1), on tangent bundle TM, i.e. $\hat{J} : T(TM^0) \longrightarrow T(TM^0)$ such that r is the smallest integer which $\hat{J}^r, \hat{J}^{r-1}, \dots, \hat{J}, Id$ are linearly independent. Let (M, F) be a Finsler manifold endowed with the polynomial structure $\hat{J}$ and $\mathbf{g}_y$ be a fundamental tensor of type $(0, 2)$. A Finsler metric F is compatible with $\hat{J}$ if it satisfies,

$$\mathbf{g}_y(\hat{J}X, Y) = \mathbf{g}_y(X, \hat{J}Y),\ \forall X, Y \in T(TM^0). \tag{3.20}$$

Definition 3.25. Let (M, F) be a Finsler manifold and let $(TM, \hat{J})$ be a golden manifold, i.e. $\hat{J}^2 = \hat{J} + Id$. We say that $(F, \hat{J})$ is an almost golden Finsler structure on TM if,

$$\mathbf{g}_y(\hat{J}X, Y) = \mathbf{g}_y(X, \hat{J}Y),\ \forall X, Y \in T(TM^0). \tag{3.21}$$

In this case, triple $(M, \hat{J}, F)$ is called an almost golden Finsler manifold.

It can be proved that the condition 3.21 is equivalent to the following condition:

$$\mathbf{g}_y(\hat{\phi}X, \hat{J}Y) = \mathbf{g}_y(\hat{J}^2X, Y) = \mathbf{g}_y((\hat{J} + I)X, Y) = \mathbf{g}_y(\hat{J}X, Y) + \mathbf{g}_y(X, Y).$$

Also, if triple $(M, \hat{P}, F)$ is a product Finsler manifold, i.e. $\hat{P}^2 = Id$ the condition 3.21 is equivalent to:

$$\mathbf{g}_y(\hat{P}X, \hat{P}Y) = \mathbf{g}_y(X, \hat{P}\hat{P}Y) = \mathbf{g}_y(X, \hat{P}^2Y) = \mathbf{g}_y(X, Y),\ \forall X, Y \in T(TM^0). \tag{3.22}$$

Similar to what has been proved in the smooth manifolds(see [3]). It can be shown that the golden structure and the product structure on tangent bundle TM^0, are related to each other by the following Lemma:

Lemma 3.26. ([4]) Let (M, F) be a Finsler manifold.

i. A product structure $\hat{P}$ on TM^0 induces a golden structure as follows:

$$\hat{\phi}_{\hat{P}} = \frac{1}{2}(I + \sqrt{5}\hat{P}). \tag{3.23}$$

We have: $\hat{J}^2 = \hat{J} + I$, which yields $\hat{J} = \frac{1+\sqrt{5}}{2} = \frac{1}{2}(I + \sqrt{5}I) = \frac{1}{2}(I + \sqrt{5}\sqrt{I}) = \frac{1}{2}(I + \sqrt{5}\hat{P})$.

ii. A golden structure $\hat{J}$ on TM^0 induces a product structure as follows:

$$\hat{P}_{\hat{J}} = \frac{1}{\sqrt{5}}(2\hat{J} - I). \tag{3.24}$$

By direct calculations of the previous proof, also this equation can be proved.

As a direct consequence of the above lemma, the following proposition can be considered.

Proposition 3.27. ([4]) Let $(M, \hat{J}, F)$ be an almost golden Finsler manifold. Then

$$N_{\hat{P}_{\hat{J}}}(X,Y) = \frac{4}{5} N_{\hat{J}}(X,Y), \quad \forall X, Y \in T(TM^0). \tag{3.25}$$

As mentioned before, a polynomial structure J is integrable if the Nijenhuis tensor N_J vanishes. In this section, this condition of integrability for almost golden Finsler manifold is studied.

Suppose M is a manifold with a Finsler metric F. We define an almost golden structure on the slit tangent bundle and find it is a golden structure if and only if the manifold M has zero flag curvature.

Using the horizontal-vertical decomposition, we consider the linear map $\hat{J} : \chi(TM^0) \longrightarrow \chi(TM^0)$ by setting :

$$\hat{\phi}\left(\frac{\delta}{\delta x^i}\right) = \frac{1}{2}\left(\frac{\delta}{\delta x^i} + \sqrt{5}\frac{\partial}{\partial y^i}\right) \ \hat{\phi}\left(\frac{\partial}{\partial y^i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial y^i} + \sqrt{5}\frac{\delta}{\delta x^i}\right) \tag{3.26}$$

for $i = 1, ..., n$. It is easy to show that the golden structure $\hat{J}$ is compatible with the Sasaki metric $\tilde{g}$.

Lemma 3.28. The structure J with the definition of:

$$J(\frac{\delta}{\delta x^i}) = A\frac{\delta}{\delta x^i} + B\frac{\partial}{\partial y^i}, \ J(\frac{\partial}{\partial y^i}) = C\frac{\delta}{\delta x^i} + D\frac{\partial}{\partial y^i}, \tag{3.27}$$

is a golden structure if and only if the following relation to be establish

$$A^2 - A + BC = 1, \ A + D = 1, \ D^2 - D + BC = 1.$$

Now we prove that the almost golden structure $\hat{J}$ is compatible with the general metric $\hat{g}$ if we have $c_2 = c_1/F^2$ and $c_3 = 0$. For this purpose, we modify $\hat{J}$ to a $\chi(TM^0)$-linear map given in the basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ as follows:

$$\hat{\phi}(\frac{\delta}{\delta x^i}) = (\alpha_1\delta_i^k + \beta_1 y_i y^k)\frac{\delta}{\delta x^k} + (\alpha_2\delta_i^k + \beta_2 y_i y^k)\frac{\partial}{\partial y^k}, \tag{3.28}$$

$$\hat{\phi}(\frac{\partial}{\partial y^i}) = (\alpha_3\delta_i^k + \beta_3 y_i y^k)\frac{\delta}{\delta x^k} + (\alpha_4\delta_i^k + \beta_4 y_i y^k)\frac{\partial}{\partial y^k}, \tag{3.29}$$

where α_i, β_i for $i = 1, ...4$ are functions on TM^0 to be determined. Then lemma 3.28 can be lead to $\alpha_1 = \alpha_4, \beta_1 = \beta_4$ and $\alpha_2 = \alpha_3, \beta_2 = \beta_3$ and also

$$\left\{ \begin{array}{c} \alpha_1 + 1 = \alpha_1^2 + \alpha_2^2 \\ \alpha_2 = 2\alpha_1\alpha_2 \end{array} \right. \quad \left\{ \begin{array}{c} \beta_1 = 2\alpha_1\beta_1 + 2\alpha_2\beta_2 + F^2(\beta_1^2 + \beta_2^2) \\ \beta_2 = 2\alpha_1\beta_2 + 2\alpha_2\beta_1 + F^2(2\beta_1\beta_2) \end{array} \right. \tag{3.30}$$

by solving of the system of Eqs (3.30), the following equations can be written as:

$$\alpha_1 = \frac{1}{2}, \ \alpha_2 = \pm\frac{\sqrt{5}}{2}, \ \beta_1 = \beta_2 = 0.$$

Also the condition of $\hat{g}(\hat{\phi}(X), Y) = \hat{g}(X, \hat{\phi}(Y))$, gives

$$c_1 = c_2 F^2, \ \beta_2 c_1 = \beta_2 c_3 + F^2(\alpha_2 c_3 + \beta_2 c_2). \tag{3.31}$$

By solving of the system of Eqs. (3.31), the coefficients C_2 and C_3 can be written as seen below:

$$c_2 = \frac{c_1}{F^2}, \ c_3 = 0. \tag{3.32}$$

Now, we can write the almost golden structure $\hat{\phi}$ and the general metric $\hat{g}$ as follows:

$$\hat{\phi}(\frac{\delta}{\delta x^i}) = \frac{1}{2}\delta_i^k \frac{\delta}{\delta x^k} \pm \frac{\sqrt{5}}{2}\delta_i^k \frac{\partial}{\partial y^k}, \tag{3.33}$$

$$\hat{\phi}(\frac{\partial}{\partial y^i}) = \frac{1}{2}\delta_i^k \frac{\partial}{\partial y^k} \pm \frac{\sqrt{5}}{2}\delta_i^k \frac{\delta}{\delta x^k} \tag{3.34}$$

and

$$\hat{g}(x,y) = c_1 g_{ij} dx^i \otimes dx^j + c_1 g_{ij} \delta y^i \otimes \delta y^j + c_2 g_{ij} dx^i \otimes \delta y^j. \tag{3.35}$$

Hence, $(TM^0, \hat{g}, \hat{\phi})$ is an almost golden manifold.

Lemma 3.29. The Nijenhuis tensor field of the almost golden structure $\hat{\phi}$ on tangent bundle TM for $\frac{\partial}{\partial y^i}\mid_y \in \mathcal{V}_y TM$ and $\frac{\delta}{\delta x^i}\mid_y \in \mathcal{H}_y TM$ is given by the following:

$$\begin{aligned} N_{\hat{\phi}}\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) &= \left[\hat{\phi}\frac{\delta}{\delta x^i}, \hat{\phi}\frac{\delta}{\delta x^j}\right] + \hat{\phi}^2\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] - \hat{\phi}\left[\hat{\phi}\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] - \hat{\phi}\left[\frac{\delta}{\delta x^i}, \hat{\phi}\frac{\delta}{\delta x^j}\right] \\
&= \left[\frac{1}{2}\left(\frac{\delta}{\delta x^i} + \sqrt{5}\frac{\partial}{\partial y^i}\right), \frac{1}{2}\left(\frac{\delta}{\delta x^j} + \sqrt{5}\frac{\partial}{\partial y^j}\right)\right] + \hat{\phi}\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] + \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] \\
&-\hat{\phi}\left[\frac{1}{2}\left(\frac{\delta}{\delta x^i} + \sqrt{5}\frac{\partial}{\partial y^i}\right), \frac{\delta}{\delta x^j}\right] - \hat{\phi}\left[\frac{\delta}{\delta x^i}, \frac{1}{2}\left(\frac{\delta}{\delta x^j} + \sqrt{5}\frac{\partial}{\partial y^j}\right)\right] = \frac{1}{4}\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] \\
&+\frac{\sqrt{5}}{4}\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] + \frac{\sqrt{5}}{4}\left[\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^j}\right] + \frac{5}{4}\left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] + \hat{\phi}\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] + \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] \\
&-\frac{1}{2}\hat{\phi}\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] - \frac{\sqrt{5}}{2}\hat{\phi}\left[\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^j}\right] - \frac{\sqrt{5}}{2}\hat{\phi}\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] - \frac{1}{2}\hat{\phi}\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] \\
&= \frac{1}{4}\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] + \frac{\sqrt{5}}{4}\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] + \frac{\sqrt{5}}{4}\left[\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^j}\right] - \frac{\sqrt{5}}{2}\hat{\phi}\left[\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^j}\right] - \frac{\sqrt{5}}{2}\hat{\phi}\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] \\
&= \left(\frac{1}{4} + 1\right)\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = \frac{5}{4}\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -\frac{5}{4}R^k{}_{ij}\frac{\partial}{\partial y^k},
\end{aligned} \tag{3.36}$$

$$N_{\hat{\phi}}\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = \frac{5}{4}R^k{}_{ij}\frac{\partial}{\partial y^k}, \tag{3.37}$$

$$N_{\hat{\phi}}\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = \frac{5}{4}R^k{}_{ij}\frac{\partial}{\partial y^k}, \tag{3.38}$$

$$N_{\hat{\phi}}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = -\frac{5}{4}R^k{}_{ij}\frac{\partial}{\partial y^k}, \tag{3.39}$$

where $R^k_{ij} = \frac{1}{3}\left\{\frac{\partial R^k_i}{\partial y^j} - \frac{\partial R^k_j}{\partial y^i}\right\}$. Thus, $R^k_i = 0$ if and only if $R^k_{ij} = 0$. This is just the condition for (M, F) to have zero flag curvature. Then we have the follow Theorem:

Proposition 3.30. The almost golden structure $\hat{\phi}$ on manifold M with Finsler metric $F : TM \longrightarrow [0, \infty)$ is integrable if and only if (M, F) has zero flag curvature.

Proof. Let $\hat{\phi}$ be integrable, then $N_{\hat{\phi}} \equiv 0$. From (3.36), (3.37), (3.38) and (3.39), we have $R^k_{ij} = 0$.

Conversely, if (M, F) has zero flag curvature, i.e. $R^k_{ij} = 0$, then from (3.36), (3.37), (3.38) and (3.39) we get $N_{\hat{\phi}} \equiv 0$. By this way, the proof can be completed. □

Remark 3.31. One says that a covariant derivative ∇ on manifold M is adapted to polynomial structure J if $\nabla J = 0$. We say (M, J, F) is an almost golden Finsler manifold if $\nabla F = 0$ and $\nabla J = 0$. In this section, it is proved that there is no an almost golden Finsler manifold with the said metric $\hat{g}$.

We shall give the following proposition.

Proposition 3.32. The golden structure $\hat{\phi}$ is compatible with covariant derivative $\hat{\nabla}$ if and only if M is a flat Riemannian manifold.

Proof. Recall that if $\hat{\phi}$ is a tensor field of type $(1, 1)$. Then $\hat{\phi}$ is compatible with covariant derivative $\hat{\nabla}$ if $\hat{\nabla}\hat{\phi} = 0$ means $\hat{\phi}\nabla_X Y = \nabla_X^{\hat{\phi}Y}$ for every X, Y vector fields on M. By this definition, two equations can be achieved from the following one:

$$\hat{\phi}\nabla_{\frac{\partial}{\partial y^i}}^{\frac{\partial}{\partial y^j}} = \nabla_{\frac{\partial}{\partial y^i}}^{\hat{\phi}\frac{\partial}{\partial y^j}}.$$

We have

$$\frac{1+\sqrt{5}}{3}C^k_{ij} + \frac{2-\sqrt{5}}{3}L^k_{ij} = \frac{1+\sqrt{5}}{3}C^k_{ij} + \frac{2+\sqrt{5}}{3}L^k_{ij} + \frac{\sqrt{5}}{3}R^k_{ij},$$
$$\frac{1+\sqrt{5}}{3}C^k_{ij} + \frac{-1+2\sqrt{5}}{3}L^k_{ij} = \frac{1-\sqrt{5}}{3}C^k_{ij} - \frac{1+\sqrt{5}}{3}L^k_{ij} + \frac{\sqrt{5}}{6}R^k_{ij}.$$

These follow that

$$L^k_{ij} = -\frac{5}{14}R^k_{ij}, \qquad C^k_{ij} = -\frac{4}{14}R^k_{ij}. \tag{3.40}$$

By a simple calculation, using the equation:

$$\hat{\phi}\nabla_{\frac{\partial}{\partial y^i}}^{\frac{\delta}{\delta x^j}} = \nabla_{\frac{\partial}{\partial y^i}}^{\hat{\phi}\frac{\delta}{\delta x^j}}$$

it is easy to check that,

$$\frac{2-\sqrt{5}}{3}C^k_{ij} + \frac{1-2\sqrt{5}}{3}L^k_{ij} - \frac{\sqrt{5}}{6}R^k_{ij} = \frac{2+\sqrt{5}}{3}C^k_{ij} + \frac{1+2\sqrt{5}}{3}L^k_{ij} + \frac{1}{3}R^k_{ij},$$
$$\frac{2\sqrt{5}-1}{3}C^k_{ij} + \frac{\sqrt{5}-2}{3}L^k_{ij} + \frac{2\sqrt{5}-1}{6}R^k_{ij} = \frac{\sqrt{5}-1}{3}C^k_{ij} - \frac{2+\sqrt{5}}{3}L^k_{ij} - \frac{1}{6}R^k_{ij}.$$

We have following

$$R^k_{ij} = 0, \quad L^k_{ij} = -\frac{1}{2}C^k_{ij}. \tag{3.41}$$

Using (3.40) and (3.41), we have $\mathbf{C} = \mathbf{R} = 0$. Hence, the proof is completed. □

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