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Research Paper

A NOTE ON ϕ-APPROXIMATE BIFLATNESS OF A SEMIGROUP ALGEBRA

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Abstract. In this note, we show that [8, Theorem 2.3] is not true. We show that $\ell^1(\mathbb{N}_{\max})$ is an unital Banach algebra which is ϕ-pseudo amenable but it is not ϕ-approximate biflat for some $\phi \in Hom(\ell^1(\mathbb{N}_{\max})$ results.

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1. Introduction and Background

The definition of amenability for Banach algebras was given by Johnson. A Banach algebra A is amenable, if every derivation $D : A \to X^*$ has a form $D(a) = a \cdot x_0 - x_0 \cdot a$, where X is any Banach A-bimodule and x_0 belongs to X^*. Also Johnson showed that the amenability of a Banach algebra is equivalent with the existence of an element $M \in (A \otimes_p A)^{**}$ such that $a \cdot M = M \cdot a$ and $\pi_A^{**}(M)a = a$, for all $a \in A$. Here $A \otimes_p A$ is denoted for the projective tensor product of A with A. Also $\pi_A : A \otimes_p A \to A$ is defined by $\pi_A(a \otimes b) = ab$ for all $a, b \in A$.

Helemskii gave a related homological notion to amenability called biflatness. He studied the geometry and the structure of Banach algebras via homological theory. One of the most important notion in homology of Banach algebras is biflatness. Indeed a Banach algebra A is biflat, if there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho(a) = a$ for all $a \in A$. It is known that a Banach algebra A is amenable if and only if A is biflat and posses a bounded approximate identity. For the history of amenability and related homological notions see [9].

Recently approximate notions of amenability like pseudo-amenability and approximate biflatness have been introduced and studied among Banach algebras, see [3], [4], [7], [10] and [11]. Also, some approximate notions of amenability with respect to a homomorphism were given. In fact, for a bounded linear map $\phi : A \to A$ which preserves the product of A or $\phi \in Hom(A)$, A is ϕ-approximate biflat if there exists a net (θ_α) of A-bimodule morphisms from A into $(A \otimes_p A)^{**}$ such that $\pi_A \circ \theta_\alpha \circ \phi(a) \to \phi(a)$ for all $a \in A$. Also A is called ϕ-pseudo amenable if there exists a net (m_α) in $A \otimes_p A$ such that $m_\alpha \phi(a) - \phi(a) m_\alpha \to 0$ and $\pi_A^{**}(m_\alpha)\phi(a) \to \phi(a)$ for all $a \in A$. In the case that ϕ is the identity map, A is called pseudo-amenable. For further information see [8].

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In this short note, we show that [8, Theorem 2.3] is not true. We show that $\ell^1(\mathbb{N}_{\max})$ is an unital Banach algebra which is ϕ-pseudo amenable but it is not ϕ-approximate biflat for some $\phi \in Hom(\ell^1(\mathbb{N}_{\max}))$.

Let A be a Banach algebra. The projective tensor product $A \otimes_p A$ is a Banach A-bimodule by the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \qquad (b \otimes c) \cdot a = b \otimes ca, \qquad (a, b, c \in A).$$

Let X and Y be Banach A-bimodules. Then the map $T : X \to Y$ is called A-bimodule morphism if

$$T(a \cdot x) = a \cdot T(x), \qquad T(x \cdot a) = T(x) \cdot a, \qquad (a \in \mathcal{A}, x \in X).$$

2. Main Results

In this section, we give a counter example among the semigroup algebras which shows that [8, Theorem 2.3] is not always true.

Example 2.1. Suppose that $S = \mathbb{N}_{\max}$. That is the semigroup $\mathbb{N}$ which is equipped to the operation max. Clearly, the related semigroup algebra $\ell^1(S)$ is a unital Banach algebra with unit δ_1. Here we put $\phi = id_{\ell^1(S)} \in Hom(\ell^1(S))$. We show that $\ell^1(S)$ is an unital ϕ-pseudo amenable Banach algebra which is not ϕ-approximate biflat.

To see this, it is known that $\ell^1(S)$ is approximately amenable [6, Example 4.6]. Since $\ell^1(S)$ is unital, so by [7, Proposition 3.2], $\ell^1(S)$ is pseudo-amenable. Thus $\ell^1(S)$ is ϕ-pseudo amenable. Now we conversely suppose that $\ell^1(S)$ is $\phi = id_{\ell^1(S)}$-approximate biflat. Thus there exists a net of $\ell^1(S)$-bimodule morphisms $\theta_\alpha : \ell^1(S) \to \ell^1(S) \otimes_p \ell^1(S)$ such that $\pi^{**}_{\ell^1(S)} \circ \theta_\alpha \circ id_{\ell^1(S)}(a) \to a$ for all $a \in \ell^1(S)$. Set $m_\alpha = \theta_\alpha(\delta_1) \in (\ell^1(S) \otimes_p \ell^1(S))^{**}$. Clearly

$$a \cdot m_\alpha = a \cdot \theta_\alpha = \theta_\alpha(a\delta_1) = \theta_\alpha(\delta_1 a) = \theta_\alpha(\delta_1) \cdot a = m_\alpha \cdot a$$

and

$$\pi^{**}_{\ell^1(S)}(m_\alpha)a = \pi^{**}_{\ell^1(S)} \circ \theta_\alpha(\delta_1)a \to \delta_1 a = a,$$

for all $a \in \ell^1(S)$. Now by [5, Lemma 1.7] it is known that there exists a bounded linear map

$$\psi : \ell^1(S)^{**} \otimes_p \ell^1(S)^{**} \to (\ell^1(S) \otimes_p \ell^1(S))^{**}$$

such that

(i) $\psi(a \otimes b) = a \otimes b$
(ii) $a \cdot \psi(m) = \psi(a \cdot m)$, $\psi(m \cdot a) = \psi(m) \cdot a$
(iii) $\pi^{**}_{\ell^1(S)}(\psi(m)) = \pi_{\ell^1(S)^{**}}(m)$,

for all $a, b \in \ell^1(S)$ and $m \in \ell^1(S) \otimes_p \ell^1(S)$. Define $M_\alpha = \psi(\pi^{**}_{\ell^1(S)}(m_\alpha) \otimes \delta_1) - m_\alpha$. Then we have

$$\begin{aligned}\pi^{**}_{\ell^1(S)}(M_\alpha) = \pi^{**}_{\ell^1(S)}(\psi(\pi^{**}_{\ell^1(S)}(m_\alpha) \otimes \delta_1) - m_\alpha) &= \pi^{**}_{\ell^1(S)}(\psi(\pi^{**}_{\ell^1(S)}(m_\alpha) \otimes \delta_1)) - \pi^{**}_{\ell^1(S)}(m_\alpha) \\ &= \pi^{**}_{\ell^1(S)^{**}}(\pi^{**}_{\ell^1(S)}(m_\alpha) \otimes \delta_1) - \pi^{**}_{\ell^1(S)}(m_\alpha) \\ &= 0.\end{aligned}$$

Thus M_α is a net in $\ker^{**} \pi_{\ell^1(S)}$. On the other hand, since $\ell^1(S)$ has a unit, $\pi_{\ell^1(S)}$ is surjective. By applying [1, A.3.48], we have

$$\overline{\ker \pi_{\ell^1(S)}}^{w^*} = (\ker \pi_{\ell^1(S)})^{**} = \ker \pi^{**}_{\ell^1(S)}.$$

We know that $\ker\pi_{\ell^1(S)}$ is a closed ideal of $\ell^1(S)\otimes_p \ell^1(S)^{op}$, where $\ell^1(S)^{op}$ is denoted for the reverse semigroup algebra. Suppose that $v=\sum_i a_i\otimes b_i\in\ker\pi_{\ell^1(S)}$, where a_i and b_i belong to $\ell^1(S)$. Now

$$\begin{aligned}
v\cdot M_\alpha &= \sum_i a_i\otimes b_i\cdot(\psi(\pi^{**}_{\ell^1(S)}(m_\alpha)\otimes\delta_1)-m_\alpha)\\
&= \sum_i a_i\otimes b_i\cdot(\psi(\pi^{**}_{\ell^1(S)}(m_\alpha)\otimes\delta_1))-\sum_i a_i\otimes b_i\cdot m_\alpha\\
&= \sum_i a_i\otimes b_i\cdot(\psi(\pi^{**}_{\ell^1(S)}(m_\alpha)\otimes\delta_1))-\sum_i a_i\cdot m_\alpha\cdot b_i\\
&\sum_i a_i\otimes b_i\cdot(\psi(\pi^{**}_{\ell^1(S)}(m_\alpha)\otimes\delta_1))-m_\alpha\cdot\sum_i a_ib_i\\
&= \sum_i a_i\otimes b_i\cdot(\psi(\pi^{**}_{\ell^1(S)}(m_\alpha)\otimes\delta_1))-0.
\end{aligned}$$

Let $g\in\ker\pi^{**}_{\ell^1(S)}$. Then we can find a net (g_β) in $\ker\pi_{\ell^1(S)}$ such that $g_\beta\xrightarrow{w^*}g$. Applying above considerations,

$$\begin{aligned}
gM_\alpha=(w^*-\lim g_\beta)M_\alpha &= w^*-\lim g_\beta\psi(\pi^{**}_{\ell^1(S)}(m_\alpha)\otimes\delta_1)\\
&=(w^*-\lim g_\beta)\psi(\pi^{**}_{\ell^1(S)}(m_\alpha)\otimes\delta_1).
\end{aligned}$$

We know that $\pi^{**}_{\ell^1(S)}(m_\alpha)=\pi^{**}_{\ell^1(S)}(m_\alpha)\delta_1\to\delta_1$. So

$$\psi(\pi^{**}_{\ell^1(S)}(m_\alpha)\otimes\delta_1)=\psi(\delta_1\otimes\delta_1)=\delta_1\otimes\delta_1.$$

Thus

$$\begin{aligned}
\lim_\alpha\sup_g ||gM_\alpha-g|| &= \lim_\alpha\sup_g ||g\psi(\pi^{**}_{\ell^1(S)}(m_\alpha)\otimes\delta_1)-g||\\
&\leq \lim_\alpha\sup_g ||g||||\psi(\pi^{**}_{\ell^1(S)}(m_\alpha)\otimes\delta_1)-\delta_1\otimes\delta_1||=0,
\end{aligned}$$

here the supremum takes place over

$$ball(\ker\pi^{**}_{\ell^1(S)})=\{x\in\ker\pi^{**}_{\ell^1(S)}|||x||\leq 1\}.$$

It gives that the mapping of right multiplication by M_α, say R_{M_α} converges to $id_{\ker\pi^{**}_{\ell^1(S)}}$ with respect to the norm topology on $ball(\ker\pi^{**}_{\ell^1(S)})$. Hence for some α, R_{M_α} is invertible. So using surjectivity of R_{M_α} we can find that $\Gamma\in\ker\pi^{**}_{\ell^1(S)}$ such that $\Gamma M_\alpha=M_\alpha$. Thus for all $f\in\ker\pi^{**}_{\ell^1(S)}$, we have

$$(f\Gamma-f)M_\alpha=f\Gamma M_\alpha-fM_\alpha=0.$$

Since R_{M_α} is injective, we have $f\Gamma - f = 0$. Therefore Γ is a right unit for $\ker \pi^{**}_{\ell^1(S)}$. Define $M = \delta_1 \otimes \delta_1 - \Gamma \in (\ell^1(S) \otimes_p \ell^1(S)^{op})^{**}$. So

$$\begin{aligned} a \cdot M - M \cdot a &= a \cdot (\delta_1 \otimes \delta_1 - \Gamma) - (\delta_1 \otimes \delta_1 - \Gamma) \cdot a \\ &= a \otimes \delta_1 - a \cdot \Gamma - \delta_1 \otimes a + \Gamma \cdot a \\ &= a \otimes \delta_1 - (a \otimes \delta_1)\Gamma - \delta_1 \otimes a + (\delta_1 \otimes a)\Gamma \\ &= (a \otimes \delta_1 - \delta_1 \otimes a)(\delta_1 \otimes \delta_1 - \Gamma) \\ &= 0. \end{aligned}$$

Also

$$\pi^{**}_{\ell^1(S)}(M) = \pi^{**}_{\ell^1(S)}(\delta_1 \otimes \delta_1) = \delta_1.$$

It follows that M is a virtual diagonal for $\ell^1(S)$. So $\ell^1(S)$ is amenable. Now applying [2, Theorem 2] the set of idempotents of S, namely $E(S)$ must be finite. But $E(S) = \mathbb{N}$ which is impossible. So $\ell^1(S)$ is not $id_{\ell^1(S)}$-approximate biflat.

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