Mathematical Analysis & Convex Optimization

Vol. 1 (2020), No. 2, 15–24
https:\\ maco.lu.ac.ir
DOI: 10.29252/maco.1.2.2

Research Paper

THE PRODUCT BETWEEN THREE BANACH ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be Banach algebras, $\alpha \in Hom(\mathcal{A}, \mathcal{B})$ and $\beta \in Hom(\mathcal{C}, \mathcal{B})$, and $\| \alpha \| \leq 1$, $\|\beta \| \leq 1$. IN this paper we define the Banach algebra $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ by new product on $\mathcal{A} \times \mathcal{B} \times \mathcal{C}$ which is a strongly splitting extension of $\mathcal{C}$ by $\mathcal{B}$. Then we show that these products from a large class of Banach algebras which contains all module extensions and triangular Banach algebras. Finally we consider spectrum, Arens regularity, amenability and weak amenability of these products.

Keywords: strongly splitting extension, triangular Banach algebras, amenability, weak amenability .
MSC(2010): 46H05; 46H25; 46H35.

1. Introduction and Background

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebra and α be a multiplicative linear functional on $\mathcal{A}$. The Lau product $\mathcal{A} \times_\alpha \mathcal{B}$ was first introduced by Lau [11] for the special case that $\mathcal{A}$ is the predual of a von Neumann algebras and α is the identity of $\mathcal{A}^*$. Monfared [12] extended the notion of Lau product $\mathcal{A} \times_\alpha \mathcal{B}$ to arbitrary Banach algebras and studied various properties of such product. In particular $\mathcal{A} \times_\alpha \mathcal{B}$ is a strongly splitting Banach algebra extension of $\mathcal{B}$ by $\mathcal{A}$. The reader can see [13, 1] for a thorough study of this question and its relation to automatic continuity and cohomology of Banach algebras.

Module extension as a good generalization of Banach algebra extensions were introduced by Gourdeau [9] and were used to show that amenability of $\mathcal{A}^{**}$ implies amenability of $\mathcal{A}$. Zhang [16] used module extensions to answer an open question regarding weak amenability, raised by Dales, Ghahramani, and Gronbaek [3]. Many researchers have become interested in this subject and have worked on it. See [14, 6] for more reading. In [4] we define (α, β) -product by the following identity, where $\alpha, \beta \in Hom(\mathcal{A}, \mathcal{B})$

$$(a, b) \cdot (a', b') = (aa', \alpha(a)b' + b\beta(a')).$$

In this paper we define the new product between three Banach algebras $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$, such that it is extension of (α, β) -product. As we will see in example 2.4, triangular Banach algebras can be easily represented in

Date: Received: August, 20, 2020 , Accepted: October, 17, 2020.

terms of an this product. moreover the above mentioned group of examples, in contrast to direct products $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ provide a wealth of counter-examples, as there are properties such as commutativity, which are satisfied by three of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ but rate by the another one.

These facts suggest that this product are worth to study. The main aim of this paper is to define and study the product between three Banach algebras, also study some homologimathcal properties of $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$, specifically, the concepts spectrum, Arens regularity, amenability, and weak amenability.

Before proceeding further, let us recall some terminology. Throughout $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are Banach algebras, $Hom(\mathcal{A}, \mathcal{B})$ denotes the set of all homomorphism from $\mathcal{A}$ into $\mathcal{B}$ and by $\Delta(\mathcal{A})$ we mean $Hom(\mathcal{A}, \mathbb{C})$.

2. **Main Results**

2.1. **Definitions and Remarks.** In this section we study some properties of the new product between three Banach algebras. We begin with a more general definition, namely $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$, as it was appeared in [4].

Definition 2.1. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be Banach algebras, $\mathcal{X}$ be a Banach $\mathcal{B}$-bimodule, $\alpha \in Hom(\mathcal{A}, \mathcal{B})$, and $\beta \in Hom(\mathcal{C}, \mathcal{B})$ such that $\|\alpha\| \leq 1$ and $\|\beta\| \leq 1$. The Banach algebra $\mathcal{A} \times_\alpha \mathcal{X} \times_\beta \mathcal{C}$ is defined by the following actions

$$(a_1, x_1, c_1) + (a_2, x_2, c_2) = (a_1 + a_2, x_1 + x_2, c_1 + c_2)$$
$$\lambda(a_1, x_1, c_1) = (\lambda a_1, \lambda x_1, \lambda c_1)$$
$$(a_1, x_1, c_1) \cdot (a_2, x_2, c_2) = (a_1 a_2, \alpha(a_1) x_2 + x_1 \beta(c_2), c_1 c_2)$$
$$\|(a, x, c)\| = \|a\| + \|x\| + \|c\|.$$

Similarly we can define the Banach algebra $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$.

Example 2.2. If $\mathcal{C}$=$\mathcal{A}$, then $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{A} \cong \mathcal{A} \times_{\alpha,\beta} \mathcal{B}$ and $\mathcal{A} \times_\alpha \mathcal{X} \times_\beta \mathcal{A} \cong \mathcal{A} \times_{\alpha,\beta} \mathcal{X}$ was defined in [4].

Example 2.3. In the above definition $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \{0\} = \mathcal{A} \times_{\alpha,0} \mathcal{B}$ and $\{0\} \times_\alpha \mathcal{B} \times_\beta \mathcal{C} = \mathcal{B} \times_{0,\beta} \mathcal{C}$.

Example 2.4. Suppose $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras and $\mathcal{X}$ is a Banach ($\mathcal{A}$, $\mathcal{B}$)-module. The triangular algebra $\mathcal{T} = \begin{pmatrix} \mathcal{A} & \mathcal{X} \\ 0 & \mathcal{B} \end{pmatrix}$ with usual matrix operations and norm

$$\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \| = \|a\|_{\mathcal{A}} + \|x\|_{\mathcal{X}} + \|b\|_{\mathcal{B}}$$

is a Banach algebra.
Now if we suppose $\alpha = \beta = id$ then one can easily see that the map

$$\theta : \mathcal{A} \times_\alpha \mathcal{X} \times_\beta \mathcal{C} \to \mathcal{T}, \ \theta(a, x, b) = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$$

is a surjective isometric algebra isomorphism.

Remark 2.5. **(i)** $\mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C}$ is a strongly splitting Banach algebra extension of $\mathcal{C}$ by $\mathcal{A}\times_{\alpha,0}\mathcal{B}$. In other words, $\mathcal{A}\times_{\alpha,0}\mathcal{B}$ is a closed ideal of $\mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C}$ and $(\mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C})/\mathcal{A}\times_{\alpha,0}\mathcal{B}$ is isometrically isomorphic to $\mathcal{C}$. Similarly the following short exact sequences are strongly splitting:

$$\Sigma_1 : 0 \to \mathcal{B}\times_{0,\beta}\mathcal{C} \xrightarrow{i} \mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C} \xrightarrow{q} \mathcal{A} \to 0$$

$$\Sigma_2 : 0 \to \mathcal{B} \xrightarrow{i} \mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C} \xrightarrow{q} \mathcal{A}\times\mathcal{C} \to 0.$$

(ii) For $\alpha,\gamma\in Hom(\mathcal{A},\mathcal{B})$, $\beta,\eta\in Hom(\mathcal{C},\mathcal{B})$, $\mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C}\cong\mathcal{A}\times_\gamma\mathcal{B}\times_\eta\mathcal{C}$ if and only if there exist $\varphi\in Hom(\mathcal{A})$ and $\psi\in Hom(\mathcal{C})$ such that $\alpha=\gamma o\varphi$, $\beta=\eta o\psi$, if and only if there exist $\varphi,\psi\in Hom(\mathcal{B})$ such that $\alpha=\varphi o\gamma$, $\beta=\psi o\eta$.

(iii) $(1_\mathcal{A},0,1_\mathcal{B})$ is an identity for $\mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C}$.

(iv) $(a_\lambda,b_\lambda,c_\lambda)_\lambda$ is a bounded approximate identity for $\mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C}$ if and only if $\|b_\lambda\|\to 0$ and $(a_\lambda)_\lambda$, $(c_\lambda)_\lambda$ are respectively bounded approximate identity for $\mathcal{A}$ and $\mathcal{C}$.

(v) The dual of the space $\mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C}$ can be identified with $\mathcal{A}^*\times\mathcal{B}^*\times\mathcal{C}^*$ naturally as in the direct products and maximum norm.

(vi) Suppose I is an ideal of $\mathcal{A}$, J is an ideal of $\mathcal{B}$, and K is an ideal of $\mathcal{C}$. Then
(a) If $I\subseteq Ker\alpha$ and $J\subseteq Ker\beta$ then $I\times J\times K$ is an ideal in $\mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C}$.
(b) If $I\not\subseteq Ker\alpha$ or $J\not\subseteq Ker\beta$, then $I\times J\times K$ is an ideal in $\mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C}$ if and only if $J=\mathcal{B}$.

Example 2.4, preceding remark and the next proposition reveal resemblance of this products to matrix products.

Proposition 2.6. Let M be an ideal of $\mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C}$ and

$$I=\{a\in\mathcal{A}:(a,b,c)\in M \quad \text{for some} \quad ,b\in\mathcal{B},c\in\mathcal{C}\},$$

$$J=\{b\in\mathcal{B}:(a,b,c)\in M \quad \text{for some} \quad a\in\mathcal{A},c\in\mathcal{C}\},$$

$$K=\{c\in\mathcal{C}:(a,b,c)\in M \quad \text{for some} \quad ,a\in\mathcal{A},b\in\mathcal{B}\}.$$

Then
(i) I is an ideal in $\mathcal{A}$ and K is an ideal in $\mathcal{C}$.
(ii) If α and β are onto, then J is an ideal of $\mathcal{B}$. Furthermore if $\mathcal{A}$and $\mathcal{B}$ have an approximate identity and M is closed, then $M=I\times J\times K$.

Proof. **(i)** Straightforward.
(ii) Let $j\in J$ and $b\in\mathcal{B}$. Then there are $a\in\mathcal{A}$ and $c\in\mathcal{C}$ such that $\alpha(a)=b$ and $\beta(c)=b$. Since M is an ideal of $\mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C}$, then $(a,b,c)(0,j,0)=(0,\alpha(a)j,0)$ and $(0,j,0)(a,b,c)=(0,j\beta(c),0)$ are both in M and hence $jb,bj\in J$.
Let $(a_\lambda)_\lambda$ and $(c_\mu)_\mu$ be a bounded approximate identity for $\mathcal{A}$ and $\mathcal{C}$ respectively, $a_0\in I$, $b_0\in J$, and $c_0\in K$. Choose $a\in I$ and $c\in K$ such that $(a,b_0,c)\in M$. Then

$$\|(a_\lambda,0,c_\mu)(a_0,0,c_0)-(a_0,0,c_0)\|=\|a_\lambda a_0-a_0\|+\|c_\mu c_0-c_0\|\to 0$$

and hence $(a_0, 0, c_0) \in M$. Similarly $(a, 0, c) \in M$. Therefore

$$(a_0, b_0, c_0) = (a_0, 0, c_0) + (a, b_0, c) - (a, 0, c) \in M.$$

$\square$

Proof of the next theorem was inspired by [12, proposition 2.4.]

Theorem 2.7. Let $\mathcal{A}$ $\mathcal{B}$, and $\mathcal{C}$ be Banach algebras with the non-empty spectrum, $\alpha \in Hom(\mathcal{A}, \mathcal{B})$, and $\beta \in Hom(\mathcal{C}, \mathcal{B})$ such that $\|\alpha\| \leq 1$ and $\|\beta\| \leq 1$. Let

$$\begin{aligned} E: &= \{(1/2\psi o(\alpha + \beta(c)), \psi, 0) : \psi \in \Delta(\mathcal{B}), c \in \mathcal{C}\} \\ &\cup \{(0, \psi, 1/2\psi o(\alpha(a) + \beta)) : \psi \in \Delta(\mathcal{B}), a \in \mathcal{A}\} \\ F: &= \{(\varphi, 0, \omega) : \varphi \in \Delta(\mathcal{A}), \omega \in \Delta(\mathcal{C})\}. \end{aligned}$$

Then E and F are disjoint, closed subsets of $(\Delta(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}), weak^*)$ and

$$\Delta(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}) = E \cup F.$$

Proof. It is easy to see that $E \cup F \subseteq \Delta(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C})$ and $E \cap F = \emptyset$. Conversely, let $(\varphi, \psi, \omega) \in \Delta(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C})$. Then for every $(a, b, c), (a', b', c') \in \mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ the identities

$$(\varphi, \psi, \omega)((a, b, c)(a', b', c')) = (\varphi, \psi, \omega)(a, b, c)(\varphi, \psi, \omega)(a', b', c')$$

imply that

$$\begin{aligned} \varphi(aa') + \psi(\alpha(a)b' + b\beta(c')) + \omega(cc') &= \varphi(a)\varphi(a') + \varphi(a)\psi(b') + \varphi(a)\omega(c') \\ &+ \psi(b)\varphi(a') + \psi(b)\psi(b') + \psi(b)\omega(c') \\ &+ \omega(c)\varphi(a') + \omega(c)\psi(b') + \omega(c)\omega(c'). \end{aligned}$$

Taking $b = b' = c = c' = 0$, we get $\varphi(aa') = \varphi(a)\varphi(a')$, $a = a' = b = b' = 0$, we get $\omega(cc') = \omega(c)\omega(c')$, and taking $a = a' = c = c' = 0$, we get $\psi(bb') = 0$. Thus

$$\begin{aligned} \psi(\alpha(a))\psi(b') + \psi(b)\psi(\beta(c')) &= \varphi(a)\psi(b') + \varphi(a)\omega(c') + \psi(b)\varphi(a') \\ &+ \psi(b)\omega(c') + \omega(c)\varphi(a') + \omega(c)\psi(b'). \end{aligned}$$

Taking $a = a', b = b'$, and $c = c'$ we get $\psi(b)(\psi(\alpha(a) + \beta(c))) = \psi(b)(2\varphi(a) + 2\omega(c)) + 2\varphi(a)\omega(c)$. So if $\psi \neq 0$ and $b \in \mathcal{B}$ is chosen so that $\psi(b) \neq 0$ and $\omega = 0$ then, $\varphi = 1/2(\psi o(\alpha + \beta(c)))$ for some $c \in \mathcal{C}$ similarly if $\psi \neq 0$ and $b \in \mathcal{B}$ is chosen so that $\psi(b) \neq 0$ and $\varphi = 0$ then, $\omega = 1/2(\psi o(\alpha(a) + \beta))$ for some $a \in \mathcal{A}$. Therefore $(\varphi, \psi, \omega) \in E$. Now if $\psi = 0$, then $(\varphi, 0, \omega) \in F$. Therefore $\Delta(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}) = E \cup F$.

Let $(1/2(\psi_0 o(\alpha + \beta(c)), \psi_0, 0) \in E$ and choose $b \in \mathcal{B}$ such that $\psi_0(b) \neq 0$. Let $\epsilon = 1/2|\psi_0(b)|$ and consider the following relative weak*-neighborhood of $(1/2(\psi_0 o(\alpha + \beta(c)), \psi_0, 0)$

$$U = \{(\varphi, \psi, \omega) \in \Delta(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}) :| \psi(b) - \psi_0(b) |< \epsilon\}.$$

If $(\varphi, 0, \omega) \in U \cap F$, then $| \psi_0(b) |< \epsilon$, which is a contradiction. Thus $U \subseteq E$. This shows that E is open in $(\Delta(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}), weak^*)$ and hence F is closed.

Suppose $(\varphi, 0, \omega) \in F \cap \overline{E}^{w*}$ and choose a net $\{(1/2(\psi_\lambda o(\alpha+\beta(c_0)), \psi_\lambda, 0)\}$ in E which is weak*-convergent to $(\varphi, 0, \omega)$, that is,

$$1/2(\psi_\lambda o(\alpha(a) + \beta(c_0)) + \psi_\lambda(b) \to \varphi(a) \qquad (a, b, c) \in \mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}.$$

Taking $a = 0$, we conclude that $1/2(\psi_\lambda o(\beta(c_0))) + \psi_\lambda(b) \to 0,\ b \in \mathcal{B}$. In particular $\psi_\lambda o(\alpha + \beta(c_0)) \overset{w*}{\to} 0$. Letting $b = 0$ we see that $1/2(\psi_\lambda o(\alpha(a) + \beta(c_0)) \to \varphi(a)$ and hence $\varphi = 0$ which is a contradiction. Therefore E is closed in $(\Delta(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}), weak^*)$. ☐

Corollary 2.8. Let $\alpha \in Hom(\mathcal{A}, \mathcal{B})$, and $\beta \in Hom(\mathcal{C}, \mathcal{B})$, $\|\alpha\| \leq 1$ and $\|\beta\| \leq 1$. Then $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ is semisimple if and only if $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are semisimple.

Proof. Suppose $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ is semisimple, and $b \in \mathcal{B}$ is such that for $\psi \in \Delta(\mathcal{B})$, $\psi(b) = 0$. Then $(1/2(\psi o(\alpha + \beta(c_0)), \psi, 0)(0, b, 0) = 0$ and $(\varphi, 0, \omega)(0, b, 0) = 0$ $(\varphi \in \Delta(\mathcal{A}), \omega \in \Delta(\mathcal{C}))$. Thus $(\varphi, \psi, \omega)(0, b, 0) = 0$ for all $(\varphi, \psi, \omega) \in \Delta(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C})$ and hence $b = 0$. Therefore $\mathcal{B}$ is semisimple. Similarly $\mathcal{A}$ and $\mathcal{B}$ are semisimple.

Conversely if $(a, b, c) \in \mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ is such that for $(\varphi, \psi, \omega) \in \Delta(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C})$, $(\varphi, \psi, \omega)(a, b, c) = 0$, then $\varphi(a) = (\varphi, 0, 0)(a, b, c) = 0$ $(\varphi \in \Delta(\mathcal{A}))$. Since $\mathcal{A}$ is semisimple, it follows that $a = 0$. Consequently $\psi(b) = 0$, $(\psi \in \Delta(\mathcal{B}))$ and $\omega(c) = 0$, $(\omega \in \Delta(\mathcal{C}))$, and hence $b = 0, c = 0$ as $\mathcal{B}$, $\mathcal{C}$ are semisimple. Therefore $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ is semisimple. ☐

Remark 2.9. Suppose $\mathcal{A}$ and $\mathcal{C}$ are commutative and for every $a \in \mathcal{A}, b \in \mathcal{B}$, and $c \in \mathcal{C}$, $\alpha(a)b = b\beta(c)$. (By this hypothesis $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ is commutative.) Since $\mathcal{B}$ is a closed ideal of $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ and $(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C})/\mathcal{B}$ is isometrically isomorphic to $\mathcal{A} \times \mathcal{C}$, it follows from [10, theorems 4.2.6 and 4.3.8], $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ is regular if and only if $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are regular.

2.2. **Arens regularity.** Let $\mathcal{A}$ be a Banach algebra. The first and second Arens multiplications on $\mathcal{A}^{**}$ that we denote by $\Box$ and $\Diamond$ respectively, are defined in three steps. For $a, b \in \mathcal{A}$, $\phi \in \mathcal{A}^*$ and $\Phi, \Psi \in \mathcal{A}^{**}$, the elements $\phi.a$, $a.\phi$, $\Phi.\phi$, $\phi.\Phi$ of $\mathcal{A}^*$ and $\Psi \Box \Phi$, $\Phi \Diamond \Psi$ of $\mathcal{A}^{**}$ are defined in the following way:

$$\begin{array}{ll} < \phi.a\ ,\ b >=< \phi\ ,\ ab > & < a.\phi\ ,\ b >=< \phi\ ,\ ba > \\ < \Phi.\phi\ ,\ b >=< \Phi\ ,\ \phi.b > & < \phi.\Phi\ ,\ a >=< \Phi\ ,\ a.\phi > \\ < \Phi \Box \Psi\ ,\ \phi >=< \Phi\ ,\ \Psi.\phi > & < \Phi \Diamond \Psi\ ,\ \phi >=< \Psi\ ,\ \phi.\Phi > . \end{array}$$

When we refer to $\mathcal{A}^{**}$ without explicit reference to any of Arens products, we mean $\mathcal{A}^{**}$ with the first Arens product. For fixed $\Psi \in \mathcal{A}^{**}$ the map $\Phi \mapsto \Phi \Box \Psi$ [resp. $\Phi \mapsto \Psi \Diamond \Phi$] is weak* $-$ weak* continuous, but the map $\Phi \mapsto \Psi \Box \Phi$ [resp. $\Phi \mapsto \Phi \Diamond \Psi$] is not necessarily weak* $-$ weak* continuous, unless Ψ is in $\mathcal{A}$. The left and right topologimathcal centers of $\mathcal{A}^{**}$ are defined by:

$$\mathcal{Z}_t^{(l)}(\mathcal{A}^{**}) = \{\Phi \in \mathcal{A}^{**} : \Phi \Box \Psi = \Phi \Diamond \Psi, \quad \Psi \in \mathcal{A}^{**}\},$$

$$\mathcal{Z}_t^{(r)}(\mathcal{A}^{**}) = \{\Phi \in \mathcal{A}^{**} : \Psi\Box\Phi = \Psi\Diamond\Phi, \quad \Psi \in \mathcal{A}^{**}\}.$$

We say that $\mathcal{A}$ is left Arens regular [resp. strongly Arens irregular] if $\mathcal{Z}_t^{(l)}(\mathcal{A}^{**}) = \mathcal{A}^{**}$ [resp. $\mathcal{Z}_t^{(l)}(\mathcal{A}^{**}) = \mathcal{A}$], right Arens regular [resp. strongly Arens irregular] if $\mathcal{Z}_t^{(r)}(\mathcal{A}^{**}) = \mathcal{A}^{**}$ [resp. $\mathcal{Z}_t^{(r)}(\mathcal{A}^{**}) = \mathcal{A}$], and Arens regular [resp. strongly Arens irregular] if it is both left and right Arens regular [resp. strongly Arens irregular].

Let $\alpha \in Hom(\mathcal{A}, \mathcal{B})$. Then both of $\alpha^{**} : (\mathcal{A}^{**}, \Box) \to (\mathcal{B}^{**}, \Box)$ and $\alpha^{**} : (\mathcal{A}^{**}, \Diamond) \to (\mathcal{B}^{**}, \Diamond)$ are continuous homomorphisms [2, page 251]. Moreover if $\|\alpha\| \leq 1$, then $\|\alpha^{**}\| \leq 1$. A similar argument applies to $\beta \in Hom(\mathcal{C}, \mathcal{B})$.

Proof of the next theorem was inspired by [4, Theorem 3.1]

Theorem 2.10. Suppose $\alpha \in Hom(\mathcal{A}, \mathcal{B})$, $\beta \in Hom(\mathcal{C}, \mathcal{B})$ and $\|\alpha\| \leq 1$, $\|\beta\| \leq 1$, and $\mathcal{B}$ is Arens regular.
(i) If $\mathcal{A}^{**}$, $\mathcal{B}^{**}$, $\mathcal{C}^{**}$ and $(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C})^{**}$ are equipped with their first [resp. second] Arens products, then $(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C})^{**}$ is isometrically algebra isomorphic to $\mathcal{A}^{**} \times_{\alpha^{**}} \mathcal{B}^{**} \times_{\beta^{**}} \mathcal{C}^{**}$.
(ii) Let $\mathcal{Z}_t$ be either of left or right topologimathcal centers. Then

$$\mathcal{Z}_t((\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C})^{**}) = \mathcal{Z}_t(\mathcal{A}^{**}) \times_{\alpha^{**}} \mathcal{B}^{**} \times_{\beta^{**}} \mathcal{Z}_t(\mathcal{C}^{**}).$$

In particular, $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ is Arens regular if and only if $\mathcal{A}$ and $\mathcal{C}$ are Arens regular.

Proof. (i) Since the underlying Banach space of both of $(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C})^{**}$ and $\mathcal{A}^{**} \times_{\alpha^{**}} \mathcal{B}^{**} \times_{\beta^{**}} \mathcal{C}^{**}$ are $\mathcal{A}^{**} \times \mathcal{B}^{**} \times \mathcal{C}^{**}$, then it is enough to show that identity map between these two algebras keeps the product. The first Arens product on $\mathcal{A}^{**} \times_{\alpha^{**}} \mathcal{B}^{**} \times_{\beta^{**}} \mathcal{C}^{**}$ is identified by the equations

$$(\Phi, \Psi, \Omega)(\Phi', \Psi', \Omega') = (\Phi\Box\Phi', \alpha^{**}(\Phi)\Box\Psi' + \Psi\Box\beta^{**}(\Omega'), \Omega\Box\Omega') \tag{2.1}$$

when $(\Phi, \Psi, \Omega), (\Phi', \Psi', \Omega') \in (\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C})^{**}$. We calculate the first Arens product on $(\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C})^{**}$. Let $(a, b, c), (a', b', c') \in \mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$, $(\varphi, \psi, \omega) \in \mathcal{A}^* \times \mathcal{B}^* \times \mathcal{C}^*$, and $(\Phi, \Psi, \Omega), (\Phi', \Psi', \Omega') \in \mathcal{A}^{**} \times \mathcal{B}^{**} \times \mathcal{C}^{**}$. Then:

$$\begin{aligned} < (\varphi, \psi, \omega) \quad &\cdot \quad (a, b, c), (a', b', c') >=< (\varphi, \psi, \omega), (a, b, c) \cdot (a', b', c') > \\ &= \quad < (\varphi, \psi, \omega), (aa', \alpha(a)b' + b\beta(c'), cc') > \\ &= \quad < \varphi, aa' > + < \psi, \alpha(a)b' + b\beta(c') > + < \omega, cc' > \\ &= \quad < \varphi \cdot a, a' > + < \psi \cdot \alpha(a), b' > \\ &+ \quad < \beta^*(\psi \cdot b), c' > + < \omega \cdot c, c' > \\ &= \quad < (\varphi \cdot a, \psi \cdot \alpha(a), (\beta^*(\psi \cdot b) + \omega \cdot c)), (a', b', c') > . \end{aligned}$$

Thus

$$(\varphi, \psi, \omega) \cdot (a, b, c) = (\varphi \cdot a, \psi \cdot \alpha(a), \beta^*(\psi \cdot b) + \omega \cdot c).$$

Also

$$\begin{aligned} <(\Phi,\Psi,\Omega) \quad &\cdot \quad (\varphi,\psi,\omega),(a,b,c)>=<(\Phi,\Psi,\Omega),(\varphi,\psi,\omega)\cdot(a,b,c)> \\ &= \quad <(\Phi,\Psi,\Omega),(\varphi\cdot a,\psi\cdot\alpha(a),\beta^*(\psi\cdot b)+\omega\cdot c)> \\ &= \quad <\Phi,\varphi\cdot a>+<\Psi,\psi\cdot\alpha(a)>+<\Omega,\beta^*(\psi\cdot b)>+<\Omega,\omega\cdot c> \\ &= \quad <\Phi\cdot\varphi,a>+<\alpha^*(\Psi\cdot\psi),a> \\ &+ \quad <\beta^{**}(\Omega)\cdot\psi,b>+<\Omega\cdot\omega,c> \\ &= \quad <(\Phi\cdot\varphi+\alpha^*(\Psi\cdot\psi),\beta^{**}(\Omega)\cdot\psi,\Omega\cdot\omega),(a,b,c)>. \end{aligned}$$

So

$$(\Phi,\Psi,\Omega)\cdot(\varphi,\psi,\omega)=(\Phi\cdot\varphi+\alpha^*(\Psi\cdot\psi),\beta^{**}(\Omega)\cdot\psi,\Omega\cdot\omega).$$

Now

$$\begin{aligned} <(\Phi,\Psi,\Omega) \quad &\Box \quad (\Phi',\Psi',\Omega'),(\varphi,\psi,\omega)>=<(\Phi,\Psi,\Omega),(\Phi',\Psi',\Omega')\cdot(\varphi,\psi,\omega)> \\ &= \quad <(\Phi,\Psi,\Omega),(\Phi'\cdot\varphi+\alpha^*(\Psi'\cdot\psi),\beta^{**}(\Omega')\cdot\psi,\Omega'\cdot\omega)> \\ &= \quad <\Phi,\Phi'\cdot\varphi+\alpha^*(\Psi'\cdot\psi)>+<\Psi,\beta^{**}(\Omega')\cdot\psi>+<\Omega,\Omega'\cdot\omega> \\ &= \quad <\Phi\Box\Phi',\varphi>+<\alpha^{**}(\Phi)\Box\Psi',\psi> \\ &+ \quad <\Psi\Box\beta^{**}(\Omega'),\psi>+<\Omega\Box\Omega',\omega> \\ &= \quad <(\Phi\Box\Phi',\alpha^{**}(\Phi)\Box\Psi'+\Psi\Box\beta^{**}(\Omega'),\Omega\Box\Omega'),(\varphi,\psi,\omega)>. \end{aligned}$$

Therefore

$$(\Phi,\Psi,\Omega)(\Phi',\Psi',\Omega')=(\Phi\Box\Phi',\alpha^{**}(\Phi)\Box\Psi'+\Psi\Box\beta^{**}(\Omega'),\Omega\Box\Omega'). \tag{2.2}$$

The result for the first Arens product follows from (1) and (2). A similar argument provides the result for the second Arens product.

(ii) Since $\mathcal{B}$ is Arens regular, then $\mathcal{B}^{**}=\mathcal{Z}_t^{(l)}(\mathcal{B}^{**})=\mathcal{Z}_t^{(r)}(\mathcal{B}^{**})$. Let

$$(\Phi,\Psi,\Omega)\in\mathcal{Z}_t^{(l)}((\mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C})^{**})=\mathcal{Z}_t^{(l)}(\mathcal{A}^{**}\times_{\alpha^{**}}\mathcal{B}^{**}\times_{\beta^{**}}\mathcal{C}^{**}).$$

Then for every $(\Phi',\Psi',\Omega')\in\mathcal{A}^{**}\times_{\alpha^{**}}\mathcal{B}^{**}\times_{\beta^{**}}\mathcal{C}^{**}$ we have

$$(\Phi,\Psi,\Omega)\Box(\Phi',\Psi',\Omega')=(\Phi,\Psi,\Omega)\Diamond(\Phi',\Psi',\Omega')$$

or equivalently

$$(\Phi\Box\Phi',\alpha^{**}(\Phi)\Box\Psi'+\Psi\Box\beta^{**}(\Omega'),\Omega\Box\Omega')=((\Phi\Diamond\Phi',\alpha^{**}(\Phi)\Diamond\Psi'+\Psi\Diamond\beta^{**}(\Omega'),\Omega\Diamond\Omega').$$

In particular $\Phi\Box\Phi'=\Phi\Diamond\Phi'$, $\Omega\Box\Omega'=\Omega\Diamond\Omega'$ and hence $\Phi\in\mathcal{Z}_t^{(l)}$, $\Omega\in\mathcal{Z}_t^{(l)}$. So

$$\mathcal{Z}_t^{(l)}(\mathcal{A}^{**}\times_{\alpha^{**}}\mathcal{B}^{**}\times_{\beta^{**}}\mathcal{C}^{**})\subseteq\mathcal{Z}_t^{(l)}(\mathcal{A}^{**})\times_{\alpha^{**}}\mathcal{B}^{**}\times_{\beta^{**}}\mathcal{Z}_t^{(l)}(\mathcal{C}^{**}).$$

Conversely let $(\Phi,\Psi,\Omega)\in\mathcal{Z}_t^{(l)}(\mathcal{A}^{**})\times_{\alpha^{**}}\mathcal{B}^{**}\times_{\beta^{**}}\mathcal{Z}_t^{(l)}(\mathcal{C}^{**})$. Arens regularity of $\mathcal{B}$ for every $(\Phi',\Psi',\Omega')\in\mathcal{A}^{**}\times_{\alpha^{**}}\mathcal{B}^{**}\times_{\alpha^{**}}\mathcal{C}^{**}$ implies that

$$(\Phi,\Psi,\Omega)\Box(\Phi',\Psi',\Omega')=(\Phi,\Psi,\Omega)\Diamond(\Phi',\Psi',\Omega')$$

and hence $(\Phi,\Psi,\Omega)\in\mathcal{Z}_t^{(l)}(\mathcal{A}^{**}\times_{\alpha^{**}}\mathcal{B}^{**}\times_{\beta^{**}}\mathcal{C}^{**})$. Therefore

$$\mathcal{Z}_t^{(l)}(\mathcal{A}^{**})\times_{\alpha^{**}}\mathcal{B}^{**}\times_{\beta^{**}}\mathcal{Z}_t^{(l)}(\mathcal{C}^{**})\subseteq\mathcal{Z}_t^{(l)}(\mathcal{A}^{**}\times_{\alpha^{**}}\mathcal{B}^{**}\times_{\beta^{**}}\mathcal{C}^{**}).$$

□

2.3. **Amenability.** In this section we show stability of several notions of amenability with respect to this product between three Banach algebras. Let $\mathcal{X}$ be a Banach $\mathcal{A}$ -bimodule. We denote the set of all bounded derivations from $\mathcal{A}$ into $\mathcal{X}$ by $\mathcal{Z}^1(\mathcal{A},\mathcal{X})$ and the set of inner derivations from $\mathcal{A}$ into $\mathcal{X}$ by $\mathcal{B}^1(\mathcal{A},\mathcal{X})$. Let

$$\mathcal{H}^1(\mathcal{A},\mathcal{X}) := \mathcal{Z}^1(\mathcal{A},\mathcal{X})/\mathcal{B}^1(\mathcal{A},\mathcal{X})$$

be the first cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}$. We say that $\mathcal{A}$ is amenable if $\mathcal{H}^1(\mathcal{A},\mathcal{X}^*) = \{0\}$ for every Banach $\mathcal{A}$ -bimodule $\mathcal{X}$ and it is weakly amenable if $\mathcal{H}^1(\mathcal{A},\mathcal{A}^*) = \{0\}$. A derivation $D : \mathcal{A} \to \mathcal{X}$ is approximately inner if there exists a net $(x_\lambda) \subseteq \mathcal{X}$ such that $D(a) = lim_\lambda(a \cdot x_\lambda - x_\lambda \cdot a)(a \in \mathcal{A})$. The algebra $\mathcal{A}$ is approximately amenable if for each Banach $\mathcal{A}$ -bimodule $\mathcal{X}$ every derivation $D : \mathcal{A} \to \mathcal{X}^*$ is approximately inner and $\mathcal{A}$ is approximately weakly amenable if every derivation $D : \mathcal{A} \to \mathcal{A}^*$ is approximately inner.

Amenability has well known hereditary properties [3, 15]. In particular, if $\mathcal{U}$ is a strongly splitting extension of $\mathcal{B}$ on $\mathcal{A}$ it is amenable (respectively contractible) if and only if both $\mathcal{A}$ and $\mathcal{B}$ are amenable (respectively contractible).

Theorem 2.11. Let $\alpha \in Hom(\mathcal{A},\mathcal{B})$, $\beta \in Hom(\mathcal{C},\mathcal{B})$, $\|\alpha\| \leq 1$ and $\|\beta\| \leq 1$. Then
(i) $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ is amenable (respectively contractible) if and only if $\mathcal{A}$,$\mathcal{B}$, and $\mathcal{C}$ are amenable (respectively contractible).
(ii) If moreover $\mathcal{B}$ has a bounded approximate identity and $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ is approximately amenable then so are $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

Proof. (i) This part follows from the fact that the short exact sequence

$$\Sigma : 0 \to \mathcal{A} \times_{\alpha,0} \mathcal{B} \xrightarrow{i} \mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C} \xrightarrow{q} \mathcal{C} \to 0$$

splits strongly and [4, Theorem 4.1].
(ii) This a consequence of 2.5 and [8, Corollary 2.1]. □

Theorem 2.12. Let $\alpha \in Hom(\mathcal{A},\mathcal{B})$, $\beta \in Hom(\mathcal{C},\mathcal{B})$, $\|\alpha\| \leq 1$ and $\|\beta\| \leq 1$.
(i) If $\mathcal{A}\mathcal{B}$, and $\mathcal{C}$ are weakly amenable then so is $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$.
(ii) If $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ is weakly amenable then $\mathcal{A}$ and $\mathcal{C}$ are weakly amenable. Moreover suppose that $\mathcal{B}$ is commutative.
(iii) If $\mathcal{A}\mathcal{B}$ and $\mathcal{C}$ are approximately weakly amenable then so is $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$.
(iv) If $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ is approximately weakly amenable then $\mathcal{A}$ and $\mathcal{C}$ are approximately weakly amenable.

Proof. (i) Since $\mathcal{B}$ is a weakly amenable closed ideal of $\mathcal{A} \times_{\alpha,0} \mathcal{B}$ and $\mathcal{A} \cong \mathcal{A} \times_{\alpha,0} \mathcal{B}/\mathcal{B}$ is weakly amenable then $\mathcal{A} \times_{\alpha,0} \mathcal{B}$ is weakly amenable. Similarly $\mathcal{A} \times_{\alpha,0} \mathcal{B}$ is a weakly amenable closed ideal of $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ and $\mathcal{C} \cong \mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}/\mathcal{A} \times_{\alpha,0} \mathcal{B}$ is weakly amenable then $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ is weakly amenable.

(ii) Let $d : \mathcal{A} \to \mathcal{A}^*$ be a bounded derivation and define $D : \mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C} \to \mathcal{A}^* \times \mathcal{B}^* \times \mathcal{C}^*$ by $D(a, b, c) = (d(a), 0, 0)$. Then D is a bounded linear map and

$$\begin{aligned} D((a,b,c)(a',b',c')) &= D(aa', \alpha(a)b' + b\beta(c'), cc') \\ &= (d(aa'), 0, 0) = (d(a)a' + ad(a'), 0, 0) \\ &= (d(a), 0, 0)(a', b', c') + (a, b, c)(d(a'), 0, 0) \\ &= D(a, b, c)(a', b', c') + (a, b, c)D(a', b', c'). \end{aligned}$$

So D is a bounded derivation and hence there is a $(\zeta_1, \zeta_2, \zeta_3) \in \mathcal{A}^* \times \mathcal{B}^* \times \mathcal{C}^*$ such that

$$D(a, b, c) = (\zeta_1, \zeta_2, \zeta_3)(a, b, c) - (a, b, c)(\zeta_1, \zeta_2, \zeta_3),\ ((a, b, c) \in \mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}).$$

So

$$\begin{aligned} d(a), 0, 0) = D(a, 0, 0) &= (\zeta_1, \zeta_2, \zeta_3)(a, 0, 0) - (a, 0, 0)(\zeta_1, \zeta_2, \zeta_3) \\ &= (\zeta_1 a - a\zeta_1, 0, 0). \end{aligned}$$

Therefore $d(a) = \zeta_1 a - a\zeta_1 \quad (a \in \mathcal{A})$. Similarly $\mathcal{C}$ is also weak amenable.

(iii) Since for commutative Banach algebras the two concepts of weak amenability and approximate weak amenability coincide, then $\mathcal{B}$ is weakly amenable. But $\mathcal{B}$ is a closed ideal of $\mathcal{A} \times_{\alpha,0} \mathcal{B}$, and $\mathcal{A} \cong \mathcal{A} \times_{\alpha,0} \mathcal{B}/\mathcal{B}$ is approximately weakly amenable. So by [5, Proposition 2.2] $\mathcal{A} \times_{\alpha,0} \mathcal{B}$ is approximately weakly amenable. Also $\mathcal{C} \cong \mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}/\mathcal{A} \times_{\alpha,0} \mathcal{B}$ is approximately weakly amenable and so by [5, Proposition 2.2] $\mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$ is approximately weakly amenable.

(iv) Let $d : \mathcal{A} \to \mathcal{A}^*$ be a bounded derivation and as in part (ii) define a bounded derivation $D : \mathcal{A}\times_\alpha\mathcal{B}\times_\beta\mathcal{C} \to \mathcal{A}^*\times\mathcal{B}^*\times\mathcal{C}^*$ by $D(a, b, c) = (d(a), 0.0)$. By assumption there exists a net $(\varphi_\lambda, \psi_\lambda, \omega_\lambda)_\lambda$ in $\mathcal{A}^* \times \mathcal{B}^* \times \mathcal{C}^*$ such that for every $(a, b, c) \in \mathcal{A} \times_\alpha \mathcal{B} \times_\beta \mathcal{C}$

$$D(a, b, c) = lim_\lambda((a, b, c)(\varphi_\lambda, \psi_\lambda, \omega_\lambda) - (\varphi_\lambda, \psi_\lambda, \omega_\lambda)(a, b, c)).$$

Now

$$\begin{aligned} < d(a), a' > &= < D(a, 0, 0), (a', 0, 0) > \\ &= < lim_\lambda((a, 0, 0)(\varphi_\lambda, \psi_\lambda, \omega_\lambda) - (\varphi_\lambda, \psi_\lambda, \omega_\lambda)(a, 0, 0)), (a', 0, 0) > \\ &= < lim_\lambda(a\varphi_\lambda - \varphi_\lambda a), a' > \end{aligned}$$

and hence $d(a) = lim_\lambda(a\varphi_\lambda - \varphi_\lambda a) \quad a \in \mathcal{A}$. By same argument $\mathcal{C}$ is also approximately weakly amenable. $\square$

2.4. **Conclusion.** By using our definition, many related concepts and theorems such as cohomological characterization, multipliers and BSE-functions Can be generalized to our new Banach algebras.

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