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Research Paper

ON KROPINA GEODESIC ORBIT SPACES

MILAD L. ZEINALI AND DARIUSH LATIFI*

Abstract. In this paper, we study Kropina spaces whose geodesics are the orbits of one-parameter subgroup of the group of isometries. Also, we study Kropina g.o. metrics on homogeneous spaces with two isotropy summands and we will investigate Kropina g.o. metrics on compact homogeneous spaces with two isotropy summands. A complete characterization of navigation data of non-Riemannian Kropina g.o. metrics is given.

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1. Introduction

The theory of Finsler spaces developed from the calculus of variations as well as Riemannian geometry. To obtain Finsler spaces instead of Riemann spaces we must replace the requirement that the space be locally Euclidean by the requirement that it be locally Minkowskian. Since a Euclidean metric is also Minkowskian, a Riemann space is also a Finsler space. In 1972, Matsumoto [20] introduced the concept of (α, β)-metrics which are the generalization of Randers metric introduced by Randers [23]. The (α, β)-metrics form an important class of Finsler metrics appearing iteratively in formulating Physics, Mechanics, Seismology, Biology, Control Theory, etc [2]. Recently, many studies have been conducted in the field of (α, β)-metrics (for more details see [1, 8, 18, 19, 27]).

An (α, β)-metric is a Finsler metric of the form $F = \alpha\varphi(s)$, $s = \frac{\beta}{\alpha}$ where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on a connected smooth n-dimensional manifold M and $\beta = b_i(x)y^i$ is a 1-form on M. The important types of (α, β)-metrics happen when $\varphi(s) = \frac{1}{s}$. In this case we have

$$F = \frac{\alpha^2}{\beta}, \tag{1.1}$$

which called Kropina metric, which was considered by Kropina firstly (see [15]).
Such a metric is of physical interest in the sense that it describes the general dynamical system represented by a Lagrangian function [3], although it has the singularity.

In 1931, the following problem studied by Zermelo [7]:

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*Corresponding author.

"Suppose a ship sails the sea and a wind comes up. How must the captain steer the ship in order to reach a given destination in the shortest time?"

Zermelo solved this problem for the Euclidean flat plane. The problem was solved by Bao, Robles and Shen for the Riemannian manifold (M,h) under the assumption $h(W,W)<1$ where W is the wind [5]. If W is a time-independent wind, they found the path minimizing are exactly the geodesics of Randers metric

$$F(x,y)=\alpha(x,y)+\beta(x,y)=\frac{\sqrt{\lambda.|y|^2+W_0^2}}{\lambda}-\frac{W_0}{\lambda},$$

where $W=W^i\frac{\partial}{\partial x^i}$ is the wind velocity, $|y|^2=h(y,y)$, $\lambda=1-|W|^2$ and $W_0=h(W,y)$.

The Randers metric F is said to solve the Zermelo's navigation problem in the case $h(W,W)<1$. The condition $h(W,W)<1$ ensures that F is a positive-definite Finsler metric.

By normalizing, we can consider an open sea represented by a Riemannian manifold (M,h) and a wind $W=W^i\frac{\partial}{\partial x^i}$ such that $h(W,W)=1$. Indeed, another description of Kropina metrics is the definition of these metrics as the solutions of the navigation problem on some Riemannian manifold (M,h) under the influence of a vector field W with

$$\|W\|_h=h(W,W)=1.$$

In this case the pair (h,W) is called the navigation data of F. In fact $F=\frac{\alpha^2}{\beta}$ is Kropina metric on manifold M if and only if

$$F=\frac{h^2}{2W_0},$$

where $h^2=e^{2\rho}\alpha^2$, $2W_0=e^{2\rho}\beta$, $e^{2\rho}b^2=4$ and $b=\|\beta\|_h$ for some functions $\rho=\rho(x)$ on M.

The Chern connection defines the covariant derivative D_VU of a vector field $U\in\chi(M)$ in the direction $V\in T_pM$. Let $\sigma:[0,r]\to M$ be a smooth curve with velocity field $T=T(t)=\dot\sigma(t)$. Suppose that U and W are vector fields defined along σ. We define D_TU with reference vector W as

$$D_TU=\left[\frac{dU^i}{dt}+U^jT^k(\Gamma^i_{jk})_{(\sigma,W)}\right]\frac{\partial}{\partial x^i}|_{\sigma(t)}.$$

A curve $\sigma:[0,r]\to M$, with velocity $T=\dot\sigma$ is a Finslerian geodesic if

$$D_T\left[\frac{T}{F(T)}\right]=0,$$

with reference vector T.

There are some important subclasses of geodesic orbit manifolds. Indeed, *g.o.* spaces may be considered as a natural generalization of Riemannian symmetric spaces. On the other hand, the class of *g.o.* spaces is much larger than the class of symmetric spaces. A Riemannian manifold (M,g) is called a geodesic orbit manifold (*g.o.* manifold) if every its geodesic is an orbit of a one-parameter group of isometries of (M,g). Every such manifold is homogeneous and can be identified with a coset space $M=G/H$ of a transitive Lie group G of isometries. A Riemannian homogeneous space $(M=G/H,g)$ of a Lie group G is called a space with homogeneous geodesics (or a geodesic orbit space, shortly, *g.o.* space), if any its geodesic is an orbit of a one-parameter subgroup of the group G. This terminology was introduced by

O. Kowalski and L. Vanhecke in the [14].

We noted that, any homogeneous space $M = G/H$ of a compact Lie group G admits a Riemannian metric g such that (M, g) is a $g.o.$ space. It suffices to take the metric g induced by a bi-invariant Riemannian metric g_0 on the Lie group G such that $(G, g_0) \to (M = G/H, g)$ is a Riemannian submersion with totally geodesic fibres. Such geodesic orbit space $(M = G/H, g)$ is called a normal homogeneous space. It should be noted also that any naturally reductive Riemannian manifold is geodesic orbit. Recall that a Riemannian manifold (M, g) is naturally reductive if it admits a transitive Lie group G of isometries with a bi-invariant pseudo-Riemannian metric g_0, which induces the metric g on $M = G/H$.

A geodesic $\delta : \mathbb{R} \to M$ on a Riemannian manifold (M, Q) is called homogeneous if there is a one-parameter group of isometries $\omega : \mathbb{R} \times M \to M$ such that

$$\delta(t) = \omega\big(t, \delta(0)\big), \quad t \in \mathbb{R}.$$

The notion of a homogeneous geodesic plays a fundamental role in the theory of homogeneous Riemannian manifold, especially in the study of $g.o.$ spaces. $g.o.$ spaces in fact are a Riemannian manifold whose geodesics are all homogeneous. Any naturally reductive Riemannian manifold is a $g.o.$ space.

Since the full group of isometries of a Finsler space is a Lie group [10], we can define a Finsler $g.o.$ space in exactly the same way as in the Riemannian case, namely, a Finsler $g.o.$ space is a space such that every geodesic is the orbit of a one-parameter group of isometries.

A homogeneous Kropina space can be written as a coset space G/H with a G-invariant Kropina metric $F = \frac{\alpha^2}{\beta}$, where both the Riemannian metric α and the form β are invariant under the action of G. In particular, the Lie algebra of G, has a decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

such that $Ad(h)(\mathfrak{m}) \subset \mathfrak{m}$, $h \in H$. Identifying $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the origin o, we get an H-invariant inner product on $\mathfrak{m}$.

2. **Preliminaries**

Let M be a smooth n- dimensional C^∞ manifold and TM be its tangent bundle. A Finsler metric on a manifold M is a non-negative function $F : TM \to \mathbb{R}$ with the following properties [4]:

(1) F is smooth on the slit tangent bundle $TM^0 := TM \backslash \{0\}$.
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$.
(3) The following bilinear symmetric form $g_y : T_xM \times T_xM \to \mathbb{R}$ is positive definite

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x, y + su + tv)|_{s=t=0}.$$

Definition 2.1. Let $\alpha = \sqrt{\widetilde{a}_{ij}(x)\, y^i y^j}$ be a norm induced by a Riemannian metric $\widetilde{a}$ and $\beta(x, y) = b_i(x)y^i$ be a 1-form on an n-dimensional manifold M. Let

$$b := \|\beta(x)\|_\alpha := \sqrt{\widetilde{a}^{ij}(x) b_i(x) b_j(x)}.$$

Now, let the function F is defined as follows

$$F := \alpha\varphi(s), \quad s = \frac{\beta}{\alpha}, \tag{2.1}$$

where $\varphi = \varphi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\varphi(s) - s\varphi'(s) + \left(b^2 - s^2\right)\varphi''(s) > 0, \qquad |s| \leq \ b < b_0.$$

Then F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (2.1) is called an (α, β)-metric.

A Finsler space having the Finsler function:

$$F(x, y) = \frac{\alpha^2(x, y)}{\beta(x, y)},$$

is called a Kropina space.

The Riemannian metric $\widetilde{a}$ induces an inner product on any cotangent space T_x^*M such that

$$\langle dx^i(x), dx^j(x)\rangle = \widetilde{a}^{ij}(x).$$

The induced inner product on T_x^*M induced a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field $\widetilde{X}$ on M such that

$$\widetilde{a}\left(y, \widetilde{X}(x)\right) = \beta(x, y).$$

Also, we have

$$\|\beta(x)\|_\alpha = \|\widetilde{X}(x)\|_\alpha.$$

Therefore we can write (α, β)-metrics as follows:

$$F(x, y) = \alpha(x, y)\,\varphi\left(\frac{\widetilde{a}(\widetilde{X}(x), y)}{\alpha(x, y)}\right), \tag{2.2}$$

where for any $x \in M$,

$$\sqrt{\widetilde{a}\left(\widetilde{X}(x), \widetilde{X}(x)\right)} = \|\widetilde{X}(x)\|_\alpha < b_0.$$

Let (M, F) be a Finsler space, where F is positively homogeneous. As in the Riemannian case, we have two types of definition of isometry on (M, F), in terms of Finsler function in the tangent space and the induced non-reversible distance function on the base manifold M. The equivalence of these two definitions in the Finsler case is a result of Deng and Hou [10]. They also prove that the group of isometries of a Finsler space is a Lie transformation group of the underlying manifold which can be used to study homogeneous Finsler spaces.

Definition 2.2. [17] A Finsler space (M, F) is called a homogeneous Finsler space if the group of isometries of (M, F), $I(M, F)$, acts transitively on M.

Now we have the following definition:

Definition 2.3. Let (M, F) be a Finsler space and $G = I(M, F)$ the full group of isometries. The space (M, F) is called a Finsler geodesic orbit space if every geodesic of (M, F) is the orbit of a one-parameter subgroup of G. That is, if γ is a geodesic, then there exist $W \in \mathfrak{g} = Lie(G)$ and $o \in M$, such that

$$\gamma(t) = \exp(tW).o.$$

Definition 2.4. Let $(G/H, F)$ be a homogeneous Finsler space, and $p = eH \in G/H$. A vector $X \in \mathfrak{g} - \{0\}$ is called a geodesic vector if the curve $\exp(tX).p$ is a geodesic.

For a geodesic vector we have the following geodesic Lemma from the second author:

Lemma 2.5. [17] *A vector $X \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if*

$$g_{X_{\mathfrak{m}}}\left(X_{\mathfrak{m}}, [X, Z]_{\mathfrak{m}}\right) = 0, \qquad \forall Z \in \mathfrak{m},$$

where the subscript $\mathfrak{m}$ means the corresponding projection, and g is the fundamental tensor of F on $\mathfrak{m}$.

The following result is obvious:

Proposition 2.6. *Let $(G/H, F)$ be a homogeneous Finsler space. Then every geodesic in G/H is an orbit of a one-parameter subgroup of G if and only if for every $X \in \mathfrak{m} - \{0\}$, there exists a vector $\zeta(X) \in \mathfrak{h}$, such that $X + \zeta(X)$ is a geodesic vector.*

Now for a isometries of Kropina space we have the following:

Proposition 2.7. [13] *Suppose that (M, F) is a Kropina space which arises from a navigation data (h, W). Then the isometry group of (M, F) is a closed subgroup of the isometry group of Riemannian manifold (M, h).*

3. Kropina geodesic orbit spaces

In this section we study some results for Kropina geodesic orbit spaces. Notice that we restrict our consideration to the domain where $\beta = b_i(x) y^i > 0$, which is equivalent to

$$W_0 = W_i(x) y^i > 0.$$

Let $h = \sqrt{h_{ij}(x) y^i y^j}$ be a Riemannian metric and $W = W^i \frac{\partial}{\partial x^i}$ a vector field on M. A Finsler metric F is of Kropina type if and only if it solves the navigation problem on some Riemannian manifold (M, h), under the influence of a wind W with $\|W\|_h = 1$. Namely, $F = \frac{\alpha^2}{\beta}$ if and only if $F = \frac{h^2}{2W_0}$, where $h^2 = e^{2\rho}\alpha^2$, $2W_0 = e^{2\rho}\beta$, $e^{2\rho}b^2 = 4$ and $b = \|\beta\|_h$, for some functions $\rho = \rho(x)$ on M. We call such a pair (h, W) the navigation data of Kropina metric F.

Now, suppose that $(M = G/H, g)$ be a compact homogeneous Riemannian manifold with G-compact and semi-simple. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of the compact Lie groups G and H, respectively. Let B be the minus Killing form of $\mathfrak{g}$. Then $\mathfrak{g}$ admits a B-orthogonal reductive decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

where $\mathfrak{m}$ is isomorphic to the tangent space of M at $o = eH$ via $X \to \frac{d}{dt}|_{t=0} exptX.o$. Any G-invariant metric g on M is in a one-to-one correspondence to an $Ad(H)$-invariant inner product $\langle , \rangle$ on $\mathfrak{m}$. Moreover, any $Ad(H)$-invariant inner product is in a one-to-one correspondence to an endomorphism $A : \mathfrak{m} \to \mathfrak{m}$ by $\langle X, Y \rangle = B(AX, Y)$, $\forall X, Y \in \mathfrak{m}$, which is called the metric endomorphism. Obviously, A is $Ad(H)$-equivariant (hence $ad(H)$-equivariant), symmetric with respect to B and positively definite.

We note that, a (local) flow on a manifold M is map $\phi : (-\epsilon, \epsilon) \times M \to M$, also denoted by $\phi_t := \phi(t, .)$, satisfying

(1) $\phi_0 = id : M \to M$,

(2) $\phi_s.\phi_t = \phi_{s+t}$, for any $s, t \in (-\epsilon, \epsilon)$ with $s + t \in (-\epsilon, \epsilon)$.

Proposition 3.1. *Suppose $(M, F = \frac{\alpha^2}{\beta})$ be a Kropina space with navigation data (h, W). Let $\tilde{a}$ be the Riemannian metric of α. Now define a vector field $\widetilde{\beta}$ by*

$$\tilde{a}\left(\widetilde{\beta}, Y\right) = \beta(Y),$$

where Y is an arbitrary vector field on M. Then a vector field X on M is a Killing field of (M, F) if and only if X is a Killing vector field of (M, h) and $[X, W] = 0$.

Proof. Since X is a Killing field of (M, F), then its flow ϕ_t is an isometry on (M, F), i.e. $\phi_t^* F = F$ for each t, where ϕ_t^* is the flow on TM defined by $\phi_t^*(x, y) := \big(\phi_t(x), \phi_{t*}(y)\big)$. Now according to Proposition (2.7), this is equivalent to $\phi_t^* h = h$ and $\phi_t^* W = W$. This proofs the assertion. □

Now we have the following Theorem for geodesic vector:

Theorem 3.2. *Suppose $(G/H, F = \frac{\alpha^2}{\beta})$ be a homogeneous Kropina space with navigation data (h, W). Then an element $X \in \mathfrak{g}$, $\mathfrak{g} = Lie(G)$, is a geodesic vector if and only if*

$$h(X_{\mathfrak{m}}, X_{\mathfrak{m}})h(X_{\mathfrak{m}}, [X_{\mathfrak{m}}, Z]_{\mathfrak{m}}) + F(X_{\mathfrak{m}})h([X_{\mathfrak{m}}, Z]_{\mathfrak{m}}, w) = 0. \tag{3.1}$$

holds for any $Z \in \mathfrak{m}$, $w = W|_o \in \mathfrak{m}$.

Proof. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the reductive decomposition, and $w = W|_o \in \mathfrak{m}$. Then for any $y \in \mathfrak{m}$ the Minkowski norm on $\mathfrak{m}$ induced by F can be expressed as

$$F(y) = \frac{h^2(y, y)}{2h(y, w)},$$

where h is the inner product on $\mathfrak{m}$ induced by Riemannian metric. Let g be the fundamental tensor of F. Then for any $x, y \in \mathfrak{m}$, we have

$$\begin{aligned} g_x(x, y) &= \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2\big(x + sx + ty\big)|_{s=t=0} \\ &= \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left(\frac{A(s,t)}{B(s,t)}\right)|_{s=t=0}, \end{aligned}$$

where

$$\begin{aligned} A(s,t) &= h^4\big(x + sx + ty,\ x + sx + ty\big), \\ B(s,t) &= 4h^2\big(x + sx + ty,\ w\big). \end{aligned}$$

A direct computation shows that

$$\begin{aligned} \frac{\partial A}{\partial s}(0,0) &= 8h^4(x,x) = 8A(0,0), \\ \frac{\partial A}{\partial t}(0,0) &= 8h^3(x,x)\,h(x,y) \end{aligned}$$

and

$$\frac{\partial^2 A}{\partial s \partial t}(0,0) = 0.$$

On the other hand we have

$$\frac{\partial B}{\partial s}(0,0) = 8h^2(x,w) = 2B(0,0),$$
$$\frac{\partial B}{\partial t}(0,0) = 8h(x,w)\,h(y,w),$$

and

$$\frac{\partial^2 B}{\partial s \partial t}(0,0) = 0.$$

Thus, we have

$$\begin{aligned} g_x(x,y) &= -\frac{1}{2}\left[\frac{\frac{\partial A}{\partial t}(0,0)\frac{\partial B}{\partial s}(0,0) + \frac{\partial A}{\partial s}(0,0)\frac{\partial B}{\partial t}(0,0)}{B^2(0,0)}\right] \\ &\quad + \left[\frac{A(0,0)\frac{\partial B}{\partial t}(0,0)\frac{\partial B}{\partial s}(0,0)}{B^3(0,0)}\right] \\ &= \frac{-2h^3(x,x)h(x,y)h(x,w) - 2h^4(x,x)h(y,w)}{h^3(x,w)} + \frac{h^4(x,x)h(y,w)}{h^3(x,w)} \\ &= \frac{h^2(x,x)\Big[-4h(x,x)h(x,y)h(x,w) - 2h^2(x,x)h(y,w)\Big]}{2h^3(x,w)} \\ &= \frac{-4F(x)}{h(x,w)}\Big[h(x,x)h(x,y) + F(x)h(y,w)\Big]. \end{aligned}$$

Now from the definition of geodesic vectors we have

$$\frac{-4F(X_{\mathfrak{m}})}{h(X_{\mathfrak{m}},w)}\Big[h(X_{\mathfrak{m}},X_{\mathfrak{m}})h(X_{\mathfrak{m}},[X_{\mathfrak{m}},Z]_{\mathfrak{m}}) + F(X_{\mathfrak{m}})h([X_{\mathfrak{m}},Z]_{\mathfrak{m}},w)\Big] = 0, \quad \forall Z \in \mathfrak{m},$$

and then

$$h(X_{\mathfrak{m}},X_{\mathfrak{m}})h(X_{\mathfrak{m}},[X_{\mathfrak{m}},Z]_{\mathfrak{m}}) + F(X_{\mathfrak{m}})h([X_{\mathfrak{m}},Z]_{\mathfrak{m}},w) = 0, \quad \forall Z \in \mathfrak{m}.$$

□

Now we have:

Theorem 3.3. *Suppose $(M = G/H, F)$ be a homogeneous Kropina space with navigation data (h, W). If $(G/H, h)$ is a Riemannian g.o. manifold and W is a Killing vector field of $(G/H, h)$, then (M, F) is a Kropina g.o. space.*

Proof. Suppose W is a Killing vector of $(G/H, h)$ and ψ_t be it's flow. Let G' be the group generated by ψ_t and G, and $W_0 \in \mathfrak{g}' = Lie(G')$ be the element corresponding to ψ_t. We know that W_0 lies in the center of $\mathfrak{g}'$. Now according to Theorem 4.2 of [21], every unit speed geodesic $\delta : (-\epsilon, \epsilon) \to G/H$ of F through $p = eH$ has the form

$$\delta(t) = exp(tW_0).\kappa(t),$$

where $\kappa(t)$ is the unit speed geodesic of $(G/H, h)$. Now since every geodesic of $(G/H, h)$ is an orbit of a one-parameter subgroup of G, there exists $Z \in \mathfrak{g}$ such that $\kappa(t) = exp(tZ).p$. Then we have

$$\delta(t) = exp(tW_0).\kappa(t) = exp(tW_0)exp(tZ).p = expt(W_0 + Z).p.$$

This implies that (M, F) is a geodesic orbit space. □

Now we have the next Proposition that proved in [11]:

Proposition 3.4. [11] *Suppose G is a connected Lie group and H is a closed subgroup such that G/H is a reductive homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Let h be a G-invariant Riemannian metric on G/H and suppose $z \in \mathfrak{m}$ is an H-fixed vector. Then the corresponding invariant vector field Z on G/H is a Killing vector field with respect to h if and only if z satisfies*

$$h([z, z_1]_{\mathfrak{m}}, z_2) + h(z_1, [z, z_2]_{\mathfrak{m}}) = 0, \quad \forall z_1, z_2 \in \mathfrak{m}.$$

Theorem 3.5. *Suppose $(M = G/H, F)$ be a homogeneous Kropina space with navigation data (h, Z). Let $(G/H, h)$ is a Riemannian g.o. manifold and $\tilde{Z}$ is the vector field on G/H generated by an $Ad(H)$-invariant vector $Z \in \mathfrak{m}$ such that $\|\tilde{Z}\|_h = 1$ and Z satisfies*

$$h([Z, Z_1]_{\mathfrak{m}}, Z_2) + h(Z_1, [Z, Z_2]_{\mathfrak{m}}) = 0, \quad \forall Z_1, Z_2 \in \mathfrak{m}.$$

Then (M, F) is a Kropina g.o. space.

Proof. According to Proposition (3.4), $\tilde{Z}$ is a Killing vector field on G/H and then by Theorem (3.3) (M, F) is a Kropina $g.o.$ space. □

We recall that, a geodesic $\gamma(t)$ on G/H through the origin $o = eH$ is called G-homogeneous if it is an orbit of a one-parameter subgroup of G, i.e.,

$$\gamma(t) = exp(tX).o, \quad t \in \mathbb{R};$$

where X is a non-zero vector in $\mathfrak{g}$ and we call X a geodesic vector. A Riemannian homogeneous space $(M = G/H, g)$ is called a G-geodesic orbit space (or $G - g.o.$ space) if all geodesics on G/H are G-homogeneous. In this case, the metric g is called G-geodesic orbit. If G is the full isometry group, then $(M = G/H, g)$ is called a geodesic orbit manifold (or $g.o.$ manifold). A homogeneous Riemannian manifold $(G/H, g)$ is called naturally reductive if there is an $Ad(H)$- invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that for all $X, Y, Z \in \mathfrak{m}$,

$$\langle [X, Y]_{\mathfrak{m}}, Z] \rangle + \langle Y, [X, Z]_{\mathfrak{m}} \rangle = 0,$$

or, equivalently, for all $X, Y \in \mathfrak{m}$,

$$\langle [X, Y]_{\mathfrak{m}}, X \rangle = 0.$$

We note that, the Kropina metric on $M = G/H$ arising from a Riemannian $g.o.$ metric h and an invariant Killing vector field W is geodesic orbit with respect to G'. In the above discussion, we assume G to be semisimple, which means G is a proper subgroup of G'. It is well-known that an extension of the isometry group may affect the naturally reductivity of the given metric (see Theorem (3.3) and it's proof).

It is well-known that naturally reductivity is equivalent to the geometrical property that for each vector $X \in \mathfrak{m}$, the orbit $\gamma(t) = \exp(tX).o$ is a geodesic. Therefore, naturally reductive Riemannian homogeneous spaces are geodesic orbit spaces, which can be also deduced from the algebraic equivalent conditions [8].

In [8], authors proved the following lemma.

Lemma 3.6. [8] *Let $(M = G/H, h)$ be a $G - g.o.$ space. Then all G-invariant vector fields on M are Killing vector fields.*

Now from this property we have the following Theorem based on the results in Theorem (3.3).

Theorem 3.7. *Let $(G/H, F)$ be a homogeneous Kropina space with navigation data (h, W). If $(G/H, h)$ is a Riemannian $G-g.o.$ space, then (M, F) is a Kropina g.o. space.*

The next Theorem give us condition for a Kropina metric to be Berwaldian:

Theorem 3.8. *Let $(M = G/H, F)$ be a homogeneous Kropina space with navigation data (h, W). Then the Kropina metric F is of Berwald type if and only if $ad(w)_{\mathfrak{m}}$, where $w = W|_o \in \mathfrak{m}$, is skew-symmetric with respect to h and $h(w, [m, m]_{\mathfrak{m}}) = 0$.*

Proof. Since F is Berwald metric, Then W is parallel with respect to h (see Theorem 3.3 in [25]). Now by the similar arguments in [16], we obtain the desired result. □

From the proof of Theorem (3.3), the Kropina metric on M arising from a Riemannian g.o. metric h and an invariant Killing vector field W is geodesic orbit with respect to G'. we assumed that G to be semisimple, which means G is a proper subgroup of G'. It is well-known that an extension of the isometry group may affect the naturally reductivity of the given metric. So we can discuss the naturally reductivity of Kropina metrics F with respect to G'. Denote by H' the isotropy subgroup in G' at the origin point $o = eH \in M$, then M is diffeomorphic to G'/H' with an $Ad(H')$-invariant decomposition $\mathfrak{g}' = \mathfrak{h}' \tilde{+} \mathfrak{m}'$, where $\mathfrak{g}'$ and $\mathfrak{h}'$ are the Lie algebras of G', H' respectively, and m' is isomorphic to the tangent space of M at o. In [8], authors shown that the decomposition $g' = h' \tilde{+} m$ is $Ad(H')$-invariant. Now we have the next Theorem like the Randers case in [8].

Theorem 3.9. *Let there is a closed intermediate subgroup $\tilde{K}$ of G such that $H \subset \tilde{K} \subset G$ and $G/\tilde{K}$ is a symmetric space of compact semisimple type with a B-orthogonal reductive decomposition*

$$\mathfrak{g} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{m}},$$

and the trivial $Ad(H)$-submodule $\mathfrak{n}$ is contained in $\tilde{\mathfrak{k}}$. Then any invariant non-Riemannian Kropina metric F with navigation data (h, W) is non-Berwaldian, hence non-naturally reductive.

4. Kropina g.o. metrics on homogeneous spaces with two isotropy summands

In this section, we assume that G/H is a compact and simply connected homogeneous space with G simple and the isotropy representation is the direct sum of two irreducible representations. We will give characterization of navigation data of non-Riemannian Kropina g.o. metrics.

The classification of the compact homogeneous spaces with two isotropy summands was done for the first time by W. Dickinson and M.M. Kerr in [12]. After them, compact simply connected geodesic orbit Riemannian spaces $(G/H, g)$ with two isotropy summands were Classified by Z. Chen and Yu. Nikonorov in [9].

Let $\mathfrak{m}$ be the tangent space of $M = G/H$ at $o = eH$, which is the B- orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$ with a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then we have the following decomposition:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where $\mathfrak{m}_1, \mathfrak{m}_2$ are irreducible $Ad(H)$-submodules in $\mathfrak{m}$ and $B(\mathfrak{m}_1, \mathfrak{m}_2) = 0$.

We have the following Proposition and Theorem that proved in [9].

Proposition 4.1. [9] *Suppose a compact homogeneous space G/H with connected compact H has two irreducible components in the isotropy representation. Then one of the following possibilities holds:*

(1) $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f}$ *and* $\mathfrak{h} = diag(\mathfrak{f}) \subset \mathfrak{g}$ *for a compact simple Lie algebra* $\mathfrak{f}$*;*
(2) $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ *and* $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$*, where* $\mathfrak{h}_i \subset \mathfrak{g}_i$ *and the pair* $(\mathfrak{g}_i, \mathfrak{h}_i)$ *is isotropy irreducible with simple compact Lie algebras* $\mathfrak{g}_i$ *for* $i = 1, 2$*;*
(3) $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{g}_1$ *for simple compact Lie algebras* $\mathfrak{g}_1$ *and* $\mathfrak{f}$*,* $\mathfrak{h} = diag(\mathfrak{f}) \oplus \mathfrak{h}_1$*, where* $\mathfrak{h}_1 \subset \mathfrak{g}_1$ *and the pair* $(\mathfrak{g}_1, \mathfrak{h}_1)$ *is isotropy irreducible;*
(4) $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{k}$*, where* $\mathfrak{l}$ *is a simple compact Lie algebra,* $\mathfrak{k}$ *is either a simple compact Lie algebra or* $\mathbb{R}$*, and there exist a Lie algebra* $\mathfrak{k}_1$ *such that* $\mathfrak{k} \oplus \mathfrak{k}_1$ *is a subalgebra in* $\mathfrak{l}$ *such that the pair* $(\mathfrak{l}, \mathfrak{k} \oplus \mathfrak{k}_1)$ *is isotropy irreducible, whereas* $\mathfrak{h} = diag(\mathfrak{k}) \oplus \mathfrak{k}_1 \subset \mathfrak{l} \oplus \mathfrak{k}$*;*
(5) $\mathfrak{g} = \mathbb{R}^2, \mathfrak{h} = 0, G/H = S^1 \times S^1$*;*
(6) $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{g}_1$*, where* $\mathfrak{g}_1$ *is a semi-simple compact Lie algebra,* $\mathfrak{h} \subset \mathfrak{g}_1$ *and the pair* $(\mathfrak{g}_1, \mathfrak{h})$ *is isotropy irreducible for* $i = 1, 2$*;*
(7) $\mathfrak{g}$ *is a simple compact Lie algebra.*

Theorem 4.2. [9] *Assume that G/H is a compact and simply connected homogeneous space with non-simple G and the isotropy representation is the direct sum of two irreducible representations (it corresponds to cases (1)–(6) in Proposition (4.1). Then G/H, supplied with any G invariant Riemannian metric, is naturally reductive, hence, geodesic orbit.*

In fact, those spaces G/H in cases (2), (3), (5), and (6) of Proposition (4.1) are normal homogeneous. It is known also that G/H in case (4) are naturally reductive. Note that for a simply connected compact homogeneous space G/H the group H is connected, but the cases (5) and (6) are impossible. Case (1) of Proposition (4.1) is more complicated that proved in [9].

Among compact simply connected spaces G/H with G simple and with two isotropy summands, only $Spin(8)/G_2$ has two equivalent irreducible submodules [12]. Except for this case, any G-invariant Riemannian metric on G=H must be in the following form:

$$\langle , \rangle = x_1 B|_{\mathfrak{m}_1} + x_2 B|_{\mathfrak{m}_1}, \quad x_1, x_2 \in \mathbb{R}_+. \tag{4.1}$$

The authors in [9], shown that if for $x_1 \neq x_2$, G/H can admit a *g.o.* metric, then there exists a subgroup K of G such that $H \subset K \subset G$ and G/K is symmetric. Also, all G-invariant Riemannian metrics on these homogeneous spaces G/H are G-geodesic orbit. On the other hand, a G-invariant metric g on G/H is G naturally reductive if and only if $x_1 = x_2$ in the metric defined in equation (4.1). In the table (1), We list all homogeneous spaces G/H in Theorem 2 in [9] along with the isotropy representation of H. In the following table, λ_i is the fundamental representation corresponding to the simple root system, id is trivial representation. $[\pi]_{\mathbb{R}}$ is a real irreducible representation which is the sum of a complex irreducible representation and its dual $\pi \oplus \pi^*$. When $\mathfrak{h}$ has a $\mathfrak{u}(1)$-factor, we use ϕ to denote the fundamental (one complex dimensional) representation. We note that, the spaces of cases (1), (3)-(9) are weakly symmetric, whereas the space of case (2) is not.

Table 1. Compact homogeneous spaces with two isotropy summands

No.	$H \subset K \subset G$	χ_1	χ_2	Remark
(1)	$G_2 \subset Spin(7) \subset Spin(8)$	λ_1	λ_1	-
(2)	$SO(2) \times G_2 \subset SO(2) \times SO(7) \subset SO(9)$	$id \otimes \lambda_1$	$\lambda_1 \otimes \lambda_1$	-
(3)	$U(n) \subset SO(2n) \subset SO(2n+1)$	$[\lambda_2 \otimes \phi]_{\mathbb{R}}$	$[\lambda_1]_{\mathbb{R}}$	$n \geqslant 2$
(4)	$SU(2n+1) \subset U(2n+2) \subset SO(4n+2)$	id	$[\lambda_2]_{\mathbb{R}}$	$n \geqslant 2$
(5)	$Spin(7) \subset SO(8) \subset SO(9)$	λ_1	λ_3	-
(6)	$SU(m) \times SU(n) \subset S(U(m)U(n)) \subset SU(m+n)$	id	$[\lambda_1 \times \lambda_{n-1}]_{\mathbb{R}}$	$m > n \geqslant 1$
(7)	$Sp(n)U(1) \subset Sp(U(2n)U(1)) \subset SU(2n+1)$	$[\lambda_2 \otimes id]_{\mathbb{R}}$	$[\lambda_2 \otimes \phi]_{\mathbb{R}}$	$n \geqslant 2$
(8)	$Sp(n)U(1) \subset Sp(n)Sp(1) \subset Sp(n+1)$	$id \otimes [\phi^2]_{\mathbb{R}}$	$\lambda_1 \otimes [\phi]_{\mathbb{R}}$	$n \geqslant 1$
(9)	$Spin(10) \subset Spin(10)SO(2) \subset E_6$	id	$[\lambda_4]_{\mathbb{R}}$	-

Now by table (1), in cases (4), (6) and (9), for the trivial $Ad(H)$- submodules we have

$$\mathfrak{n}_1 = \mathfrak{m}_1 \quad and \quad \mathfrak{n}_2 = \{0\}.$$

Now by Theorem (3.7), a Kropina metric on G/H with navigation data (h, W) is a non-Riemannian Kropina $G' - g.o.$ metric if and only if h is a Riemannian $G - g.o.$ metric on M and W is induced by any non-zero $z \in \mathfrak{m}_1$ satisfying $\|z\|_h = 1$. Since G/K is a compact symmetric space and the trivial Ad(H)-submodule $\mathfrak{m} = \mathfrak{n}$ is contained in $\mathfrak{k}$, by Theorem (3.9), the above G'-invariant Kropina metrics with navigation data (h, W) are non-naturally reductive.

When all Kropina $g.o.$ metrics are Riemannian, by Theorem (3.7), we have no trivial $Ad(H)$- submodule contained in $\mathfrak{m}_1$ or $\mathfrak{m}_2$ for the spaces of cases (1)-(3), (5), (7) and (8). Thus we have the next Theorem:

Theorem 4.3. *Let $M = G/H$ is a simply connected compact homogeneous space with two isotropy summands and let $(M = G/H, F)$ is a Kropina space with navigation data (h, W). Then M can admit non-Riemannian and non-naturally reductive Kropina geodesic orbit metrics with respect to G' if and only if M is one of the following cases:*

(1) $SO(4n+2)/SU(2n+1), \quad n \geqslant 2;$
(2) $SU(m+n)/SU(m) \times SU(n), \quad m > n \geqslant 1;$
(3) $E_6/Spin(10)$.

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(Milad L. Zeinali) Department of Mathematics, Faculty of Basic Sciences, University of Mohaghegh Ardabili, p.o.box. 5619911367, Ardabil-Iran.
Email address: `miladzeinali@gmail.com`

(Dariush Latifi) Department of Mathematics, Faculty of Basic Sciences, University of Mohaghegh Ardabili, p.o.box. 5619911367, Ardabil-Iran.
Email address: `latifi@uma.ac.ir`