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Research Paper

SOME PROPERTIES OF 4-DIMENSIONAL FINSLER MANIFOLDS

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ABSTRACT. In this paper, we study Cartan torsion, mean Cartan torsion and mean Landsberg curvature of 4-dimensional Finsler metrics. First, we find the necessary and sufficient condition under which a 4-dimensional Finsler manifold has bounded Cartan torsion and mean Cartan torsion. Then, we show that a 4-dimensional Finsler manifold has relatively isotropic mean Landsberg curvature if and only if it is Riemannian or the main scalars of Finsler metric satisfy the certain conditions.

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Keywords: Cartan torsion, mean Cartan torsion, mean Landsberg curvature, Landsberg curvature.

1. Introduction

In dimension four, in marked contrast with lower dimensions, topological and smooth manifolds are quite different. Indeed, there exist some topological 4-dimensional manifolds which admit no smooth structure, and even if there exists a smooth structure, it need not be unique. This means that there are smooth 4-dimensional manifolds which are homeomorphic but not diffeomorphic. In Riemannian geometry, 4-dimensional Riemannian manifolds are important in Physics because of their applications in general relativity and spacetime.

As Chern said "Finsler geometry is just Riemannian geometry without the quadratic restriction". The topic of 4-dimensional Finsler manifolds has a very interesting history because one of the most important Finsler metrics was discovered in this dimension of Finsler manifolds. In 1941, Randers published a paper concerned with an asymmetric metric in the four-space of general relativity. His metric is in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is gravitation field and $\beta = b_i(x)y^i$ is the electromagnetic field. He regarded these metrics not as Finsler metrics but as "affinely connected Riemannian metrics". This metric was first recognized as a kind of Finsler metric in 1957 by Ingarden, who first named them Randers metrics.

It is an important problem in Finsler geometry is whether or not every smooth Finsler manifold can be isometrically immersed into a Minkowski space. The answer is affirmative for Riemannian manifolds [9]. For Finsler manifolds, the problem under some conditions was considered by Burago-Ivanov, Gu and Ingarden (see [2][5][6][7][8]). However, Shen showed that every Finsler manifold with unbounded Cartan torsion can not be isometrically imbedded into any Minkowski space [13]. Thus the norm of Cartan torsion plays an important role for studying of immersion theory in Finsler geometry. For a Finsler manifold (M, F), the second

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and third order derivatives of $\mathcal{F} := \frac{1}{2}F_x^2$ at $y \in T_xM_0$ are fundamental form $\mathbf{g}_y$ and the Cartan torsion $\mathbf{C}_y$ on T_xM, respectively. The Cartan torsion was first introduced by Finsler [4] and emphased by Cartan [3]. For the Finsler metric F, the norm of the Cartan torsion is defined as follows

$$||\mathbf{C}|| = \sup_{F(y)=1, v\neq 0} \frac{|\mathbf{C}_y(v,v,v)|}{[\mathbf{g}_y(v,v)]^{\frac{3}{2}}}.$$

The bound for two dimensional Randers metrics $F = \alpha + \beta$ is verified by Lackey [1]. Then, Shen proved that the Cartan torsion of Randers metrics on a manifold M of dimension $n \geq 3$ is uniformly bounded by $3/\sqrt{2}$ [14]. In [8], Mo-Zhou extend his result to a general Finsler metrics, $F = (\alpha+\beta)^m/\alpha^{m-1}$ ($m \in [1,2]$). In [15], Tayebi-Sadeghi obtained a relation between the norm of Cartan and mean Cartan torsions of Finsler metrics defined by a Riemannian metric and a 1-form on a manifold . They prove that generalized Kropina metrics $F = \alpha^{m+1}/\beta^m$, ($m \neq 0$) have bounded Cartan torsion. It results that every C-reducible metric has bounded Cartan torsion. In [11], Rajabi studied the norm of Cartan torsion of Ingarden-Támassy metric

$$F = \alpha + \frac{\beta^2}{\alpha}$$

and Arctangent Finsler metric

$$F = \alpha + \beta \arctan(\beta/\alpha) + \epsilon\beta$$

that are special (α,β)-metrics. In [12], Sadeghi found a condition under which an (α,β)-metric has bounded Cartan torsion. More precisely, he proved that an (α,β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, on a manifold M has bounded Cartan torsion if and only if the function

$$A(s) := \frac{3s(\phi'^2 + \phi\phi'') - 3\phi\phi' - (b^2 - s^2)(\phi\phi''' + 3\phi'\phi'')}{2\phi^{\frac{1}{2}}(b^2-s^2)^{-\frac{1}{2}}\big(\phi - s\phi' + (b^2-s^2)\phi''\big)^{\frac{3}{2}}} \tag{1.1}$$

is a bounded function for $|s| < b$ where $b := \|\beta_x\|_\alpha$.

Taking a trace of Cartan torsion gives us the mean Cartan torsion, i.e., $\mathbf{I} := trace(\mathbf{C})$. By Deicke theorem, a positive-definite Finsler metric is Riemannian if and only if the mean Cartan torsion vanishes. For the Finsler metric F, the norm of the mean Cartan torsion is defined as follows

$$||\mathbf{I}|| = \sup_{F(y)=1, v\neq 0} \frac{|\mathbf{I}_y(v)|}{[\mathbf{g}_y(v,v)]^{\frac{3}{2}}}.$$

In this paper, we consider 4-dimensional Finsler manifolds and find the necessary and sufficient condition under which a 4-dimensional Finsler manifold has bounded Cartan torsion and mean bounded Cartan torsion. More precisely, we prove the following.

Theorem 1.1. *Let (M,F) be a 4-dimensional positive-definite Finsler manifold. Then the following hold*

- *(i) F has bounded Cartan torsion if and only if the following holds*
$$\mathcal{A}^2 + 3(\mathcal{B}^2 + \mathcal{C}^2 + 2\mathcal{D}^2) + 4(\mathcal{E}^2 + \mathcal{F}^2 + \mathcal{G}^2) + 2(3\mathcal{H}^2 + 3\mathcal{D}\mathcal{E} + \mathcal{F}\mathcal{G}) < \infty.$$
- *(ii) F is Riemannian if and only if $\mathcal{A} + \mathcal{B} + \mathcal{C} = 0$;*
- *(iii) F has bounded mean Cartan torsion if and only if $\mathcal{A} + \mathcal{B} + \mathcal{C} < \infty$,*

where $\mathcal{A} = \mathcal{A}(x,y)$, $\mathcal{B} = \mathcal{B}(x,y)$, $\mathcal{C} = \mathcal{C}(x,y)$, $\mathcal{D} = \mathcal{D}(x,y)$, $\mathcal{E} = \mathcal{E}(x,y)$ and $\mathcal{F} = \mathcal{F}(x,y)$ are scalar functions on TM and called the main scalars of F.

The mean Landsberg curvature $\mathbf{J}_y$ is the rate of change of $\mathbf{I}_y$ along geodesics for any $y \in T_xM_0$. By definition, $\mathbf{J}/\mathbf{I}$ is regarded as the relative rate of change of $\mathbf{I}$ along Finslerian geodesics. A Finsler metric F has relatively isotropic mean Landsberg curvature if it satisfies $\mathbf{J} = cF\mathbf{I}$, where $c = c(x)$ is a scalar function on M. In this paper, we show that a 4-dimensional Finsler manifold has relatively isotropic mean Landsberg curvature if and only if it is Riemannian or the main scalars of Finsler metric satisfy the certain conditions.

Theorem 1.2. *Let (M, F) be a 4-dimensional Finsler manifold. Then F has relatively isotropic mean Landsberg curvature*

$$\mathbf{J} = cF\mathbf{I}, \tag{1.2}$$

if and only if it is Riemannian or the main scalars of F satisfy following

$$(\mathcal{A}' + \mathcal{B}' + \mathcal{C}') = cF(\mathcal{A} + \mathcal{B} + \mathcal{C}), \quad h_0 = j_0 = 0. \tag{1.3}$$

where $c = c(x)$ is a scalar function on M.

2. Preliminary

Let M be an n-dimensional C^∞ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent bundle and $TM_0 := TM - \{0\}$ the slit tangent bundle. A Finsler structure on M is a function $F : TM \to [0, \infty)$ with the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, i.e., $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$; (iii) The following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is positively defined on TM_0

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Then the pair (M, F) is called a Finsler manifold.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y) = 0$ and $\mathbf{I}_{\lambda y} = \lambda^{-1}\mathbf{I}_y$, $\lambda > 0$. Therefore, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

Also, the Landsberg curvature of F can be defined by following

$$\mathbf{L}_y(u, v, w) := \frac{d}{dt}\Big[\mathbf{C}_{\dot\sigma(t)}\big(U(t), V(t), W(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U(t), V(t), W(t)$ are linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$. Then the

Landsberg curvature $\mathbf{L}_y$ is the rate of change of $\mathbf{C}_y$ along geodesics for any $y \in T_xM_0$. A Finsler metric F is called a Landsberg metric if $\mathbf{L} = 0$.

For $y \in T_xM$, define $\mathbf{J}_y : T_xM \to \mathbb{R}$ by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$\mathbf{J}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{L}_y(u, \partial_i, \partial_j).$$

The non-Riemannian quantity $\mathbf{J}$ is called the *mean Landsberg curvature* or $\mathbf{J}$-curvature of F. We say that F is a weakly Landsberg metric if $\mathbf{J} = 0$. It is easy to see that the mean Landsberg curvature of F is also given by

$$\mathbf{J}_y(u) := \frac{d}{dt}\Big[\mathbf{I}_{\dot\sigma(t)}\big(U(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, σ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and U is linearly parallel vector field along σ with $U(0) = u$. Thus the mean Landsberg curvature $\mathbf{J}_y$ is the rate of change of $\mathbf{I}_y$ along geodesics for any $y \in T_xM_0$.

Throughout this paper, we use the Berwald connection on Finsler manifolds. The h- and v-covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

3. Proof of Theorems

In this section, we are going to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1: Let (M, F) be a 4-dimensional Finsler manifold. Suppose that $\ell_i := F_{y^i}$ is the unit vector along the element of support, m_i is the unit vector along mean Cartan torsion I_i, i.e., $m_i := I_i/||\mathbf{I}||$, where $||\mathbf{I}|| := \sqrt{I_iI^i}$, and n_i and p_i are unit vectors orthogonal to the vectors ℓ_i and m_i. Then the quadruple (ℓ_i, m_i, n_i, p_i) is called the Miron frame. In this frame, we have

$$g_{ij} = \ell_i\ell_j + m_im_j + n_in_j + p_ip_j, \tag{3.1}$$

$$g^{ij} = \ell^i\ell^j + m^im^j + n^in^j + p^ip^j. \tag{3.2}$$

Thus

$$h_{ij} = m_im_j + n_in_j + p_ip_j.$$

Taking a vertical derivative of (3.1) yields the Cartan torsion as follows

$$\begin{aligned} FC_{ijk} &= \mathcal{A}m_im_jm_k + \mathcal{B}(m_in_jn_k + n_im_jn_k + n_in_jm_k) + \mathcal{C}(m_ip_jp_k + p_im_jp_k + p_ip_jm_k) \\ &+\mathcal{D}(m_im_jn_k + m_in_jm_k + n_im_jm_k) + \mathcal{E}n_in_jn_k + \mathcal{F}(m_im_jp_k + m_ip_jm_k + p_im_jm_k) \\ &+\mathcal{G}(n_in_jp_k + n_ip_jn_k + p_in_jn_k) + \mathcal{H}(m_in_jp_k + m_ip_jn_k + n_im_jp_k + n_ip_jm_k \\ &+p_im_jn_k + p_in_jm_k) - (\mathcal{D} + \mathcal{E})(n_ip_jp_k + p_in_jp_k + p_ip_jn_k) - (\mathcal{F} + \mathcal{G})p_ip_jp_k, \end{aligned} \tag{3.3}$$

where $\mathcal{A} = \mathcal{A}(x, y)$, $\mathcal{B} = \mathcal{B}(x, y)$, $\mathcal{C} = \mathcal{C}(x, y)$, $\mathcal{D} = \mathcal{D}(x, y)$, $\mathcal{E} = \mathcal{E}(x, y)$ and $\mathcal{F} = \mathcal{F}(x, y)$ are scalar functions on TM and called the main scalars of F. On the other hand, multiplying

(3.3) with $g^{ip}g^{jq}g^{ks}$ implies that

$$\begin{aligned} FC^{ijk} &= \mathcal{A}m^im^jm^k + \mathcal{B}(m^in^jn^k + n^im^jn^k + n^in^jm^k) + \mathcal{C}(m^ip^jp^k + p^im^jp^k + p^ip^jm^k) \\ &+\mathcal{D}(m^im^jn^k + m^in^jm^k + n^im^jm^k) + \mathcal{E}n^in^jn^k + \mathcal{F}(m^im^jp^k + m^ip^jm^k + p^im^jm^k) \\ &+\mathcal{G}(n^in^jp^k + n^ip^jn^k + p^in^jn^k) + \mathcal{H}(m^in^jp^k + m^ip^jn^k + n^im^jp^k + n^ip^jm^k \\ &+p^im^jn^k + p^in^jm^k) - (\mathcal{D}+\mathcal{E})(n^ip^jp^k + p^in^jp^k + p^ip^jn^k) - (\mathcal{F}+\mathcal{G})p^ip^jp^k. \end{aligned} \tag{3.4}$$

(3.3) × (3.4) gives us

$$||\mathbf{C}|| = \sqrt{\mathcal{A}^2 + 3\mathcal{B}^2 + 3\mathcal{C}^2 + 3\mathcal{D}^2 + \mathcal{E}^2 + 3\mathcal{F}^2 + 3\mathcal{G}^2 + 6\mathcal{H}^2 + 3(\mathcal{D}+\mathcal{E})^2 + (\mathcal{F}+\mathcal{G})^2}. \tag{3.5}$$

By (3.5), we get the proof of part (i).

By (3.3), we have

$$FI_k = (\mathcal{A}+\mathcal{B}+\mathcal{C})m_k. \tag{3.6}$$

According to (3.6), F is Riemannian if and only if $\mathcal{A}+\mathcal{B}+\mathcal{C} = 0$. This is the proof of part (ii).

Now, contracting (3.6) with g^{ik} gives us

$$FI^k = (\mathcal{A}+\mathcal{B}+\mathcal{C})m^k. \tag{3.7}$$

(3.6) × (3.7) implies that

$$||\mathbf{I}|| = \frac{1}{F}\big(\mathcal{A}+\mathcal{B}+\mathcal{C}\big). \tag{3.8}$$

By (3.8), we get the proof of (iii). □

Proof of Theorem 1.2: The horizontal derivation of Miron frame are given by following

$$\ell_{i|j} = 0, \quad m_{i|j} = h_jn_i + j_jp_i, \quad n_{i|j} = k_jp_i - h_jm_i, \quad p_{i|j} = -j_jm_i - k_jn_i,$$

where $h_s = h_s(x,y)$, $j_s = j_s(x,y)$ and $k_s = k_s(x,y)$ are called the h-connection vectors (for more details, see [10]). Thus

$$m'_i := m_{i|j}y^j = h_0n_i + j_0p_i, \quad n'_i := n_{i|j}y^j = k_0p_i - h_0m_i, \quad p'_i := p_{i|j}y^j = -j_0m_i - k_0n_i,$$

where $h_0 := h_sy^s$, $j_0 := j_sy^s$ and $k_0 := k_sy^s$. Taking a horizontal derivation of (3.6) along geodesics implies

$$FJ_k = (\mathcal{A}' + \mathcal{B}' + \mathcal{C}')m_k + (\mathcal{A}+\mathcal{B}+\mathcal{C})(h_0n_k + j_0p_k). \tag{3.9}$$

By assumption $J_i = cFI_i$, where $c = c(x)$ is a scalar function on M. Then (3.6), (3.9) and the mentioned assumption give us

$$\Big[(\mathcal{A}' + \mathcal{B}' + \mathcal{C}')m_k + (\mathcal{A}+\mathcal{B}+\mathcal{C})(h_0n_k + j_0p_k)\Big] = cF(\mathcal{A}+\mathcal{B}+\mathcal{C})m_k. \tag{3.10}$$

Contracting (3.10) with m^k and considering $n_km^k = p_km^k = 0$ implies that

$$(\mathcal{A}' + \mathcal{B}' + \mathcal{C}') = cF(\mathcal{A}+\mathcal{B}+\mathcal{C}). \tag{3.11}$$

Putting (3.11) in (3.10) yields

$$(\mathcal{A}+\mathcal{B}+\mathcal{C})(h_0n_k + j_0p_k) = 0. \tag{3.12}$$

By (3.12), we have two cases: if $\mathcal{A}+\mathcal{B}+\mathcal{C}=0$ then by part (ii) of Theorem 1.1, F is Riemannian. Now, suppose that F is not a Riemannian metric. Then (3.12) reduces to

$$h_0 n_k + j_0 p_k = 0. \tag{3.13}$$

Multiplying (3.12) with n^k and using $p_k n^k = 0$ implies $h_0 = 0$. Also, contracting (3.12) with p^k and using $n_k p^k = 0$ implies $j_0 = 0$. This completes the proof. □

Remark 3.1. The following holds

$$\mathcal{A}'(t) := \frac{d}{dt}\Big[\mathcal{A}_{\dot\sigma(t)}\big(U(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, $\sigma=\sigma(t)$ is the geodesic with $\sigma(0)=x$, $\dot\sigma(0)=y$ and $U=U(t)$ is a linearly parallel vector field along σ with $U(0)=u$. Then the $\mathcal{A}'$ is the rate of change of $\mathcal{A}$ along geodesics for any $y \in T_xM_0$. In this case, by (3.11) we get

$$(\mathcal{A}+\mathcal{B}+\mathcal{C})(t) = \exp\left[\int_0^s c\ dt\right](\mathcal{A}+\mathcal{B}+\mathcal{C})(0).$$

By putting $c=0$ in Theorem 1.2, we conclude the following.

Corollary 3.2. *Let (M,F) be a 4-dimensional Finsler manifold. Then F is weakly Landsbergian if and only if it is Riemannian or satisfies following*

$$\mathcal{A}'+\mathcal{B}'+\mathcal{C}'=0, \quad h_0=0, \quad j_0=0. \tag{3.14}$$

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