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Research Paper

ADJOINTATIONS OF OPERATOR INEQUALITIES FOR SECTOR MATRICES

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Abstract. In this paper, we first extend the well-known inequalities to the case of sector matrices. We also explore the adjointness of operator inequalities with binary operations for sector matrices. As a result of our exploration, we establish four distinct inequalities: a matrix inequality, a unitarily invariant norm inequality, a singular value inequality, and a determinant inequality. For example, we demonstrate that if σ_1 and σ_2 are non-zero connections, and if A, B, and C belong to $\mathcal{S}_\alpha$, such that

$$\mathcal{R}\left(A\sigma_1(B\sigma_2 C)\right) \leq \cos^4(\alpha)\ \mathcal{R}\left((A\sigma_1 B)\sigma_2(A\sigma_1 C)\right),$$

then

$$\mathcal{R}\left(A\sigma_1^*(B\sigma_2^* C)\right) \geq \cos^4(\alpha)\ \mathcal{R}\left((A\sigma_1^* B)\sigma_2^*(A\sigma_1^* C)\right).$$

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1. Introduction and Background

Let $(H, \langle .,. \rangle)$ be a complex Hilbert space and let $B(H)$ be the C^*- algebra of bounded linear operators acting on H. A selfadjoint operator $A \in B(H)$ is called positive if $\langle Ax, x\rangle \geqslant 0$ for all vectors $x \in \mathbb{C}^n$. We write $A \geq 0$ if A is positive. For selfadjoint operators $A, B \in B(H)$ a partial order is defined as $A \geq B$ if $A - B \geq 0$. Let $\mathbb{M}_n$ denote the set of all $n \times n$ complex matrices. A Hermitian matrix $A \in \mathbb{M}_n$ is said to be positive semidefinite, denoted by $A \geq 0$, if $\langle Ax, x\rangle \geqslant 0$ for all vectors $x \in \mathbb{C}^n$. It is positive definite if it is positive semidefinite and invertible, we will write $A > 0$, the class of positive definite matrices is denoted by $\mathbb{M}_n^+$. An operator $A \in B(H)$ is called accretive if in its Cartesian or Toeplitz decomposition, $A = \mathcal{R}A + i\mathcal{I}A$, $\mathcal{R}A$ is positive definite ($\mathcal{R}A > 0$), where $\mathcal{R}A = \frac{A+A^*}{2}$, $\mathcal{I}A = \frac{A-A^*}{2i}$. The numerical range of $A \in \mathbb{M}_n$ is defined as

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ \|x\| = 1\}.$$

The sector region $\mathcal{S}_\alpha$ is defined as follows:

$$\mathcal{S}_\alpha = \{z \in \mathbb{C} : \mathcal{R}z > 0,\ |\mathcal{I}z| \leq (\mathcal{R}z)\tan\alpha\}.$$

The given statement $W(A) \subset \mathcal{S}_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$, says that if the numerical range of a matrix A is a subset of a sector region $\mathcal{S}_\alpha$ in the complex plane, then A is called a sectorial matrix and is denoted by $A \in \mathcal{S}_\alpha$. Furthermore, since $0 \notin \mathcal{S}_\alpha$, then each member of $\mathcal{S}_\alpha$ is

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invertible. It is also stated that a sectorial matrix is necessarily accretive.
An operator mean σ in the sense of Kubo-Ando [8] is defined by an operator monotone function $f : (0, \infty) \to (0, \infty)$ with $f(1) = 1$ (briefly we write $f \in \boldsymbol{m}$) as

$$A\sigma B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}, \tag{1.1}$$

for positive invertible operators A and B. The function f that satisfies the conditions for the operator mean σ is called the "representing function" of σ. Some important operator means are given as follows:

- Arithmetic mean: $A\nabla B = (A + B)/2$.
- t-Weighted arithmetic mean: $A\nabla_t B = (1 - t)A + tB$. $(0 < t < 1)$
- Harmonic mean: $A!B = \left((A^{-1} + B^{-1})/2\right)^{-1}$.
- t-Weighted harmonic mean: $A!_t B = [(1 - t)A^{-1} + tB^{-1}]^{-1}$. $(0 < t < 1)$
- Geometric mean: $A\sharp B = A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}$.
- t-Weighted geometric mean: $A\sharp_t B = A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2}$. $(0 < t < 1)$

Definition 1.1. ([14]), Let σ be an operator mean with representing function f. The operator mean with representing function $f(t^{-1})^{-1}$ is called the adjoint of σ and denoted by σ^*. Formula (1.1) gives an explicit form on the adjoint,

$$A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1} \qquad \textit{for invertible } A \textit{ and } B.$$

For a function $f : (0, \infty) \to (0, \infty)$, we define the adjoint of f by

$$f^*(x) = \frac{1}{f(1/x)}, \quad x > 0.$$

It states that if a nonzero connection σ is associated with an operator monotone function f, then the adjoint of σ, denoted by σ^*, is associated with the operator monotone function f^*. This means that the adjoint of a connection preserves the operator monotonicity of the function associated with it.
Recall that, if $f \in \boldsymbol{m}$, then $f^* \in \boldsymbol{m}$ and so if $A \in \mathcal{S}_\alpha$, then $f^*(A) \in \mathcal{S}_\alpha$.
The adjoint formation is involutive, $(\sigma^*)^* = \sigma$. The adjoint mean of the t-weighted arithmetic mean is t-weighted harmonic mean, i.e. $\nabla_t^* = !_t$ and the t-weighted geometric mean is self adjoint, i.e. $(\sharp_t)^* = \sharp_t$. Where $0 < t < 1$.

We will use the following lemmas in our main results proof:

Lemma 1.2. *([2]), Let $A, B \in \mathcal{S}_\alpha$. Then $A\sigma B \in \mathcal{S}_\alpha$ and*

$$\mathcal{R}A\sigma\mathcal{R}B \leq \mathcal{R}(A\sigma B) \leq \sec^2(\alpha)\ (\mathcal{R}A\sigma\mathcal{R}B).$$

Lemma 1.3. *([2]), Let $f \in \boldsymbol{m}$ and $A \in \mathcal{S}_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$. Then*

$$f(\mathcal{R}A) \leq \mathcal{R}(f(A)) \leq \sec^2(\alpha)\ f(\mathcal{R}A).$$

The article refers the reader to other articles (cited as [5, 10, 11, 12, 13, 15]) for further information on this topic.

Lemma 1.4. *([7]), Let $\alpha \geq 1$. Then*

(i) If $f : (0, \infty) \to (0, \infty)$ is an operator monotone function, then $f(\alpha t) \leq \alpha f(t)$.

(ii) If $g : (0, \infty) \to (0, \infty)$ is an operator monotone decreasing function, then $g(\alpha t) \geq \dfrac{1}{\alpha} g(t)$.

In [4], Chansangiam has proven that if f is increasing or decreasing, it has influenced f^* and certain operator inequalities are adjointable where $A, B > 0$:

Lemma 1.5. *([4]), Let σ_1 and σ_2 be binary operations for invertible positive operators and $f, g, h : (0,\infty) \to (0,\infty)$ be continuous functions. Then the following statements are equivalent:*

(i) $f(A\sigma_1 B) \leq g(A)\sigma_2 h(B)$ for all $A \geq B > 0$.
(ii) $f^*(A\sigma_1^* B) \geq g^*(A)\ \sigma_2^*\ h^*(B)$ for all $A \geq B > 0$.

Lemma 1.6. *([4]), Let σ_1 and σ_2 be nonzero connections. Then the following statements are equivalent:*

(i) $A\sigma_1(B\sigma_2 C) \leq (A\sigma_1 B)\sigma_2(A\sigma_1 C)$ for all $A, B, C \geq 0$.
(ii) $A\sigma_1^*(B\sigma_2^* C) \geq (A\sigma_1^* B)\sigma_2^*(A\sigma_1^* C)$ for all $A, B, C \geq 0$.

Lemma 1.7. *([1]), Let A and B be two strictly positive operators and $0 \leq t \leq 1$. If $f : (0,\infty) \to (0,\infty)$ is an operator monotone decreasing function, then*

$$f(A)!_t f(B) \geq f(A\nabla_t B).$$

Lemma 1.8. *([7]), Let A, B be two strictly positive operators, $0 \leq t \leq 1$ and σ_t be an arbitrary mean between ∇_t and $!_t$. If $f : (0,\infty) \to (0,\infty)$ is an operator monotone decreasing function, then*

$$f(A)\sigma_t f(B) \leq f(A!_t B).$$

2. MAIN RESULTS

This section is started by an accretive version of Lemma 1.5. The proofs of Theorems 2.1 and 2.4 involve using certain properties of accretive operators and applying them to the accretive versions of the inequalities.

Theorem 2.1. *Let σ_1 and σ_2 be binary operations for invertible accretive operators and $f, g, h \in \boldsymbol{m}$ be continuous functions. If $A, B \in \mathcal{S}_\alpha$ such that*

$$\sec^4(\alpha)\ \mathcal{R}\left(f(A\sigma_1 B)\right) \leq \mathcal{R}\left(g(A)\sigma_2 h(B)\right), \tag{2.1}$$

then

$$\mathcal{R}\left(g^*(A)\ \sigma_2^*\ h^*(B)\right) \leq \sec^4(\alpha)\ \mathcal{R}\left(f^*(A\ \sigma_1^*\ B)\right).$$

Proof. Assume (2.1) and consider $A, B \in \mathcal{S}_\alpha$

$$\begin{aligned}
f\left(\mathcal{R}A\sigma_1\mathcal{R}B\right) &\leq f\left(\mathcal{R}\left(A\sigma_1 B\right)\right) && (\text{ by Lemma } 1.2)\\
&\leq \mathcal{R}\left(f\left(A\sigma_1 B\right)\right) && (\text{ by Lemma } 1.3)\\
&\leq \cos^4(\alpha)\ \mathcal{R}\left(g(A)\sigma_2 h(B)\right) && (\text{ by } (2.1))\\
&\leq \cos^2(\alpha)\ \left(\mathcal{R}(g(A))\sigma_2\mathcal{R}(h(B))\right) && (\text{ by Lemma } 1.2)\\
&\leq g(\mathcal{R}A)\sigma_2 h(\mathcal{R}B) && (\text{ by Lemma } 1.2).
\end{aligned}$$

Now by Lemma 1.5,

$$f^*\left(\mathcal{R}A\ \sigma_1^*\ \mathcal{R}B\right) \geq g^*(\mathcal{R}A)\ \sigma_2^*\ h^*(\mathcal{R}B). \tag{2.2}$$

Then

$$\begin{aligned}
\mathcal{R}\left(f^*(A\sigma_1^* B)\right) &\geq f^*\left(\mathcal{R}(A\sigma_1^* B)\right) && (\text{ by Lemma 1.3}) \\
&\geq f^*(\mathcal{R}A\ \sigma_1^*\ \mathcal{R}B) && (\text{ by Lemma 1.2}) \\
&\geq g^*(\mathcal{R}A)\ \sigma_2^*\ h^*(\mathcal{R}B) && (\text{ by (2.2)}) \\
&\geq \cos^2(\alpha)\ \ (\mathcal{R}(g^*(A))\ \sigma_2^*\ \mathcal{R}(h^*(B))) && (\text{ by Lemma 1.3}) \\
&\geq \cos^4(\alpha)\ \mathcal{R}\left(g^*(A)\ \sigma_2^*\ h^*(B)\right) && (\text{ by Lemma 1.2}).
\end{aligned}$$

□

According to the first part of the proof of Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let σ_1 and σ_2 be binary operations for invertible accretive operators and $f, g, h \in \boldsymbol{m}$ be continuous functions. If $A, B \in \mathcal{S}_\alpha$ such that*

$$f(\mathcal{R}A\sigma_1\mathcal{R}B) \leq g(\mathcal{R}A)\sigma_2 h(\mathcal{R}B),$$

then

$$\mathcal{R}\left(g^*(A)\ \sigma_2^*\ h^*(B)\right) \leq \sec^4(\alpha)\ \mathcal{R}\left(f^*(A\ \sigma_1^*\ B)\right).$$

Remark 2.3. Let $A, B \in \mathcal{S}_\alpha$ and $0 \leq t \leq 1$. If $f : (0, \infty) \to (0, \infty)$ is an operator monotone decreasing function, then f^* is also. By applying Lemma 1.7 for f^*, we have

$$f^*((1-t)\mathcal{R}A + t\,(\mathcal{R}B)) \leq \left[(1-t)(f^*(\mathcal{R}A))^{-1} + t(f^*(\mathcal{R}B))^{-1}\right]^{-1},$$

therefore by Corollary 2.2, we get

$$\mathcal{R}\left((1-t)f(A)\ + t\ f(B)\right) \leq \sec^4(\alpha)\ \mathcal{R}\left(f([(1-t)A^{-1} + tB^{-1}]^{-1})\right).$$

Therefore, Theorem 2.1 is an extension of [1, Remark 2.6].

Theorem 2.1 can be considered an extension of Theorem 2 in [4]. By applying similar strategies, we can easily prove the following theorem, which is the reverse of Theorem 2.1.

Theorem 2.4. *Let σ_1 and σ_2 be binary operations for invertible accretive operators and $f, g, h \in \boldsymbol{m}$ be continuous functions. If $A, B \in \mathcal{S}_\alpha$ such that*

$$\sec^4(\alpha)\ \mathcal{R}\left(g(A)\sigma_2 h(B)\right) \leq \mathcal{R}(f\left(A\sigma_1 B\right))\ \ \textit{for all}\ A, B \in \mathcal{S}_\alpha, \tag{2.3}$$

then

$$\mathcal{R}\left(f^*(A\sigma_1^* B)\right) \leq \sec^4(\alpha)\ \mathcal{R}\left(g^*(A)\ \sigma_2^*\ h^*(B)\right).$$

Proof. Assume (2.3) and consider $A, B \in \mathcal{S}_\alpha$,

$$\begin{aligned}
g(\mathcal{R}A)\sigma_2 h(\mathcal{R}B) &\leq \mathcal{R}\left(g(A)\right)\sigma_2\mathcal{R}\left(h(B)\right) && (\text{by Lemma 1.3}) \\
&\leq \mathcal{R}\left(g(A)\sigma_2 h(B)\right) && (\text{by Lemma 1.2}) \\
&\leq \cos^4(\alpha)\mathcal{R}\left(f(A\sigma_1 B)\right) && (\text{by (2.3)}) \\
&\leq \cos^2(\alpha)f(\mathcal{R}\left(A\sigma_1 B\right)) && (\text{by Lemma 1.3}) \\
&\leq f(\mathcal{R}A\sigma_1\mathcal{R}B) && (\text{by Lemmas 1.4 and 1.2}),
\end{aligned}$$

then by Lemma 1.5,

$$f^*(\mathcal{R}A\ \sigma_1^*\ \mathcal{R}B) \leq g^*(\mathcal{R}A)\ \sigma_2^*\ h^*(\mathcal{R}B). \tag{2.4}$$

and therefore

$$\begin{aligned}
\mathcal{R}\left(f^*(A\sigma_1^* B)\right) &\le \sec^2(\alpha)\ f^*\left(\mathcal{R}(A\sigma_1^* B)\right) && \text{(by Lemma 1.3)}\\
&\le \sec^2(\alpha)\ f^*\left(\sec^2(\alpha)\ (\mathcal{R}A\ \sigma_1^*\ \mathcal{R}B)\right) && \text{(by Lemma 1.2)}\\
&\le \sec^4(\alpha)\ f^*\left(\mathcal{R}A\ \sigma_1^*\ \mathcal{R}B\right) && \text{(by Lemma 1.4)}\\
&\le \sec^4(\alpha)\ \left(g^*(\mathcal{R}A)\ \sigma_2^*\ h^*(\mathcal{R}B)\right) && \text{(by (2.4))}\\
&\le \sec^4(\alpha)\ \left(\mathcal{R}(g^*(A))\ \sigma_2^*\ \mathcal{R}(h^*(B))\right) && \text{(by Lemma 1.3)}\\
&\le \sec^4(\alpha)\ \mathcal{R}\left(g^*(A)\ \sigma_2^*\ h^*(B)\right) && \text{(by Lemma 1.2).}
\end{aligned}$$

□

According to the first part of the proof of Theorem 2.4, we get the following useful corollary.

Corollary 2.5. *Let σ_1 and σ_2 be binary operations for invertible accretive operators and $f, g, h \in \boldsymbol{m}$ be continuous functions. If $A, B \in \mathcal{S}_\alpha$ such that*

$$g(\mathcal{R}A)\sigma_2 h(\mathcal{R}B) \le f\left(\mathcal{R}A\sigma_1\mathcal{R}B\right),$$

then

$$\mathcal{R}\left(f^*(A\sigma_1^* B)\right) \le \sec^4(\alpha)\ \mathcal{R}\left(g^*(A)\ \sigma_2^*\ h^*(B)\right).$$

Proposition 2.6. *Let $A, B \in \mathcal{S}_\alpha$, $0 \le t \le 1$, and σ_t be an arbitrary mean between ∇_t and $!_t$. If $f : (0, \infty) \to (0, \infty)$ is an operator monotone decreasing function, then*

$$\mathcal{R}\left(f((1-t)A + tB)\right) \le \sec^4(\alpha)\ \mathcal{R}\left(f(A)\ \sigma_t^*\ f(B)\right). \tag{2.5}$$

Proof. Since f is an operator monotone decreasing, so is f^*. Applying Lemma 1.8 for f^*, we get

$$f^*(\mathcal{R}A)\sigma_t f^*(\mathcal{R}B) \le f^*\left(\left[(1-t)\mathcal{R}^{-1}A + t\mathcal{R}^{-1}B\right]^{-1}\right),$$

therefore by Corollary 2.5, we have

$$\mathcal{R}\left(f((1-t)A + tB)\right) \le \sec^4(\alpha)\ \mathcal{R}\left(f(A)\ \sigma_t^*\ f(B)\right).$$

□

Inequality (2.5) can be regarded as an extension of the known fact in [7, Lemma 2.4]. The following inequality has been proved by Lin [10]:

$$\mathcal{R}\left(A!_t B\right) \le \sec^2(\alpha)\ \left(\mathcal{R}A!_t\mathcal{R}B\right), \quad 0 < t < 1. \tag{2.6}$$

Where $A, B \in \mathcal{S}_\alpha$.

The following proposition is an extension of (2.6).

Proposition 2.7. *Let $A, B \in \mathcal{S}_\alpha$ and $f \in \boldsymbol{m}$. Then*

$$\mathcal{R}\left(f(A!_t B)\right) \le \sec^4(\alpha)\ \mathcal{R}\left(f(A)!_t f(B)\right), \tag{2.7}$$

where $0 < t < 1$.

Proof. By [2, (6.12)], recall that if $f \in \boldsymbol{m}$, then

$$(1-t)f(\mathcal{R}A) + tf(\mathcal{R}B) \le f\left((1-t)\mathcal{R}A + t\mathcal{R}B\right).$$

If $f \in \boldsymbol{m}$, then $f^* \in \boldsymbol{m}$. Consequently

$$(1-t)f^*(\mathcal{R}A) + tf^*(\mathcal{R}B) \le f^*\left((1-t)\mathcal{R}A + t\mathcal{R}B\right).$$

Finally by Corollary 2.5, we have

$$\mathcal{R}\left(f(A!_t B)\right) \leq \sec^4(\alpha)\ \mathcal{R}\left(f(A)!_t f(B)\right),$$

where $0 < t < 1$. □

Next, we will give a relation between the harmonic and geometric means.

Proposition 2.8. *Let $A, B \in \mathcal{S}_\alpha$ and $f \in \boldsymbol{m}$. Then*

$$\mathcal{R}\left(f(A!B)\right) \leq \sec^4(\alpha)\mathcal{R}\left(f(A)\sharp f(B)\right).$$

Proof. It is well known that if $f \in \boldsymbol{m}$ is a continuous function and $A, B \in \mathbb{M}_n^+$, then

$$f(A)\sharp f(B) \leq f\left(A \nabla B\right). \tag{2.8}$$

Let $A, B \in \mathcal{S}_\alpha$. Using the (2.8) for f^*, we get

$$f^*(\mathcal{R}A)\sharp f^*(\mathcal{R}B) \leq f^*\left(\mathcal{R}A \nabla \mathcal{R}B\right).$$

Therefore by Corollary 2.5, we have

$$\mathcal{R}\left(f(A!B)\right) \leq \sec^4(\alpha)\mathcal{R}\left(f(A)\sharp f(B)\right).$$

□

Finally, we have proved the following inequality for mixed operator means.

Theorem 2.9. *Let σ_1 and σ_2 be nonzero connections and $A, B, C \in \mathcal{S}_\alpha$. If*

$$\mathcal{R}\left(A\sigma_1(B\sigma_2 C)\right) \leq \cos^4(\alpha)\ \mathcal{R}\left((A\sigma_1 B)\sigma_2(A\sigma_1 C)\right), \tag{2.9}$$

then

$$\mathcal{R}\left(A\sigma_1^*(B\sigma_2^* C)\right) \geq \cos^4(\alpha)\ \mathcal{R}\left((A\sigma_1^* B)\sigma_2^*(A\sigma_1^* C)\right).$$

Proof. The property (2.9) and Lemma 1.2 together imply that,

$$\begin{aligned}
\mathcal{R}A\sigma_1\left(\mathcal{R}B\sigma_2\mathcal{R}C\right) &\leq \mathcal{R}A\sigma_1\left(\mathcal{R}(B\sigma_2 C)\right) \\
&\leq \mathcal{R}\left(A\sigma_1(B\sigma_2 C)\right) \\
&\leq \cos^4(\alpha)\ \mathcal{R}\left((A\sigma_1 B)\sigma_2(A\sigma_1 C)\right) \\
&\leq \cos^2(\alpha)\ \left(\mathcal{R}(A\sigma_1 B)\sigma_2\mathcal{R}(A\sigma_1 C)\right) \\
&\leq \left(\mathcal{R}A\sigma_1\mathcal{R}B\right)\sigma_2\left(\mathcal{R}A\sigma_1\mathcal{R}C\right),
\end{aligned}$$

now by Lemma 1.6,

$$\mathcal{R}A\ \sigma_1^*\ \left(\mathcal{R}B\ \sigma_2^*\ \mathcal{R}C\right) \geq \left(\mathcal{R}A\ \sigma_1^*\ \mathcal{R}B\right)\ \sigma_2^*\ \left(\mathcal{R}A\ \sigma_1^*\ \mathcal{R}C\right) \tag{2.10}$$

and consequently by Lemma 1.2 and (2.10), we get

$$\begin{aligned}
\mathcal{R}\left(A\sigma_1^*(B\sigma_2^* C)\right) &\geq \mathcal{R}A\ \sigma_1^*\ \mathcal{R}(B\ \sigma_2^*\ C) \\
&\geq \mathcal{R}A\ \sigma_1^*\ \left(\mathcal{R}B\ \sigma_2^*\ \mathcal{R}C\right) \\
&\geq \left(\mathcal{R}A\ \sigma_1^*\ \mathcal{R}B\right)\sigma_2^*\ \left(\mathcal{R}A\ \sigma_1^*\ \mathcal{R}C\right) \\
&\geq \cos^2(\alpha)\ \left(\mathcal{R}(A\sigma_1^* B)\sigma_2^*\mathcal{R}(A\sigma_1^* C)\right) \\
&\geq \cos^4(\alpha)\ \mathcal{R}\left((A\sigma_1^* B)\sigma_2^*(A\sigma_1^* C)\right).
\end{aligned}$$

□

Corollary 2.10. *Let σ_1 and σ_2 be nonzero connections and $A, B > 0$. If*

$$A\sigma_1(A\sigma_2 B) \leq A\sigma_2(A\sigma_1 B),$$

then

$$A\sigma_1^*(A\sigma_2^* B) \geq A\sigma_2^*(A\sigma_1^* B).$$

Proof. If we put A instead of B and B instead of C in Theorem 2.9, the desired result will be obtained. ☐

3. Applications

Let us derive operator inequalities involving operator means by using the previous theorems. The following lemma, helps us present the norm version of Proposition 2.6.

Lemma 3.1. *([3, 16]), Let $A \in \mathcal{S}_\alpha$. Then*

$$\|\mathcal{R}A\| \leq \|A\| \leq sec(\alpha)\ \|\mathcal{R}A\|.$$

for any unitarily invariant norm $\|.\|$ on $B(H)$.

Theorem 3.2. *Let $A, B \in \mathcal{S}_\alpha$, $0 \leq t \leq 1$ and σ_t be an arbitrary mean between ∇_t and $!_t$. If $f : (0, \infty) \to (0, \infty)$ is an operator monotone decreasing function, then*

$$\|f((1-t)A + tB)\| \leq \sec^5(\alpha)\ \|f(A)\ \sigma_t^*\ f(B)\|,$$

for any unitarily invariant norm $\|.\|$ on $B(H)$.

Proof. Lemma 3.1 and Proposition 2.6, together imply that

$$\begin{aligned}\|f((1-t)A + tB)\| &\leq \sec(\alpha)\ \|\mathcal{R}\left(f((1-t)A + tB)\right)\| \\ &\leq \sec^5(\alpha)\ \|\mathcal{R}\left(f(A)\ \sigma_t^*\ f(B)\right)\| \\ &\leq \sec^5(\alpha)\ \|f(A)\ \sigma_t^*\ f(B)\|.\end{aligned}$$

☐

By applying the following lemma, we obtain the determinant version of Proposition 2.6.

Lemma 3.3. *([6, 9]), If $A \in \mathcal{S}_\alpha$, then*

$$det(\mathcal{R}A) \leq |det(A)| \leq sec^n(\alpha)\ det(\mathcal{R}A).$$

Theorem 3.4. *Let $A, B \in \mathcal{S}_\alpha$, $0 \leq t \leq 1$ and σ_t be an arbitrary mean between ∇_t and $!_t$. If $f : (0, \infty) \to (0, \infty)$ is an operator monotone decreasing function, then*

$$|det\left(f((1-t)A + tB)\right)| \leq \sec^{5n}(\alpha)\ \ |det\left(f(A)\ \sigma_t^*\ f(B)\right)|.$$

Proof. By applying Lemma 3.3, Proposition 2.6, and the determinant definition, we get

$$\begin{aligned}|det\left(f((1-t)A + tB)\right)| &\leq \sec^n(\alpha)\ \det\left(\mathcal{R}\left(f((1-t)A + tB)\right)\right) \\ &\leq \sec^{5n}(\alpha)\ \det\left(\mathcal{R}\left(f(A)\ \sigma_t^*\ f(B)\right)\right) \\ &\leq \sec^{5n}(\alpha)\ \ |det\left(f(A)\ \sigma_t^*\ f(B)\right)|.\end{aligned}$$

☐

Using the following lemma, singular value versions of Proposition 2.6 can be obtained.

Lemma 3.5. *([7]), Let $A \in \mathcal{S}_\alpha$. Then*

$$\lambda_j(\mathcal{R}A) \leq s_j(A) \leq sec^2(\alpha)\ \lambda_j(\mathcal{R}A), \qquad j = 1, \ldots, n,$$

where s_j and λ_j denote the jth largest singular value and eigenvalue of a matrix.

Theorem 3.6. *Let $A, B \in \mathcal{S}_\alpha$, $0 \leq t \leq 1$ and σ_t be an arbitrary mean between ∇_t and $!_t$. If $f : (0, \infty) \to (0, \infty)$ is an operator monotone decreasing function, then*

$$s_j\left(f((1-t)A + tB)\right) \leq \sec^6(\alpha)\ s_j\left(f(A)\ \sigma_t^*\ f(B)\right).$$

Proof. Lemma 3.5 and Proposition 2.6, together imply that

$$\begin{aligned} s_j\left(f((1-t)A + tB)\right) &\leq \sec^2(\alpha)\ \lambda_j\left(\mathcal{R}\left(f((1-t)A + tB)\right)\right) \\ &\leq \sec^6(\alpha)\ \lambda_j\left(\mathcal{R}\left(f(A)\ \sigma_t^*\ f(B)\right)\right) \\ &\leq \sec^6(\alpha)\ s_j\left(f(A)\ \sigma_t^*\ f(B)\right). \end{aligned}$$

□

It is well known that the numerical radius $\omega(A)$ of $A \in \mathbb{M}_n$ is defined by

$$\omega(A) = \sup\{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}.$$

When $A \in \mathcal{S}_0$, we have $\omega(A) = \|A\|$, and therefore $\omega(\mathcal{R}A) = \|\mathcal{R}A\|$. Bedrani et al. [2] showed that if $A \in \mathcal{S}_\alpha$, then

$$\omega(\mathcal{R}A) \leq \omega(A) \leq \sec(\alpha)\ \omega(\mathcal{R}A). \tag{3.1}$$

Theorem 3.7. *Let $A, B \in \mathcal{S}_\alpha$, $0 \leq t \leq 1$ and σ_t be an arbitrary mean between ∇_t and $!_t$. If $f : (0, \infty) \to (0, \infty)$ is an operator monotone decreasing function, then*

$$\omega\left(f((1-t)A + tB)\right) \leq \sec^5(\alpha)\ \omega\left(f(A)\ \sigma_t^*\ f(B)\right).$$

Proof. By Proposition 2.6 and inequality (3.1), we get

$$\begin{aligned} \omega\left(f((1-t)A + tB)\right) &\leq \sec(\alpha)\ \omega\left(\mathcal{R}(f((1-t)A + tB))\right) \\ &\leq \sec^5(\alpha)\ \omega\left(\mathcal{R}(f(A)\ \sigma_t^*\ f(B))\right) \\ &\leq \sec^5(\alpha)\ \omega\left(f(A)\ \sigma_t^*\ f(B)\right). \end{aligned}$$

□

References

[1] T. Ando, F. Hiai, Operator log-convex functions and operator means, *Math. Ann*, **350**:611-630, 2011.
[2] Y. Bedrani, F. Kittaneh and M. Sababeh, From positive to accretive matrices, *Positivity*, **25**(4):1601-1629, 2021.
[3] R. Bhatia, *Matrix Analysis*, Vol. 169, Springer-Verlag., New York, (1997).
[4] P. Chansangiam, Adjointations of operator inequalities and characterizations of operator monotonicity via operator means, *Communications in Mathematics and Applications*, **7**:93-103, 2016.
[5] A. Ghazanfari and S. Malekinejad, Heron means and Polya inequality for sector matrices, *Bull. Math. Soc. Sci. Math. Roumanie. Tome*, **64**:329-339, 2021.
[6] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press., Cambridge, 2013.
[7] H. Jafarmanesh, M. Khosravi and A. Sheikhhosseini, Some operator inequalities involving operator monotone functions, *Bull. Sci. Math*, **166**:102938, 2021.
[8] F. Kubo and T. Ando, Means of positive linear operators, *Math. Ann*, **246**:205-224, 1980.
[9] M. Lin, Extension of a result of Hanynsworth and Hartfiel,*Arch. Math*, **104**:93-100, 2015.
[10] M. Lin, Some inequalities for sector matrices,*Oper. Matrices*, **10**:915-921, 2016.

[11] J.T. Liu and Q.W. Wang, More inequalities for sector matrices, *Bulletin of the Iranian Mathematical Society*, **44**:1059-1066, 2018.

[12] S. Malekinejad, M. Khosravi and A. Sheikhhosseini, Mean inequalities for sector matrices involving positive linear maps, *Positivity*, **26**(44):1-17, 2022.

[13] Y. Mao, Y. Mao, Inequalities for the Hienz Mean of Sector Matrices, *Bulletin of the Iranian Mathematical Society*, **46**:1767-1774, 2020.

[14] J. E. pečarić, T. Furuta, J. M. Hot and Y. Seo, *Mond-pečarić Method in operator inequalities*, Element., Zagreb, (2005).

[15] M. Raissouli, M.S. Moslehian and S.furuichi, Relative entropy and Tsallis entropy of two accretive operators, *C. R. Acad. Sci. Paris Ser I*, **355**:687-693, 2017.

[16] F. Zhan, A matrix decomposition and its applications, *Linear Multilinear Algebra*, **63**:2033-2042, 2015.

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