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Research Paper

THE CURVATURES OF R-QUADRATIC FINSLER METRICS

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ABSTRACT. This paper presents a study of R-quadratic Finsler spaces and a new class of Finsler metrics called $\bar{D}$-metrics. The non-Riemannian curvatures of R-quadratic Finsler spaces and their special case, the R-quadratic generalized (α, β)-metrics, are analyzed to gain insights into their behavior. The paper then introduces the $\bar{D}$-metrics, which are shown to be a proper subset of the class of GDW-metrics and contain the class of Douglas metrics. This paper contributes to the understanding of R-quadratic Finsler spaces and their properties, and presents a novel class of Finsler metrics with potential applications in the field. **MSC(2010):** 53B40; 53C60
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1. Introduction

Finsler geometry contains several intriguing curvatures, and the Riemann curvature is one of the most important among them. In a Finsler space (M, F), the Riemann curvature is a family of linear transformations $\mathbf{R}y : T_xM \to T_xM$, where $y \in T_xM$, that measures the failure of parallel transport to return to its original position in the tangent space TM. For a Finsler space (M, F), the Riemann curvature is a family of linear transformations

$$\mathbf{R}_y : T_xM \to T_xM,$$

where $y \in T_xM$, with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\forall \lambda > 0$. A Finsler metric (M, F) is called *R-quadratic* if its Riemann curvature $\mathbf{R}_y$ is quadratic in $y \in T_xM$. R-quadratic Finsler spaces form a rich class of Finsler spaces.

Numerous R-quadratic Finsler metrics exist that are non-Riemannian. It is evident that all Berwald metrics belong to this category. The Berwald curvature for Finsler metrics was initially explored by L. Berwald, who demonstrated that the third-order derivatives of spray coefficients give rise to an invariant known as the Douglas curvature [17]. A Finsler metric (M, F) is called Berwald metric if its Geodesic coefficients are quadratic in $y \in T_xM$ for any $x \in M$ or equivalently the Berwald curvature vanishes. Put

$$L_{jkl} = -\frac{1}{2} g_{im} y^m B_j{}^i{}_{kl},$$

as a Landsberg curvature of Finsler metric F. A Finsler metric is called landsberg metric if its Landsberg curvature vanishes. One of the main open problems in Finsler geometry is the so-called Landsberg Unicorn problem, that is to say, to find a Finsler metric which

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is Landsberg but not Berwald.
Taking a trace of Berwald curvature give us the mean Berwald curvature. A Finsler metric is called weak Berwald (or WB) if the mean Berwald curvature vanishes.
The class of R-quadratic Finsler metrics was introduced by Z. Shen and could be considered as a generalization of Berwald metrics. In [15], it is also proved that every R-quadratic Finsler metric is a generalized Douglas-Weyl metric or GDW-metrics. Finsler geometry encompasses numerous well-known projective invariants, one of which is the class of GDW-metrics.
This article focuses on the study of R-quadratic Finsler metrics. Apart from the Riemann curvature, Finsler geometry encompasses several other non-Riemannian quantities, including the S-curvature, E-curvature, H-curvature, and Douglas curvature, which vanish in Riemannian metrics. Investigating these non-Riemannian curvatures of R-quadratic Finsler metrics would be intriguing to determine the extent of this class of Finsler metrics. The present paper examines these curvatures of R-quadratic Finsler metrics.
A new class of Finsler metrics, named $\bar{D}$-metrics, is introduced in this study, based on the Douglas curvature of these metrics. This class includes all the Douglas metrics and is demonstrated to be a proper subset of the GDW-metrics. The paper also examines the properties of this crucial class of Finsler metrics. Furthermore, some noteworthy and non-trivial $\bar{D}$-metrics are presented in the following.

Example 1.1. [8] Put

$$\Omega=\{(x,y,z)\in R^3|x^2+y^2<1\},\quad p=(x,y,z)\in\Omega,\quad y=(u,v,w)\in T_p\Omega.$$

Define the Randers metric $F=\alpha+\beta$ by

$$\alpha=\frac{\sqrt{(-yu+xv)^2+(u^2+v^2+w^2)(1-x^2-y^2)}}{1-x^2-y^2},\quad \beta=\frac{-yu+xv}{1-x^2-y^2}.$$

The above Randers metric has vanishing flag curvature $K=0$ and S-curvature $S=0$. β is not closed then F is not of Douglas type. According to Corollary 3.7, one see that F is a non-trivial $\bar{D}$-metric.

The following example presents a $\bar{D}$-metric which is not of Douglas type, too.

Example 1.2. Consider the following Randers metric defined nearby the origin

$$F=\frac{\sqrt{|y|^2-(|xQ|^2|y|^2-<y,xQ>^2)}}{1-|xQ|^2}-\frac{<y,xQ>}{1-|xQ|^2},$$

where $Q=\left(q^i{}_j\right)$ is an anti-symmetric matrix. $R^i{}_k=0$ for F but it is not a Berwald metric when $Q\neq 0$. β is not closed and then F is not Douglas metric. On the other hands, as stated in [12], for this metric we have $e_{ij}=0$ which by Lemma 3.1 in [8] one finds that $S=0$. Then $D_j{}^i{}_{kl|m}y^m=B_j{}^i{}_{kl|m}y^m=R_j{}^i{}_{ml.k}=0$ which shows that F is a $\bar{D}$-metric.

It is clear that every $\bar{D}$-metric is a GDW-metric. In the following, an example is presented that shows the class of $\bar{D}$-metrics is a proper subset of the class of GDW-metrics. Then one could see that

$$\{Douglas\quad metrics\}\subsetneqq\{\bar{D}-metrics\}\subsetneqq\{GDW-metrics\}$$

It is evident that there is no overlap between the non-trivial $\bar{D}$-metrics and the D-recurrent metrics discussed in [2].

Example 1.3. [4] The family of Randers metrics on S^3 constructed by Bao-Shen are weakly Berwald which are not Berwaldian. Denote generic tangent vectors on S^3 as

$$u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}.$$

The Finsler function for Bao-Shen's Randers space is given by

$$F(x,y,z;u,v,w)=\alpha(x,y,z;u,v,w)+\beta(x,y,z;u,v,w),$$

with

$$\alpha=\frac{\sqrt{\lambda(cu-zv+yw)^2+(zu+cv-xw)^2+(-yu+xv+cw)^2}}{1+x^2+y^2+z^2},$$

$$\beta=\frac{\pm\sqrt{\lambda-1}(cu-zv+yw)}{1+x^2+y^2+z^2},$$

where $\lambda>1$ is a real constant. The above Randers metric has vanishing S-curvature and with positive constant flag curvature 1. Then one has

$$D_j{}^i{}_{kl|m}-D_j{}^i{}_{km|l}=B_j{}^i{}_{kl|m}-B_j{}^i{}_{km|l}=R_j{}^i{}_{lm.k}=2(C_{jkl}\delta^i{}_m-C_{jkm}\delta^i{}_l)\neq 0,$$

Then $D_j{}^i{}_{kl|0}=2C_{jkl}y^i$, it was observed that F satisfies the conditions of being a GDW-metric, but does not meet the criteria to be considered a $\bar{D}$-metric.

There exist several compelling classes of Finsler metrics that are subsets of the GDW-metrics, such as Berwald metrics, R-quadratic Finsler metrics, Douglas and $\bar{D}$-metrics. In this paper, we introduce and study a new class of Finsler metrics called $\bar{D}$-metrics, which contains the class of Douglas metrics and is a proper subset of the class of GDW metrics. Our study of these metrics is motivated by their potential applications in the field of Finsler geometry. We first consider R-quadratic Finsler spaces, then focus on the special case of R-quadratic generalized (α,β)-metrics, which have been used in a variety of applications such as in physics and biology.
To better understand the properties of these metrics, we explore their non-Riemannian curvatures. Our study provides insights into the behavior of these curvatures for R-quadratic Finsler spaces and the R-quadratic generalized (α,β)-metrics. Overall, this paper contributes to the understanding of R-quadratic Finsler spaces and their properties, as well as introducing and studying a novel class of Finsler metrics, referred to as $\bar{D}$-metrics.
Throughout this paper, the symbols "$_.$" and "$_|$" denote the vertical and horizontal derivatives with respect to Berwald connection, while the symbol "$_{|0}$" is used to represent the horizontal covariant derivative along Finsler geodesic of F, which is denoted by $_{|m}y^m$.

2. Preliminaries

A Finsler metric on a manifold M is a non-negative function F on TM having the following properties

(a) F is C^∞ on $TM\setminus\{0\}$;
(b) $F(\lambda y)=\lambda F(y)$, $\forall\lambda>0,\ \ y\in TM$;

(c) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\Big[F^2(y+su+tv)\Big]\Big|_{s,t=0}, \qquad u,v \in T_xM. \tag{2.1}$$

At each point $x \in M$, $F_x := F|_{T_xM}$ is an Euclidean norm if and only if $\mathbf{g}_y$ is independent of $y \in T_xM \setminus \{0\}$. (α,β)-metrics are the well-known examples of Finsler metrics. In the study of Finsler geometry, we often encounter complicated calculations. Then, some special classes of Finsler metrics such as (α,β)-metrics and in special case, Randers metrics, are notable spaces to study the problems in Finsler geometry. It is natural to wonder if the result of this study can be extended to the arbitrary Finsler spaces.

The (α,β)-metrics are of the form $F = \alpha\varphi(s)$, where φ is a C^∞ positive function and $s = \frac{\beta}{\alpha}$. A new class of Finsler metrics, called general (α,β)-metrics was introduced in [22]. It is given by $F = \alpha\varphi(b^2,s)$, where φ is C^∞ positive function and $b^2 := \|\beta\|^2_\alpha$. This class of Finsler metrics not only generalize (α,β)-metrics in a natural way, but also includes spherically symmetric Finsler metrics [16]. In [22], it is proved that a general (α,β)-metric $F = \alpha\varphi(b^2,s)$ satisfies

$$\varphi - s\varphi_s > 0, \quad \varphi - s\varphi_s + (b^2-s^2)\varphi_{ss} > 0, \quad for \quad n \geq 3,$$

or

$$\varphi - s\varphi_s + (b^2-s^2)\varphi_{ss} > 0, \qquad for \quad n = 2,$$

where s and b are arbitrary numbers with $|s| \leq b < b_0$. Here φ_s denotes the differentiation of φ with respect to s. Let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_j = b^m r_{mj}, \qquad s_j = b^m s_{mj},$$

where $b_{i|j}$ denote the covariant derivatives of b_i with respect to α.

To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to R$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]\Big|_{t=0}, \qquad u,v,w \in T_xM. \tag{2.2}$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM\setminus\{0\}}$ is called the *Cartan torsion*. A curve $c = c(t)$ is called a *geodesic* if it satisfies

$$\frac{d^2c^i}{dt^2} + 2G^i(c,\dot{c}) = 0, \tag{2.3}$$

where $\dot{c} = \frac{dc}{dt}$ and $G^i(x,y)$ are local functions on TM given by

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\{\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\}, \quad y \in T_xM. \tag{2.4}$$

and called the coefficients of the associated spray to (M,F). The projection of an integral curve of G^i is called a geodesic in M. F is called a Berwald metric if $G^i(x,y)$ are quadratic in $y \in T_xM$ for all $x \in M$. For $y \in T_xM_0$, define

$$B_y : T_xM \times T_xM \times T_xM \to T_xM$$
$$B_y(u,v,w) = B_j{}^i{}_{kl}u^j v^k w^l \frac{\partial}{\partial x^i},$$

where $B_j{}^i{}_{kl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$. Put

$$E_y : T_xM \times T_xM \to R$$

$$E_y(u,v) = E_{jk} u^j v^k,$$

where $E_{jk} = \frac{1}{2} B_j{}^m{}_{km}$, $u = u^i \frac{\partial}{\partial x^i}$, $v = v^i \frac{\partial}{\partial x^i}$ and $w = w^i \frac{\partial}{\partial x^i}$. B and E are called the Berwald curvature and mean Berwald curvature, respectively and F is called a Berwald metric and Weakly Berwald (WB) metric if $B = 0$ and $E = 0$, respectively [18]. A Finsler metric (M, F) is called to have isotropic mean Berwald curvature if

$$E_{ij} = \frac{n+1}{2} c F^{-1} h_{ij},$$

for some scalar function $c = c(x)$ on M, where h_{ij} is the angular metric. By means of E-curvature, we can define $\bar{E}$-curvature as follows

$$\bar{E}_y : T_xM \times T_xM \times T_xM \longrightarrow \mathbb{R}$$

$$\bar{E}_y(u,v,w) := \bar{E}_{jkl}(y) u^i v^j w^k = E_{ij|k} u^i v^j w^k.$$

It is remarkable that, $\bar{E}_{ijk}$ is not totally symmetric in all three of its indices.
The S-curvature $S(x,y)$ was introduced as follows [18]

$$S(x,y) = \frac{d}{dt}[\tau(\gamma(t), \gamma'(t)]_{|t=0},$$

where $\tau(x,y)$ is the distortion of the metric F and $\gamma(t)$ is the geodesic with $\gamma(0) = x$ and $\gamma'(0) = y$ on M. It is considerable that [17]

$$E_{ij} = \frac{1}{2} S_{.i.j}, \tag{2.5}$$

where $.i$ denotes the differential with respect to y^i. The non-Riemannian quantity Ξ-curvature is denoted by $\Xi = \Xi_j dx^j$ and is defined as [18]

$$\Xi_j = S_{.j|m} y^m - S_{|j}. \tag{2.6}$$

The Finsler metric F is said to have almost vanishing Ξ-curvature if

$$\Xi_i = -(n+1) F^2 \big(\frac{\theta}{F}\big)_{.i}, \tag{2.7}$$

where θ is a 1-form on M and $n = dimM$. The H-curvature was introduced by Akbar-Zadeh which is closely related to the S-curvature [1]. The H-curvature is defined as

$$H_y : T_xM \times T_xM \longrightarrow \mathbb{R}$$

$$H_y(u,v) = H_{jk} u^j v^k,$$

where

$$H_{ij} = \frac{1}{4}\big(\Xi_{i.j} + \Xi_{j.i}\big), \tag{2.8}$$

$u = u^i \frac{\partial}{\partial x^i}$ and $v = v^i \frac{\partial}{\partial x^i}$. One says that F has almost vanishing H-curvature if

$$H_{ij} = \frac{n+1}{2} \theta F_{.i.j}.$$

Let

$$D_j{}^i{}_{kl} = B_j{}^i{}_{kl} - \frac{1}{n+1}\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}(\frac{\partial G^m}{\partial y^m} y^i).$$

It is easy to verify that $D := D_j{}^i{}_{kl} dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i}$ is a well-defined tensor on slit tangent space TM_0. We call D the Douglas tensor. The Douglas tensor D is a non-Riemannian projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent,

$$G^i = \bar{G}^i + Py^i,$$

where $P = P(x, y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is same as that of $\bar{F}$ [7], [17]. One could easily show that

$$D_j{}^i{}_{kl} = B_j{}^i{}_{kl} - \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk.l}y^i\}. \tag{2.9}$$

Douglas curvature, $D_j{}^i{}_{kl}$, is a projective invariant constructed from the Berwald curvature. Finsler metrics with $D_j{}^i{}_{kl} = 0$ are called Douglas metrics. The metrics with the following condition are called GDW-metric which are projective invariant.

$$D_j{}^i{}_{kl|m} y^m = T_{jkl} y^i,$$

for some tensors T_{jkl}.

To follow, we will be presenting an innovative category of Finsler metrics known as $\bar{D}$-metrics. These incorporate all Douglas metrics and are proven to be a proper subset of GDW-metrics.

2.1. **$\bar{D}$-metrics.** A Finsler metric is called $\bar{D}$-metric if $\bar{D}_j{}^i{}_{klm} = 0$, where

$$\bar{D}_j{}^i{}_{klm} = D_j{}^i{}_{kl|m} - D_j{}^i{}_{km|l}. \tag{2.10}$$

It is evident that this category of metrics encompasses all Douglas metrics. Nonetheless, as evidenced by the examples provided in the preceding section, there exist numerous $\bar{D}$-metrics that do not fall under the category of Douglas-type metrics (non-trivial $\bar{D}$-metrics).

3. R-quadratic Finsler metrics

This section delves into the examination of non-Riemannian curvatures for R-quadratic Finsler metrics. While some studies have explored these curvatures for specific metrics, such as Randers metrics ([12], [19], [20], [21] and etc.), this section aims to generalize the previous findings. Additionally, we introduce and study a new class of Finsler metrics, known as $\bar{D}$-metrics.

3.1. ***E*-curvature of *R*-quadratic Finsler metrics.** The E-curvature of R-quadratic Finsler metrics has been the subject of previous research, with notable contributions from [20]. Specifically, they demonstrated that R-quadratic Finsler metrics with Douglas curvature satisfying $D_j{}^i{}_{kl|0} = 0$ have an E-curvature that satisfies $E_{jk|l} = 0$. Building on this work, we have extended the understanding of the E-curvature of R-quadratic Finsler metrics by examining its behavior in a broader range of circumstances. Through our analysis, we have identified the following proposition and contribute to a deeper understanding of the behavior of R-quadratic Finsler metrics and their E-curvature.

Proposition 3.1. The E-curvature of R-quadratic Finsler metrics satisfies

$$E_{jk|l} = E_{jl|k}, \quad E_{jkl|0} = -E_{jk|l}. \tag{3.1}$$

Proof. The curvature form of Berwald connection is as follows

$$\Omega^i_j = d\omega^i_j - \omega^k_j \wedge \omega^i_k = \frac{1}{2} R^i_{jkl}\omega^k \wedge \omega^l - B^i_{jkl}\omega^k \wedge \omega^{n+l}. \tag{3.2}$$

For Berwald connection we have

$$dg_{ij} - g_{jk}\Omega^k_i - g_{ik}\Omega^k_j = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}, \tag{3.3}$$

where $L_{ijk} := C_{ijk|m}y^m$ is the Landsberg curvature. By differentiating (3.3) one gets

$$\begin{aligned} g_{pj}\Omega^p_i + g_{pi}\Omega^p{}_j &= -2L_{ijk|l}\omega^k \wedge \omega^l - 2L_{ijk.l}\omega^k \wedge \omega^{n+l} \\ &-2C_{ijk|l}\omega^k \wedge \omega^{n+l} - 2C_{ijk.l}\omega^{n+k} \wedge \omega^{n+l} - 2C_{ijk}\Omega^k{}_l y^l. \end{aligned} \tag{3.4}$$

Differentiating (3.2) yields that

$$d\Omega^i_j + \Omega^k{}_j \wedge \omega^i{}_k - \omega^k{}_j \wedge \Omega^i{}_k = 0. \tag{3.5}$$

Now we have

$$dB_j{}^i{}_{kl} - B_m{}^i{}_{kl}\omega^m{}_j - B_j{}^i{}_{ml}\omega^m{}_k - B_j{}^i{}_{km}\omega^m{}_l + B_j{}^m{}_{kl}\omega^i{}_m = B_j{}^i{}_{kl|m}\omega^m + B_j{}^i{}_{kl.m}\omega^{n+m}, \tag{3.6}$$

and

$$dR_j{}^i{}_{kl} - R_m{}^i{}_{kl}\omega^m{}_j - R_j{}^i{}_{ml}\omega^m{}_k - R_j{}^i{}_{km}\omega^m{}_l + R_j{}^m{}_{kl}\omega^i{}_m = R_j{}^i{}_{kl|m}\omega^m + R_j{}^i{}_{kl.m}\omega^{n+m}. \tag{3.7}$$

Noting (3.4), (3.5), (3.6) and (3.7) one finds that

$$B_j{}^i{}_{kl|m} - B_j{}^i{}_{mk|l} = R_j{}^i{}_{ml.k}. \tag{3.8}$$

Now assume that F is of R-quadratic type. Then based on (3.8) we get

$$B_j{}^i{}_{kl|m} - B_j{}^i{}_{mk|l} = 0. \tag{3.9}$$

By contracting the equation above by y^m and referencing (2.9), we find

$$D_j{}^i{}_{kl|0} + \frac{2}{n+1}\{E_{jk|0}\delta^i_l + E_{jl|0}\delta^i_k + E_{kl|0}\delta^i_j + E_{jkl|0}y^i\} = 0. \tag{3.10}$$

We knows that every R-quadratic Finsler metric is of GDW type. Then there is a tensor T_{jkl} on TM such that $D_j{}^i{}_{kl|0} = T_{jkl}y^i$. When inserted into equation (3.10), it is shown that

$$E_{jk|0} = 0, \quad \textit{and} \quad D_j{}^i{}_{kl|0} = T_{jkl}y^i = -\frac{2}{n+1}E_{jkl|0}y^i. \tag{3.11}$$

One could easily find that

$$D_j{}^i{}_{kl} = -\frac{2}{n+1}E_{jkl}y^i + d_j{}^i{}_{kl}, \tag{3.12}$$

where $d_j{}^i{}_{kl|0} = 0$. Then according to (2.9) one gets

$$d_j{}^i{}_{kl} = B_j{}^i{}_{kl} - \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{kl}\delta^i_j + E_{jl}\delta^i_k\}. \tag{3.13}$$

On the other hand we have the following Ricci identity for E-curvature

$$E_{jk.l|m} - E_{jk|m.l} = E_{pk}B_j{}^p{}_{ml} + E_{jp}B_k{}^p{}_{ml}, \tag{3.14}$$

which its contracting by y^m yields

$$E_{jk|m.l}y^m - E_{jk.l|0} = 0.$$

Since $E_{jk|0} = 0$ then one could easily find that $E_{jkl|0} = E_{jk|m.l}y^m = -E_{jk|l}$, in other words,

$$E_{jk|l} = E_{jl|k} = E_{kl|j}, \qquad E_{jkl|0} = -E_{jk|l}. \tag{3.15}$$

This completes the proof. □

The corollary, easily derived from the equations (3.12) and (3.15), is a result presented as the main theorem in [20].

Corollary 3.2. [20] For every R-quadratic Finsler metric with Douglas curvature $D_j{}^i{}_{kl|0} = 0$, the E-curvature satisfies $E_{jk|l} = 0$.

3.2. **S-curvature of R-quadratic Finsler metrics.** Previous research has shown that R-quadratic Randers metrics have a constant S-curvature, as demonstrated by Li and Shen [12]. Building on this work, we extend the study of S-curvature to arbitrary R-quadratic Finsler metrics. Our analysis reveals new insights into the behavior of S-curvature in R-quadratic Finsler metrics. Specifically, we identify

Theorem 3.3. The S-curvature of a R-quadratic Finsler metric (M, F) satisfies

$$S_{|k.l} - S_{|l.k} = f_{lk}(x), \tag{3.16}$$

where $f_{lk}(x)$ is a scalar function on M with $f_{lk}(x) = -f_{kl}(x)$.

Proof. In the previous section, it has been proved that the E-curvature of R-quadratic Finsler metric F satisfies

$$E_{jk|l} = E_{jl|k}. \tag{3.17}$$

In other words, based on (2.5), we have

(3.18)

$$S_{|k.l.j} = (\frac{\partial S}{\partial x^k} - N^r_k \frac{\partial S}{\partial y^r})_{.l.j} = 2(\frac{\partial E_{jl}}{\partial x^k} - N^r_k \frac{\partial E_{jl}}{\partial y^r} - E_{rl}\Gamma^r_{jk} - E_{rj}\Gamma^r_{lk}) - S_{.r}B_j{}^r{}_{kl} = 2E_{jl|k} - S_{.r}B_j{}^r{}_{kl},$$

where $\Gamma^r_{jk} = \frac{\partial^3 G^r}{\partial y^j \partial y^k}$ and $N^r_k = \frac{\partial G^r}{\partial y^k}$ are called the connection coefficients and Christoffel symbols of G, respectively. Based on (3.17) and the above equation one gets

$$0 = S_{|k.l.j} - S_{|l.k.j} = (S_{|k.l} - S_{|l.k})_{.j}.$$

Then one easily concludes (3.16). □.

In [21], it is proven that the Ξ-curvature of general (α, β)-metrics satisfying equation

$${}^{\alpha}R^i_k = \mu(\alpha^2\delta^i_k - y^i y_k), \quad b_{i|j} = c(x)a_{ij},$$

always vanishes. Furthermore, the subsequent corollary determines the exact value of the Ξ-curvature for all Finsler metrics of R-quadratic form.

Corollary 3.4. The Ξ-curvature of every R-quadratic Finsler metric is given by

$$\Xi_j = f_{jm}(x)y^m,$$

where f_{jm} is a scalar function on M with $f_{jm}(x) = -f_{mj}(x)$.

Proof. It is concluded by (2.6) and the above theorem.

3.3. **Douglas Curvature of R-quadratic Finsler metrics.** It has been established that every R-quadratic Finsler metric is a member of the larger class of Generalized Douglas Weyl (GDW)-metrics, as demonstrated in prior research by authors including [19]. This relationship is important in the study of Finsler geometry, as the class of GDW-metrics includes both Douglas and Weyl metrics, which are two important subclasses of Finsler metrics. Based on (3.8) and (3.15), the following equation would be satisfied for Douglas curvature of R-quadratic Finsler metrics.

$$D_j{}^i{}_{kl|m} - D_j{}^i{}_{km|l} = -\frac{2}{n+1}[E_{jk|m}\delta^i_l - E_{jk|l}\delta^i_m + (E_{jkl|m} - E_{jkm|l})y^i]. \tag{3.19}$$

One could easily find that

$$D_j{}^i{}_{kl|0} = \frac{2}{n+1}E_{jkl|0}y^i,$$

which means that F is of GDW-metric.
The equation (3.19) guides the creation of a novel group of Finsler metrics known as $\bar{D}$-metrics, as discussed in earlier sections. It is evident that each Douglas metric falls under the category of $\bar{D}$-metrics. The proposition following from equation (3.19) is as follows.

Proposition 3.5. Let F be a R-quadratic Finsler metric. Then it is $\bar{D}$-metric if and only if $E_{jk|l} = 0$.

Proposition 3.6. Let F be a GDW-metric. If F is of isotropic S-curvature Then it is $\bar{D}$-metric if and only if

$$(L_{jkl} + cFC_{jkl})_{|0} = 0.$$

Proof. Given that F possesses isotropic S-curvature and taking into account equation (2.5), it follows that $E_{jk} = \frac{n+1}{2}cF_{jk}$. As observed in equation (2.9), one can conclude

$$D_j{}^i{}_{kl} = B_j{}^i{}_{kl} - c(F_{jk}\delta^i_l + F_{jl}\delta^i_k + F_{kl}\delta^i_j + F_{jkl}y^i), \tag{3.20}$$

Assume that

$$B_j{}^i{}_{kl|0} = \beta_j{}^i{}_{kl} + b_{jkl}y^i,$$

for some tensors $\beta_j{}^i{}_{kl}$ and b_{jkl}. Contracting the above equation by y_i yields

$$b_{jkl} = -\frac{1}{F^2}(2L_{jkl|0} + y_m\beta_j{}^m{}_{kl}). \tag{3.21}$$

As assumed, the metric F is considered to be a GDW, indicating the presence of a tensor T_{jkl} such that $D_j{}^i{}_{kl|0} = T_{jkl}y^i$; subsequently, inserting this information along with (3.21) into (3.20) results in

$$D_j{}^i{}_{kl|0} = (b_{jkl} - c_{|0}F_{jkl} - \frac{2}{F}cL_{jkl})y^i, \tag{3.22}$$

and

$$\beta_j{}^i{}_{kl} = c_{|0}(F_{jk}\delta^i_l + F_{jl}\delta^i_k + F_{kl}\delta^i_j). \tag{3.23}$$

Above, we have incorporated the following equations in our analysis

$$F_{jkl|m} = F_{jk|m.l} - F_{jr}B_k{}^r{}_{lm} - F_{rk}B_j{}^r{}_{lm}$$

and $-2L_{jkl} = g_{jk|l} = FF_{jk|l}$ which conclude

$$F_{jk|0} = 0, \quad and \quad F_{jkl|0} = \frac{2}{F}L_{jkl}.$$

Utilizing the equation (3.21) inside (3.22) while referencing (3.23) allows for the discovery that

$$D_j{}^i{}_{kl|0} = -\frac{2}{F^2}(L_{jkl} + cFC_{jkl})_{|0}y^i, \tag{3.24}$$

thereby establishing the validity of the Theorem.□

According to the Theorem stated above, it is evident that a *GDW* metric with almost isotropic S-curvature cannot be classified as a $\bar{D}$-metric when $(L_{jkl} + cFC_{jkl})_{|0} \neq 0$. By referencing (3.19), it becomes clear that

Corollary 3.7. Every non-Berwald R-quadratic Finsler metric with vanishing S-curvature is a non-trivial $\bar{D}$-metric.

According to Theorem 1-1 from the reference [19], any regular R-quadratic non-Randers (α, β)-metric with zero S-curvature is classified as a Berwald metric. Then every metric conforming to the conditions outlined in the corollary above, is considered as a Randers metric. The non-trivial $\bar{D}$-metrics presented in the previous sections.

4. R-quadratic general (α, β)-metrics

The objective of this section is to investigate R-quadratic general (α, β)-metrics, where β is a closed and conformal 1-form, and α possesses a constant sectional curvature. This entails that α and β must adhere to the following criteria

$${}^{\alpha}R^i_k = \mu(\alpha^2\delta^i_k - y^i y_k), \quad b_{i|j} = c(x)a_{ij}, \tag{4.1}$$

where ${}^{\alpha}R^i_k$ denotes the Riemann curvature of the Riemannian metric α and μ is the Ricci constant.

Lemma 4.1. The Ξ-curvature of every R-quadratic general (α, β)-metric satisfying (4.1), vanishes.

Proof. The Ξ-curvature of these general (α, β)-metrics is calculated in [21] as follows

$$\Xi_k = -\frac{\kappa}{3}(\alpha b_k - s y_k),$$

and

$$\Xi_{k.l} = -\frac{\kappa_s}{3\alpha^2}(\alpha b_k - s y_k)(\alpha b_l - s y_l) - \frac{\kappa}{3}\Big(\frac{b_k y_l - b_l y_k}{\alpha} + \frac{s}{\alpha^2}y_k y_l - s a_{kl}\Big),$$

where κ is given as

$$\kappa = (n+1)(R_1)_s + 3(b^2 - s^2)(R_2)_s + 2(n+1)R_3, \tag{4.2}$$

and R_1, R_2 and R_3 is given in [21], $(R_1)_s = \frac{\partial R_1}{\partial s}$, $(R_2)_s = \frac{\partial R_2}{\partial s}$ and $\kappa_s = \frac{\partial \kappa}{\partial s}$. Observing corollary (3.4), one easily finds that $\Xi_{k.l} = f_{kl}(x)$ where $\Xi_{k.l} = -\Xi_{l.k}$ which considering the above equation yields

$$\Xi_{k.l} - \Xi_{l.k} = 2f_{kl}(x) = -\frac{2}{3\alpha}\kappa(b_k y_l - b_l y_k).$$

The right half of the above equation contains only the coefficients for y_l and y_k, whereas a scalar function stands alone on the left side. Then one finds that κ, $f_{kl}(x)$ and then Ξ_k vanish. □

Corollary 4.2. For the S-curvature of R-quadratic general (α, β)-metric satisfying (4.1), we have

$$S_{|k.l} = S_{|l.k}. \tag{4.3}$$

Proof. Based on the above lemma and (3.16), Instantly, one finds (4.3).□

Theorem 4.3. For the Riemann curvature of R-quadratic general (α, β)-metrics satisfying (4.1), one has

$$I_m R^m_k = 0, \tag{4.4}$$

Moreover, its Riemann curvature is given by

$$R^i_k = \kappa(r)\xi_p{}^i{}_{kq} y^p y^q,$$

where $\xi_p{}^i{}_{kq} := \xi_p{}^i{}_{kq}(x) = (b^2 a_{pk} - b_p b_k)\delta^i_q - (b^2 a_{pq} - b_p b_q)\delta^i_k + (b_k a_{pq} - b_p a_{kq})b^i$.

Proof. Suppose F represents an R-quadratic Finsler metric. By employing equation (10-11) from reference [17], given as

$$L_{jkl|m} - L_{jkm|l} = -\frac{1}{2} y_p R_k{}^p{}_{lm.j},$$

the following expression can be derived easily,

$$L_{jkl|m} = L_{jkm|l}.$$

Contracting the above equation by g^{jl} yields that $J_{k|m} = J_{m|k}$ and then

$$J_{k|0} = 0, \tag{4.5}$$

Furthermore, the association between Ξ-curvature and mean Landsberg curvature in a specific Finsler metric (M, F) is detailed below [9]

$$\Xi_k = -\frac{1}{3}(2R^m{}_{k.m} + R^m{}_{m.k}) = J_{k|0} + I_m R^m{}_k.$$

As per the preceding lemma and reference to (4.5), one can determine that $I_m R^m{}_k = 0$, effectively demonstrating the first component of the Theorem.

To establish the second part of the Theorem, it is important to recognize the calculated Riemann curvature of general (α, β) in accordance with condition (4.1) as outlined in [21]

$$R^i{}_k = R_1(\alpha^2\delta^i{}_k - y_k y^i) + \alpha R_2(\alpha b_k - s y_k)b^i + R_3(\alpha b_k - s y_k)y^i, \tag{4.6}$$

where

$$R_1 = \mu(1 + s\Psi) + c^2[\Psi^2 - 2s\Psi_{b^2} - \Psi_s + 2\chi(1 + s\Psi + u\Psi_s)],$$
$$R_2 = -\mu(2\chi - s\chi_s) + c^2[2(2\Psi_{b^2} - s\Psi_{b^2 s}) - \chi_{ss} + 2\chi(2\chi - s\chi_s) + (2\chi\chi_{ss} - \chi_s^2)],$$

$R_3 = -\mu(2\Psi - s\Psi_s) + c^2[2(2\Psi_{b^2} - s\Psi_{b^2 s}) - \Psi\Psi_s - \Psi_{ss} + 2\chi(\Psi - s\Psi_s + u\Psi_{ss}) - \chi_s(1 + s\Psi + u\Psi_s),$

with

$$\chi = \frac{\varphi_{ss} - 2(\varphi_{b^2} - s\varphi_{b^2 s})}{2(\varphi - s\varphi_s) + (b^2 - s^2)\varphi_{ss}}, \quad \Psi = \frac{\varphi_s + 2s\varphi_{b^2}}{2\varphi} - \frac{\chi}{2\varphi}[s\varphi + (b^2 - s^2)\varphi_s], \quad u = b^2 - s^2.$$

Here, $\varphi_{b^2} = \frac{\partial\varphi}{\partial b^2}$, $\varphi_s = \frac{\partial\varphi}{\partial s}$, $\varphi_{b^2 s} = \frac{\partial^2\varphi}{\partial b^2 \partial s}$ and $\varphi_{ss} = \frac{\partial^2\varphi}{\partial s^2}$. The mean Cartan torsion of the general (α, β) is given by

$$I_k = \frac{1}{\alpha}\Lambda(b_k - \frac{s}{\alpha}y_k), \tag{4.7}$$

where $\Lambda = \frac{1}{2}\big((n+1)\frac{\varphi_s}{\varphi} - (n-2)\frac{s\varphi_{ss}}{\varphi - s\varphi_s} + \frac{(b^2-s^2)\varphi_{sss} - 3s\varphi_{ss}}{\varphi - s\varphi_s + (b^2-s^2)\varphi_{ss}}\big)$. Based on (4.4) one could find that

$$\alpha\Lambda(R_1 + (b^2 - s^2)R_2)(b_k - \frac{s}{\alpha}y_k) = \alpha^2(R_1 + (b^2 - s^2)R_2)I_k = 0.$$

Then for the non-Riemannian metrics, one has

$$R_1 + (b^2 - s^2)R_2 = 0. \tag{4.8}$$

However, lemma (4.1) and equation (4.2) both result in the conclusion that

$$(n+1)(R_1)_s + 3(b^2 - s^2)(R_2)_s + 2(n+1)R_3 = 0.$$

According to the aforementioned two equations, we can see that

$$R_3 = -sR_2 + \frac{n-2}{2(n+1)}(b^2 - s^2)(R_2)_s. \tag{4.9}$$

Given that $F_{.p}R^p{}_k = 0$, we can deduce that $\big(\alpha_{.p}\varphi + \varphi_s(b_p - \frac{s}{\alpha}y_p)\big)R^p{}_k = 0$. The equations (4.4) and (4.7) further show that $\alpha_{.p}R^p{}_k = 0$. This implies $\alpha^2(R_3 + sR_2)(b_k - \frac{s}{\alpha}y_k) = 0$, subsequently

$$R_3 + sR_2 = 0. \tag{4.10}$$

Upon deducing from equations (4.9) and (4.10) that $(R_2)_s = 0$, it follows that $R_2 = \kappa(r)$ and one can conclude

$$R_1 = -(b^2 - s^2)\kappa(r), \quad R_3 = -s\kappa(r), \tag{4.11}$$

which by incorporating (4.6), the final result is determined □

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