Research Paper

SOME PROPERTIES OF K-FRAME IN N-HILBERT SPACE

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ABSTRACT. The notion of a K-frame in n-Hilbert space is presented and some of their characterizations are given. We establish stability condition of K-frame in n-Hilbert space under some perturbations. We verify that sum of two K-frames is also a K-frame in n-Hilbert space. Also, the concept of tight K-frame in n-Hilbert space is described and some properties of its are going to be established.

1. Introduction and Background

In 1952, Duffin and Schaeffer introduced frames in Hilbert spaces in their fundamental paper [6], they used frames as a tool in the study of nonharmonic Fourier series. Later in 1986, the formal definition of frame in the abstract Hilbert spaces were given by Daubechies, Grossman, Meyer [2]. A frame for a Hilbert space is a generalization of an orthonormal basis and this is such a tool that also allows each vector in this space can be written as a linear combination of elements from the frame but, linear independence among the frame elements is not required. Such frames play an important role in Gabor and wavelet analysis. Several generalizations of frames namely, g-frame, K-frames etc. have been introduced in recent times. K-frames for a separable Hilbert spaces were introduced by Lara Gavruta [13] to study the basic notions about atomic system for a bounded linear operator. In recent times, K-frame was presented to reconstruct elements from the range of a bounded linear operator K in a separable Hilbert space. K-frames are more generalization than the ordinary frames and many properties of ordinary frames may not holds for such generalization of frames. Generalized atomic subspaces for operators in Hilbert spaces were studied by P. Ghosh and T. K. Samanta [9] and they were also presented the stability of dual g-fusion frames in Hilbert spaces in [8].

The concept of 2-inner product space was first introduced by Diminnie, Gahler and White [4] in 1970's. In 1989, A. Misiak [15] developed the generalization of a 2-inner product space for $n \geq 2$. Frame in n-Hilbert space was presented by P. Ghosh and T. K. Samanta [10]. They also discussed frame in tensor product of n-Hilbert spaces [11].

In this paper, we shall present the notion of a K-frame relative to n-Hilbert space and discuss some properties. Further it will be seen that the family of all K-frames is closed with respect to addition in n-Hilbert space. We also give the notion of a tight K-frame in n-Hilbert space.

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Throughout this paper, H will denote a separable Hilbert space with the inner product $\langle \cdot , \cdot \rangle$ and $\mathcal{B}(H)$ denote the space of all bounded linear operator on H. We also denote $\mathcal{R}(T)$ for range set of T where $T \in \mathcal{B}(H)$ and $l^2(\mathbb{N})$ denote the space of square summable scalar-valued sequences with index set $\mathbb{N}$.

2. Preliminaries

Theorem 2.1. *(Douglas' factorization theorem)*[5] *Let $U, V \in \mathcal{B}(H)$. Then the following conditions are equivalent:*

(i) $\mathcal{R}(U) \subseteq \mathcal{R}(V)$.

(ii) $U U^* \leq \lambda^2 V V^*$ *for some $\lambda > 0$.*

(iii) $U = V W$ *for some $W \in \mathcal{B}(H)$.*

Theorem 2.2. [7] *Let $S, T, U \in \mathcal{B}(H)$. Then the following are equivalent:*

(i) $\mathcal{R}(S) \subseteq \mathcal{R}(T) + \mathcal{R}(U)$.

(ii) $S S^* \leq \lambda^2 (T T^* + U U^*)$ *for some $\lambda > 0$.*

(iii) $S = T A + U B$ *for some $A, B \in \mathcal{B}(H)$.*

Theorem 2.3. *([1]) Let H_1, H_2 be two Hilbert spaces and $U : H_1 \to H_2$ be a bounded linear operator with closed range $\mathcal{R}_U$. Then there exists a bounded linear operator $U^\dagger : H_2 \to H_1$ such that $U U^\dagger x = x \ \forall \, x \in \mathcal{R}_U$.*

Remark 2.4. The operator $U^\dagger$ defined in Theorem (2.3), is called the pseudo-inverse of U.

Definition 2.5. [1] A sequence $\{ f_i \}_{i=1}^\infty \subseteq H$ is said to be a frame for H if there exist constants $0 < A \leq B < \infty$ such that

$$(2.1) \qquad A \, \| f \|^2 \leq \sum_{i=1}^\infty |\langle f, f_i \rangle|^2 \leq B \, \| f \|^2 \ \forall \, f \in H.$$

The constants A and B are called frame bounds. If $\{ f_i \}_{i=1}^\infty$ satisfies the right inequality of (2.1), it is called a Bessel sequence with bound B.

Definition 2.6. [1] Let $\{ f_i \}_{i=1}^\infty$ be a frame for H. Then the bounded linear operator $T : l^2(\mathbb{N}) \to H$, defined by $T \{ c_i \} = \sum_{i=1}^\infty c_i f_i$, is called pre-frame operator and its adjoint operator $T^* : H \to l^2(\mathbb{N})$, given by $T^*(f) = \{ \langle f, f_i \rangle \}_{i=1}^\infty$ is called the analysis operator. The operator $S : H \to H$ defined by $S(f) = T T^*(f) = \sum_{i=1}^\infty \langle f, f_i \rangle \, f_i \ \forall \, f \in H$ is called the frame operator.

Definition 2.7. [13] Let $K \in \mathcal{B}(H)$. Then a sequence $\{ f_i \}_{i=1}^\infty$ in H is said to be a K-frame for H if there exist constants $0 < A \leq B < \infty$ such that

$$A \, \| K^* f \|^2 \leq \sum_{i=1}^\infty |\langle f, f_i \rangle|^2 \leq B \, \| f \|^2 \ \ \forall \, f \in H.$$

Theorem 2.8. [17] *Let K be a bounded linear operator on H. Then a Bessel sequence $\{ f_i \}_{i=1}^\infty$ in H is a K-frame if and only if there exists $\lambda > 0$ such that $S \geq \lambda \, K K^*$, where S is the frame operator for $\{ f_i \}_{i=1}^\infty$.*

Definition 2.9. [14] A real valued function $\| \cdot, \cdots, \cdot \| : X^n \to \mathbb{R}$ is called a n-norm on X if the following conditions hold:

 (*i*) $\| x_1, x_2, \cdots, x_n \| = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent,

 (*ii*) $\| x_1, x_2, \cdots, x_n \|$ is invariant under permutations of $x_1, x_2, \cdots, x_n,$

 (*iii*) $\| \alpha x_1, x_2, \cdots, x_n \| = | \alpha | \, \| x_1, x_2, \cdots, x_n \|,$

 (*iv*) $\| x + y, x_2, \cdots, x_n \| \leq \| x, x_2, \cdots, x_n \| + \| y, x_2, \cdots, x_n \|.$

The pair $(X, \| \cdot, \cdots, \cdot \|)$ is then called a linear n-normed space.

Definition 2.10. [15] Let $n \in \mathbb{N}$ and X be a linear space of dimension greater than or equal to n over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field. A function $\langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle : X^{n+1} \to \mathbb{K}$ is satisfying the following five properties:

 (*i*) $\langle x_1, x_1 | x_2, \cdots, x_n \rangle \geq 0$ and $\langle x_1, x_1 | x_2, \cdots, x_n \rangle = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent,

 (*ii*) $\langle x, y | x_2, \cdots, x_n \rangle = \langle x, y | x_{i_2}, \cdots, x_{i_n} \rangle$ for every permutation $(i_2, \cdots, i_n)$ of $(2, \cdots, n),$

 (*iii*) $\langle x, y | x_2, \cdots, x_n \rangle = \overline{\langle y, x | x_2, \cdots, x_n \rangle},$

 (*iv*) $\langle \alpha x, y | x_2, \cdots, x_n \rangle = \alpha \langle x, y | x_2, \cdots, x_n \rangle,$ for all $\alpha \in \mathbb{K},$

 (*v*) $\langle x + y, z | x_2, \cdots, x_n \rangle = \langle x, z | x_2, \cdots, x_n \rangle + \langle y, z | x_2, \cdots, x_n \rangle.$

is called an *n*-inner product on X and the pair $(X, \langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle)$ is called *n*-inner product space.

Theorem 2.11. [12] *For n-inner product space* $(X, \langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle),$

$$| \langle x, y | x_2, \cdots, x_n \rangle | \leq \| x, x_2, \cdots, x_n \| \, \| y, x_2, \cdots, x_n \|$$

hold for all $x, y, x_2, \cdots, x_n \in X.$

Theorem 2.12. [15] *For every n-inner product space* $(X, \langle \cdot, \cdot | \cdot, \cdots, \cdot \rangle),$

$$\| x_1, x_2, \cdots, x_n \| = \sqrt{\langle x_1, x_1 | x_2, \cdots, x_n \rangle}$$

defines a n-norm for which

$$\langle x, y | x_2, \cdots, x_n \rangle = \frac{1}{4} \left(\| x + y, x_2, \cdots, x_n \|^2 - \| x - y, x_2, \cdots, x_n \|^2 \right), \&$$

$$\| x + y, x_2, \cdots, x_n \|^2 + \| x - y, x_2, \cdots, x_n \|^2$$
$$= 2 \left(\| x, x_2, \cdots, x_n \|^2 + \| y, x_2, \cdots, x_n \|^2 \right)$$

hold for all $x, y, x_1, x_2, \cdots, x_n \in X.$

Definition 2.13. [14] A sequence $\{ x_k \}$ in a linear *n*-normed space X is said to be convergent to some $x \in X$ if for every $x_2, \cdots, x_n \in X$

$$\lim_{k \to \infty} \| x_k - x, x_2, \cdots, x_n \| = 0 \quad \text{and it is called a Cauchy sequence if}$$

$\lim\limits_{l, k \to \infty} \| x_l - x_k, x_2, \cdots, x_n \| = 0$ for every $x_2, \cdots, x_n \in X.$ The space X is said to be complete if every Cauchy sequence in this space is convergent in $X.$ A *n*-inner product space is called *n*-Hilbert space if it is complete with respect to its induce norm.

Remark 2.14. [11] Let L_F denote the linear subspace of X spanned by the non-empty finite set $F = \{a_2, a_3, \cdots, a_n\}$, where $a_2, a_3, \cdots, a_n$ are fixed elements in X. Then the quotient space X / L_F is a normed linear space with respect to the norm, $\| x + L_F \|_F = \| x, a_2, \cdots, a_n \|$ for every $x \in X$. Let M_F be the algebraic complement of L_F, then $X = L_F \oplus M_F$. Define $\langle x, y \rangle_F = \langle x, y \,|\, a_2, \cdots, a_n \rangle$ on X. Then $\langle \cdot, \cdot \rangle_F$ is a semi-inner product on X and this semi-inner product induces an inner product on X / L_F which is given by

$$\langle x + L_F, y + L_F \rangle_F = \langle x, y \rangle_F = \langle x, y \,|\, a_2, \cdots, a_n \rangle \quad \forall \; x, y \in X.$$

By identifying X / L_F with M_F in an obvious way, we obtain an inner product on M_F. Now for every $x \in M_F$, we define $\| x \|_F = \sqrt{\langle x, x \rangle_F}$ and it can be easily verify that $(M_F, \| \cdot \|_F)$ is a norm space. Let X_F be the completion of the inner product space M_F.

For the remaining part of this paper, $(X, \langle \cdot, \cdot \,|\, \cdots, \cdot \rangle)$ is consider to be a n-Hilbert space and I will denote the identity operator on X_F.

Definition 2.15. [11] A sequence $\{f_i\}_{i=1}^{\infty}$ in X is said to be a frame associated to $(a_2, \cdots, a_n)$ for X if there exist constants $0 < A \leq B < \infty$ such that

$$(2.2) \qquad A \, \| f, a_2, \cdots, a_n \|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n \rangle|^2 \leq B \, \| f, a_2, \cdots, a_n \|^2$$

for all $f \in X$. The constants A and B are called the frame bounds. If the sequence $\{f_i\}_{i=1}^{\infty}$ satisfies the right inequality of (2.2), is called a Bessel sequence associated to $(a_2, \cdots, a_n)$ in X with bound B.

Theorem 2.16. [11] *Let* $\{f_i\}_{i=1}^{\infty}$ *be a sequence in* X. *Then* $\{f_i\}_{i=1}^{\infty}$ *is a frame associated to* $(a_2, \cdots, a_n)$ *with bounds* A & B *if and only if it is a frame for the Hilbert space* X_F *with bounds* A & B.

Definition 2.17. [11] Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence associated to $(a_2, \cdots, a_n)$ for X. Then the bounded linear operator $T : l^2(\mathbb{N}) \to X_F$, defined by $T\{c_i\} = \sum_{i=1}^{\infty} c_i f_i$, is called pre-frame operator and its adjoint operator $T^* : X_F \to l^2(\mathbb{N})$, given by $T^*(f) = \{\langle f, f_i \,|\, a_2, \cdots, a_n \rangle\}_{i=1}^{\infty} \; \forall f \in X_F$ is called the analysis operator. The operator $S : X_F \to X_F$ defined by $S_F(f) = T T^*(f) = \sum_{i=1}^{\infty} \langle f, f_i \,|\, a_2, \cdots, a_n \rangle f_i \; \forall f \in X_F$ is called the frame operator.

If $\{f_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for X, then the frame operator S_F is bounded, positive, self-adjoint and invertible. For more details on frame relative to n-Hilbert space one can go through the paper [10].

3. K-FRAME IN n-HILBERT SPACE

Definition 3.1. Let K be a bounded linear operator on X_F. Then a sequence $\{f_i\}_{i=1}^{\infty} \subseteq X$ is said to be a K-frame associated to $(a_2, \cdots, a_n)$ for X if there exist constants $0 < A \leq B < \infty$ such that

$$A \, \| K^* f, a_2, \cdots, a_n \|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n \rangle|^2 \leq B \, \| f, a_2, \cdots, a_n \|^2,$$

for all $f \in X_F.$ In particular, if $K = I$, then by Theorem (2.16), $\{f_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for $X.$ Obviously, every K-frame associated to $(a_2, \cdots, a_n)$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$ in $X.$

Remark 3.2. In general, the frame operator of a K-frame associated to $(a_2, \cdots, a_n)$ is not invertible. But, if $K \in \mathcal{B}(X_F)$ has closed range, then $S_F : \mathcal{R}(K) \to S_F(\mathcal{R}(K))$ is an invertible operator. For $f \in \mathcal{R}(K)$, we have

$$\|f, a_2, \cdots, a_n\|^2 = \left\|(K^\dagger)^* K^* f, a_2, \cdots, a_n\right\|^2 \leq \left\|K^\dagger\right\|^2 \|K^* f, a_2, \cdots, a_n\|^2.$$

Therefore, if $\{f_i\}_{i=1}^{\infty}$ is a K-frame associated to $(a_2, \cdots, a_n)$ for X, by definition (3.1), we get

$$A\left\|K^\dagger\right\|^{-2} \|f, a_2, \cdots, a_n\|^2 \leq \langle S_F f, f \,|\, a_2, \cdots, a_n \rangle \leq B \|f, a_2, \cdots, a_n\|^2$$

and furthermore for each $f \in S_F(\mathcal{R}(K))$, we have

$$B^{-1} \|f, a_2, \cdots, a_n\|^2 \leq \langle S_F^{-1} f, f \,|\, a_2, \cdots, a_n \rangle \leq A^{-1}\left\|K^\dagger\right\|^2 \|f, a_2, \cdots, a_n\|^2.$$

Theorem 3.3. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a K-frame associated to* $(a_2, \cdots, a_n)$ *for* X *and* $T \in \mathcal{B}(X_F)$ *with* $\mathcal{R}(T) \subset \mathcal{R}(K)$. *Then* $\{f_i\}_{i=1}^{\infty}$ *is a T-frame associated to* $(a_2, \cdots, a_n)$ *for* X.

Proof. Suppose $\{f_i\}_{i=1}^{\infty}$ is a K-frame associated to $(a_2, \cdots, a_n)$ for X. Then for each $f \in X_F$, there exist constants $A, B > 0$ such that

$$A \|K^* f, a_2, \cdots, a_n\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n \rangle|^2 \leq B \|f, a_2, \cdots, a_n\|^2.$$

Since $R(T) \subset R(K)$, by Theorem (2.1), there exists $\lambda > 0$ such that $T T^* \leq \lambda^2 K K^*$. Thus,

$$\frac{A}{\lambda^2} \|T^* f, a_2, \cdots, a_n\|^2 = \frac{A}{\lambda^2} \langle T T^* f, f \,|\, a_2, \cdots, a_n \rangle$$

$$= \left\langle \frac{A}{\lambda^2} T T^* f, f \,|\, a_2, \cdots, a_n \right\rangle \leq \langle A K K^* f, f \,|\, a_2, \cdots, a_n \rangle$$

$$= A \|K^* f, a_2, \cdots, a_n\|^2.$$

Therefore, for each $f \in X_F$,

$$\frac{A}{\lambda^2} \|T^* f, a_2, \cdots, a_n\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \,|\, a_2, \cdots, a_n \rangle|^2 \leq B \|f, a_2, \cdots, a_n\|^2.$$

Hence, $\{f_i\}_{i=1}^{\infty}$ is a T-frame associated to $(a_2, \cdots, a_n)$ for X. $\qquad \square$

Theorem 3.4. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a K-frame associated to* $(a_2, \cdots, a_n)$ *for* X *with bounds* A, B *and* $T \in \mathcal{B}(X_F)$ *be an invertible with* $T K = K T$, *then* $\{T f_i\}_{i=1}^{\infty}$ *is a K-frame associated to* $(a_2, \cdots, a_n)$ *for* X.

Proof. Since T is invertible, for each $f \in X_F$,

$$\|K^* f, a_2, \cdots, a_n\|^2 = \left\|(T^{-1})^* T^* K^* f, a_2, \cdots, a_n\right\|^2$$

$$\leq \left\|(T^{-1})^*\right\|^2 \|T^* K^* f, a_2, \cdots, a_n\|^2.$$

(3.1) $\Rightarrow \left\| T^{-1} \right\|^{-2} \left\| K^* f, a_2, \cdots, a_n \right\|^2 \leq \left\| T^* K^* f, a_2, \cdots, a_n \right\|^2.$

Also, since $\{ f_i \}_{i=1}^{\infty}$ is a K-frame associated to $(a_2, \cdots, a_n)$, for $f \in X_F,$

$$\sum_{i=1}^{\infty} |\langle f, T f_i | a_2, \cdots, a_n \rangle|^2 = \sum_{i=1}^{\infty} |\langle T^* f, f_i | a_2, \cdots, a_n \rangle|^2$$

$$\geq A \left\| K^* T^* f, a_2, \cdots, a_n \right\|^2 = A \left\| T^* K^* f, a_2, \cdots, a_n \right\|^2 \ [\text{ since } TK = KT]$$

$$\geq A \left\| T^{-1} \right\|^{-2} \left\| K^* f, a_2, \cdots, a_n \right\|^2 \ [\text{ using (3.1)}].$$

On the other hand, for all $f \in X_F,$

$$\sum_{i=1}^{\infty} |\langle f, T f_i | a_2, \cdots, a_n \rangle|^2 = \sum_{i=1}^{\infty} |\langle T^* f, f_i | a_2, \cdots, a_n \rangle|^2$$

$$\leq B \left\| T^* f, a_2, \cdots, a_n \right\|^2 \leq B \left\| T \right\|^2 \left\| f, a_2, \cdots, a_n \right\|^2.$$

Hence, $\{ T f_i \}_{i=1}^{\infty}$ is a K-frame associated to $(a_2, \cdots, a_n)$ for X. $\square$

Theorem 3.5. *Let $\{ f_i \}_{i=1}^{\infty}$ be a K-frame associated to $(a_2, \cdots, a_n)$ for X with bounds A, B and $T \in \mathcal{B}(X_F)$ such that $T T^* = I$ with $T K = K T$. Then $\{ T f_i \}_{i=1}^{\infty}$ is a K-frame associated to $(a_2, \cdots, a_n)$ for X.*

Proof. Since $T T^* = I$, for $f \in X_F$, $\left\| T^* f, a_2, \cdots, a_n \right\|^2 = \left\| f, a_2, \cdots, a_n \right\|^2$ and this implies that $\left\| T^* K^* f, a_2, \cdots, a_n \right\|^2 = \left\| K^* f, a_2, \cdots, a_n \right\|^2$. Also, since $\{ f_i \}_{i=1}^{\infty}$ is a K-frame associated to $(a_2, \cdots, a_n)$, for each $f \in X_F,$

$$\sum_{i=1}^{\infty} |\langle f, T f_i | a_2, \cdots, a_n \rangle|^2 = \sum_{i=1}^{\infty} |\langle T^* f, f_i | a_2, \cdots, a_n \rangle|^2$$

$$\geq A \left\| K^* T^* f, a_2, \cdots, a_n \right\|^2 = A \left\| T^* K^* f, a_2, \cdots, a_n \right\|^2$$

$$= A \left\| K^* f, a_2, \cdots, a_n \right\|^2.$$

Thus, we see that $\{ T f_i \}_{i=1}^{\infty}$ satisfies lower K-frame condition. Following the proof of the Theorem (3.4), it can be shown that it also satisfies upper K-frame condition and therefore it is a K-frame associated to $(a_2, \cdots, a_n)$ for X. $\square$

Theorem 3.6. *Let $\{ f_i \}_{i=1}^{\infty}$ be a sequence in X. Then $\{ f_i \}_{i=1}^{\infty}$ is a K-frame associated to $(a_2, \cdots, a_n)$ for X if and only if there exists a bounded linear operator $T : l^2(\mathbb{N}) \to X_F$ such that $f_i = T e_i$ and $\mathcal{R}(K) \subset \mathcal{R}(T)$, where $\{ e_i \}_{i=1}^{\infty}$ is an F-orthonormal basis for $l^2(\mathbb{N})$.*

Proof. First we suppose that $\{ f_i \}_{i=1}^{\infty}$ is a K-frame associated to $(a_2, \cdots, a_n)$. Then, for each $f \in X_F$, there exist constants $A, B > 0$ such that

$$A \left\| K^* f, a_2, \cdots, a_n \right\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \leq B \left\| f, a_2, \cdots, a_n \right\|^2.$$

Now, we consider the linear operator $L : X_F \to l^2(\mathbb{N})$ defined by

$$L(f) = \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n \rangle e_i \ \forall f \in X_F.$$

Since $\{e_i\}_{i=1}^{\infty}$ is an F-orthonormal basis for $l^2(\mathbb{N})$, we can write

$$\|L(f)\|_{l^2}^2 = \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \le B \|f, a_2, \cdots, a_n\|^2.$$

Thus, L is well-defined and bounded linear operator on X_F. So, the adjoint operator $L^* : l^2(\mathbb{N}) \to X_F$ exists and then for each $f \in X_F$, we get

$$\langle L^* e_i, f | a_2, \cdots, a_n \rangle = \langle e_i, L(f) | a_2, \cdots, a_n \rangle$$

$$= \left\langle e_i, \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n \rangle e_i | a_2, \cdots, a_n \right\rangle$$

$$= \overline{\langle f, f_i | a_2, \cdots, a_n \rangle} = \langle f_i, f | a_2, \cdots, a_n \rangle.$$

The above calculation shows that, $L^*(e_i) = f_i$. Also, from the definition (3.1), we get $A\|K^* f\|_F^2 \le \|L(f)\|_{l^2}^2$ and this implies that

$$\langle A K K^* f, f | a_2, \cdots, a_n \rangle \le \langle L^* L f, f | a_2, \cdots, a_n \rangle$$

$$\Rightarrow A K K^* \le T T^*, \text{ where } T = L^*$$

and hence from the Theorem (2.1), $\mathcal{R}(K) \subset \mathcal{R}(T)$.

Conversely, suppose that $T : l^2(\mathbb{N}) \to X_F$ be a bounded linear operator such that $f_i = T e_i$ and $\mathcal{R}(K) \subset \mathcal{R}(T)$. We have to show that $\{f_i\}_{i=1}^{\infty}$ is a K-frame associated to $(a_2, \cdots, a_n)$. Let $g \in l^2(\mathbb{N})$ then $g = \sum_{i=1}^{\infty} c_i e_i$, where $c_i = \langle g, e_i | a_2, \cdots, a_n \rangle$. Now, for all $g \in l^2(\mathbb{N})$, we have

$$\langle T^* f, g | a_2, \cdots, a_n \rangle = \left\langle T^* f, \sum_{i=1}^{\infty} c_i e_i | a_2, \cdots, a_n \right\rangle$$

$$= \sum_{i=1}^{\infty} \overline{c_i} \langle f, T e_i | a_2, \cdots, a_n \rangle = \sum_{i=1}^{\infty} \overline{c_i} \langle f, f_i | a_2, \cdots, a_n \rangle$$

$$= \sum_{i=1}^{\infty} \overline{\langle g, e_i | a_2, \cdots, a_n \rangle} \langle f, f_i | a_2, \cdots, a_n \rangle$$

$$= \sum_{i=1}^{\infty} \langle e_i, g | a_2, \cdots, a_n \rangle \langle f, f_i | a_2, \cdots, a_n \rangle$$

$$= \left\langle \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n \rangle e_i, g | a_2, \cdots, a_n \right\rangle.$$

$$\Rightarrow T^*(f) = \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n \rangle e_i \ \forall f \in X_F.$$

Thus, for all $f \in X_F$, $\sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2$

$$= \sum_{i=1}^{\infty} |\langle f, T e_i | a_2, \cdots, a_n \rangle|^2 = \sum_{i=1}^{\infty} |\langle T^* f, e_i | a_2, \cdots, a_n \rangle|^2$$

$$= \| T^* f, a_2, \cdots, a_n \|^2 \ [\text{ since } \{ e_i \}_{i=1}^{\infty} \text{ is an } F\text{-orthonormal basis }]$$

$$\leq \| T^* \|^2 \| f, a_2, \cdots, a_n \|^2 = \| T \|^2 \| f, a_2, \cdots, a_n \|^2$$

$$\Rightarrow \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \leq \| T \|^2 \| f, a_2, \cdots, a_n \|^2 \ \forall f \in X_F.$$

Thus, $\{ f_i \}_{i=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$. Since $\mathcal{R}(K) \subset \mathcal{R}(T)$, from Theorem (2.1), there exists $A > 0$ such that $A K K^* \leq T T^*$. Hence following the proof of the Theorem (3.3), for all $f \in X_F$

$$A \| K^* f, a_2, \cdots, a_n \|^2 \leq \| T^* f, a_2, \cdots, a_n \|^2 = \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2.$$

Hence, $\{ f_i \}_{i=1}^{\infty}$ is a K-frame associated to $(a_2, \cdots, a_n)$ for X. $\qquad\square$

Theorem 3.7. *Let $\{ f_i \}_{i=1}^{\infty}$ and $\{ g_i \}_{i=1}^{\infty}$ be K-frames associated to $(a_2, \cdots, a_n)$ for X with the corresponding pre frame operators T and L, respectively. If $T L^*$ and $L T^*$ are positive operators, then $\{ f_i + g_i \}_{i=1}^{\infty}$ is also a K-frame associated to $(a_2, \cdots, a_n)$ for X.*

Proof. Let $\{ f_i \}_{i=1}^{\infty}$ and $\{ g_i \}_{i=1}^{\infty}$ be two K-frames associated to $(a_2, \cdots, a_n)$ for X. Then by Theorem (3.6), there exist bounded linear operators T and L such that $T e_i = f_i$, $L e_i = g_i$ and $\mathcal{R}(K) \subset \mathcal{R}(T)$, $\mathcal{R}(K) \subset \mathcal{R}(L)$, where $\{ e_i \}_{i=1}^{\infty}$ is an F-orthonormal basis for $l^2(\mathbb{N})$. Now, we have $\mathcal{R}(K) \subset \mathcal{R}(T) + \mathcal{R}(L)$. By Theorem (2.2), $K K^* \leq \lambda^2 (T T^* + L L^*)$, for some $\lambda > 0$. Now, for $f \in X_F$,

$$\sum_{i=1}^{\infty} |\langle f, f_i + g_i | a_2, \cdots, a_n \rangle|^2 = \sum_{i=1}^{\infty} |\langle f, T e_i + L e_i | a_2, \cdots, a_n \rangle|^2$$

$$= \sum_{i=1}^{\infty} |\langle (T + L)^* f, e_i | a_2, \cdots, a_n \rangle|^2$$

$$= \| (T + L)^* f, a_2, \cdots, a_n \|^2 \ [\text{ since } \{ e_i \} \text{ is F-orthonormal }]$$

$$= \langle (T + L)^* f, (T + L)^* f | a_2, \cdots, a_n \rangle$$

$$= \langle T^* f + L^* f, T^* f + L^* f | a_2, \cdots, a_n \rangle$$

$$= \langle T^* f, T^* f | a_2, \cdots, a_n \rangle + \langle L^* f, T^* f | a_2, \cdots, a_n \rangle +$$

$$+ \langle T^* f, L^* f | a_2, \cdots, a_n \rangle + \langle L^* f, L^* f | a_2, \cdots, a_n \rangle$$

$$= \langle T T^* f, f | a_2, \cdots, a_n \rangle + \langle T L^* f, f | a_2, \cdots, a_n \rangle +$$

$$+ \langle L T^* f, f | a_2, \cdots, a_n \rangle + \langle L L^* f, f | a_2, \cdots, a_n \rangle$$

$$\geq \langle (T T^* + L L^*) f, f | a_2, \cdots, a_n \rangle \ [\text{ since } T L^*, L T^* \text{ are positive }]$$

$$\geq \frac{1}{\lambda^2} \langle K K^* f, f | a_2, \cdots, a_n \rangle \ [\text{ since } K K^* \leq \lambda^2 (T T^* + L L^*)]$$

$$= \frac{1}{\lambda^2} \langle K^* f, K^* f | a_2, \cdots, a_n \rangle = \frac{1}{\lambda^2} \| K^* f, a_2, \cdots, a_n \|^2.$$

Therefore, for each $f \in X_F$,

$$(3.2) \qquad \frac{1}{\lambda^2} \| K^* f, a_2, \cdots, a_n \|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_i + g_i | a_2, \cdots, a_n \rangle|^2.$$

On the other hand, using the Minkowski's inequality, for each $f \in X_F$, we have

$$\left(\sum_{i=1}^{\infty} |\langle f, f_i + g_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2}$$

$$\leq \left(\sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2} + \left(\sum_{i=1}^{\infty} |\langle f, g_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2}$$

$$\leq \sqrt{A} \, \| f, a_2, \cdots, a_n \| + \sqrt{B} \, \| f, a_2, \cdots, a_n \| = \left(\sqrt{A} + \sqrt{B} \right) \| f, a_2, \cdots, a_n \|.$$

This implies that

$$(3.3) \qquad \sum_{i=1}^{\infty} |\langle f, f_i + g_i | a_2, \cdots, a_n \rangle|^2 \leq \left(\sqrt{A} + \sqrt{B} \right)^2 \| f, a_2, \cdots, a_n \|^2.$$

From (3.2) and (3.3), $\{ f_i + g_i \}_{i=1}^{\infty}$ is a K-frame associated to $(a_2, \cdots, a_n)$ for X. $\square$

Theorem 3.8. *Let* $\{ f_i \}_{i=1}^{\infty}$ *be K-frame associated to* $(a_2, \cdots, a_n)$ *for* X *and* $U :$ $X_F \rightarrow X_F$ *be a positive operator. Then* $\{ f_i + U f_i \}_{i=1}^{\infty}$ *is also a K-frame associated to* $(a_2, \cdots, a_n)$ *for* X.

Proof. Let $\{ f_i \}_{i=1}^{\infty}$ be K-frame associated to $(a_2, \cdots, a_n)$ for X with frame operator S_F. Then for each $f \in X_F$, there exist $A, B > 0$ such that

$$A \, \| K^* f, a_2, \cdots, a_n \|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \leq B \, \| f, a_2, \cdots, a_n \|^2.$$

It is easy to verify that $\langle S_F f, f | a_2, \cdots, a_n \rangle = \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2$. Thus,

$$A \, \| K^* f, a_2, \cdots, a_n \|^2 \leq \langle S_F f, f | a_2, \cdots, a_n \rangle \leq B \, \| f, a_2, \cdots, a_n \|^2.$$

This implies that $A K K^* \leq S_F \leq B I$. Now, for each $f \in X_F$,

$$\sum_{i=1}^{\infty} \langle f, f_i + U f_i | a_2, \cdots, a_n \rangle \, (f_i + U f_i)$$

$$= \sum_{i=1}^{\infty} \langle f, (I + U) f_i | a_2, \cdots, a_n \rangle \, (I + A) f_i$$

$$= (I + U) \sum_{i=1}^{\infty} \langle (I + U)^* f, f_i | a_2, \cdots, a_n \rangle \, f_i = (I + U) S_F (I + U)^* f.$$

This shows that the corresponding frame operator for $\{ f_i + U f_i \}_{i=1}^{\infty}$ is $(I + U) S_F (I + U)^*$. Since S_F, U are positive operators, $(I + U) S_F (I + U)^* \geq S_F \geq A K K^*$, by Theorem (2.8), $\{ f_i + U f_i \}_{i=1}^{\infty}$ is a K-frame associated to $(a_2, \cdots, a_n)$ for X. $\square$

Theorem 3.9. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a K-frames associated to* $(a_2, \cdots, a_n)$ *for* H_1 *with bounds* A *and* B. *If there exists a constant* $M > 0$ *such that*

$$\sum_{i=1}^{\infty} |\langle f, f_i - g_i | a_2, \cdots, a_n \rangle|^2$$

(3.4) $$\leq M \min \left\{ \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2, \sum_{i=1}^{\infty} |\langle f, g_i | a_2, \cdots, a_n \rangle|^2 \right\},$$

for all $f \in H_F$. *Then* $\{g_i\}_{i=1}^{\infty}$ *is a K-frames associated to* $(a_2, \cdots, a_n)$ *for* H_1.

Proof. For each $f \in H_F$, we have

$$\left(\sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2} = \| \{ \langle f, f_i | a_2, \cdots, a_n \rangle \} \|$$

$$\leq \| \{ \langle f, f_i - g_i | a_2, \cdots, a_n \rangle \} \| + \| \{ \langle f, g_i | a_2, \cdots, a_n \rangle \} \|$$

$$\leq \left(\sqrt{M} + 1 \right) \| \{ \langle f, g_i | a_2, \cdots, a_n \rangle \} \|$$

$$\Rightarrow \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \leq \left(\sqrt{M} + 1 \right)^2 \sum_{i=1}^{\infty} |\langle f, g_i | a_2, \cdots, a_n \rangle|^2$$

$$\Rightarrow \sum_{i=1}^{\infty} |\langle f, g_i | a_2, \cdots, a_n \rangle|^2 \geq \frac{A}{\left(\sqrt{M} + 1 \right)^2} \| K^* f, a_2, \cdots, a_n \|^2.$$

On the other hand, for each $f \in H_F$, we have

$$\left(\sum_{i=1}^{\infty} |\langle f, g_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2}$$

$$\leq \left(\sum_{i=1}^{\infty} |\langle f, f_i - g_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2} + \left(\sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2}$$

$$\leq \left(\sqrt{M} + 1 \right) \left(\sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2}$$

$$\Rightarrow \sum_{i=1}^{\infty} |\langle f, g_i | a_2, \cdots, a_n \rangle|^2 \leq B \left(\sqrt{M} + 1 \right)^2 \| f, a_2, \cdots, a_n \|^2.$$

Thus, $\{g_i\}_{i=1}^{\infty}$ is a K-frames associated to $(a_2, \cdots, a_n)$ for H_1.

Converse part: Suppose $\{g_i\}_{i=1}^{\infty}$ is a frames associated to $(a_2, \cdots, a_n)$ for H_1 with bounds C and D. Now, for each $f \in H_F$, we have

$$
\left(\sum_{i=1}^{\infty} |\langle f, f_i - g_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2}
$$

$$
\leq \left(\sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2} + \left(\sum_{i=1}^{\infty} |\langle f, g_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2}
$$

$$
\leq \left(\sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2} + \sqrt{D} \, \| f, a_2, \cdots, a_n \|
$$

$$
\leq \left(1 + \frac{\sqrt{D}}{\sqrt{A}} \right) \left(\sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2}.
$$

On the other hand, for each $f \in H_F$, we have

$$
\left(\sum_{i=1}^{\infty} |\langle f, f_i - g_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2}
$$

$$
\leq \left(\sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2} + \left(\sum_{i=1}^{\infty} |\langle f, g_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2}
$$

$$
\leq \sqrt{B} \, \| f, a_2, \cdots, a_n \| + \left(\sum_{i=1}^{\infty} |\langle f, g_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2}
$$

$$
\leq \left(1 + \frac{\sqrt{B}}{\sqrt{C}} \right) \left(\sum_{i=1}^{\infty} |\langle f, g_i | a_2, \cdots, a_n \rangle|^2 \right)^{1/2}.
$$

Now, if we take

$$
M = \min \left\{ \left(1 + \frac{\sqrt{B}}{\sqrt{C}} \right), \left(1 + \frac{\sqrt{D}}{\sqrt{A}} \right) \right\},
$$

then the condition (3.4) holds. This completes the proof. $\qquad\square$

Theorem 3.10. *Let* $\{f_i\}_{i=1}^{\infty}$ *be* K_1-*frame and* K_2-*frame associated to* $(a_2, \cdots, a_n)$ *for* H_1. *Then for any scalars* α, β, $\{f_i\}_{i=1}^{\infty}$ *is* $\alpha K_1 + \beta K_2$-*frame and* $K_1 K_2$-*frame associated to* $(a_2, \cdots, a_n)$ *for* H_1.

Proof. Since $\{f_i\}_{i=1}^{\infty}$ be an atomic system associated to $(a_2, \cdots, a_n)$ for K_1 and K_2 in H_1, there exist positive constants $A_n, B_n > 0$, $n = 1, 2$ such that

$$
A_n \, \| K_n^* f, a_2, \cdots, a_n \|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2
$$

(3.5)
$$
\leq B_n \, \| f, a_2, \cdots, a_n \|^2,
$$

for all $f \in H_F$. Now, for each $f \in H_F$, we have

$$\| K_1^* f, a_2, \cdots, a_n \|^2 = \frac{1}{|\alpha|^2} \| \alpha K_1^* f, a_2, \cdots, a_n \|^2$$

$$= \frac{1}{|\alpha|^2} \| (\alpha K_1^* + \beta K_2^*) f - \beta K_2^* f, a_2, \cdots, a_n \|^2$$

$$\geq \frac{1}{|\alpha|^2} \left(\| (\alpha K_1^* + \beta K_2^*) f, a_2, \cdots, a_n \|^2 - \| \beta K_2^* f, a_2, \cdots, a_n \|^2 \right).$$

This implies that

$$\| (\alpha K_1^* + \beta K_2^*) f, a_2, \cdots, a_n \|^2$$

$$\leq |\alpha|^2 \| K_1^* f, a_2, \cdots, a_n \|^2 + |\beta|^2 \| K_2^* f, a_2, \cdots, a_n \|^2$$

$$\leq \max \left\{ |\alpha|^2, |\beta|^2 \right\} \left\{ \| K_1^* f, a_2, \cdots, a_n \|^2 + \| K_2^* f, a_2, \cdots, a_n \|^2 \right\}.$$

Thus, for each $f \in H_F$, we have

$$\frac{1}{\max \left\{ |\alpha|^2, |\beta|^2 \right\}} \| (\alpha K_1^* + \beta K_2^*) f, a_2, \cdots, a_n \|^2$$

$$\leq \left(\frac{1}{A_1} + \frac{1}{A_2} \right) \sum_{i=1}^{\infty} | \langle f, f_i | a_2, \cdots, a_n \rangle |^2.$$

On the other hand, from 3.5, we get

$$\sum_{i=1}^{\infty} | \langle f, f_i | a_2, \cdots, a_n \rangle |^2 \leq \frac{B_1 + B_2}{2} \| f, a_2, \cdots, a_n \|^2 \ \forall f \in H_F.$$

Hence, $\{ f_i \}_{i=1}^{\infty}$ is a $\alpha K_1 + \beta K_2$-frame associated to $(a_2, \cdots, a_n)$ for H_1.
Now, for each $f \in H_F$, we have

$$\| (K_1 K_2)^* f, a_2, \cdots, a_n \|^2 = \| K_2^* K_1^* f, a_2, \cdots, a_n \|^2$$

$$\leq \| K_2^* \|^2 \| K_1^* f, a_2, \cdots, a_n \|^2.$$

Thus, for each $f \in H_F$, we have

$$\frac{A_1}{\| K_2 \|^2} \| (K_1 K_2)^* f, a_2, \cdots, a_n \|^2 \leq A_1 \| K_1^* f, a_2, \cdots, a_n \|^2$$

$$\leq \sum_{i=1}^{\infty} | \langle f, f_i | a_2, \cdots, a_n \rangle |^2 \leq B_1 \| f, a_2, \cdots, a_n \|^2.$$

Therefore, $\{ f_i \}_{i=1}^{\infty}$ is a $K_1 K_2$-frame associated to $(a_2, \cdots, a_n)$ for H_1. This completes the proof. $\qquad \square$

Theorem 3.11. *Let $K_1 \in \mathcal{B}(H_F)$ and $\{ f_i \}_{i=1}^{\infty}$ be a K_1-frame associated to $(a_2, \cdots, a_n)$ for H_1. Suppose $K_2 \in \mathcal{B}(H_F)$ and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_2 < 1$ such that*

$$\| (K_1^* - K_2^*) f, a_2, \cdots, a_n \| \leq \lambda_1 \| K_1^* f, a_2, \cdots, a_n \| +$$

$$+ \lambda_2 \| K_2^* f, a_2, \cdots, a_n \|,$$

for all $f \in H_F$. Then $\{ f_i \}_{i=1}^{\infty}$ is a K_2-frame associated to $(a_2, \cdots, a_n)$ for H_1.

Proof. Let $\{f_i\}_{i=1}^{\infty}$ be a K_1-frame associated to $(a_2, \cdots, a_n)$ for H_1 with bounds A and B. Now, for each $f \in H_F$, we get

$$\|K_2^* f, a_2, \cdots, a_n\| \leq \|(K_1^* - K_2^*) f, a_2, \cdots, a_n\| + \|K_1^* f, a_2, \cdots, a_n\|$$
$$\leq (1 + \lambda_1) \|K_1^* f, a_2, \cdots, a_n\| + \lambda_2 \|K_2^* f, a_2, \cdots, a_n\|$$
$$\Rightarrow (1 - \lambda_2) \|K_2^* f, a_2, \cdots, a_n\| \leq (1 + \lambda_1) \|K_1^* f, a_2, \cdots, a_n\|.$$

Thus, for each $f \in H_F$, we have

$$A \left(\frac{1 - \lambda_2}{1 + \lambda_1}\right)^2 \|K_2^* f, a_2, \cdots, a_n\| \leq A \|K_1^* f, a_2, \cdots, a_n\|$$

$$\leq \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 \leq B \|f, a_2, \cdots, a_n\|^2.$$

Hence, $\{f_i\}_{i=1}^{\infty}$ is a K_2-frame associated to $(a_2, \cdots, a_n)$ for H_1. This completes the proof. $\square$

Corollary 3.12. *Let $K_1, K_2 \in \mathcal{B}(H_F)$ and $0 \leq \lambda_1, \lambda_2 < 1$ such that*

$$\|(K_1^* - K_2^*) f, a_2, \cdots, a_n\| \leq \lambda_1 \|K_1^* f, a_2, \cdots, a_n\| +$$
$$+ \lambda_2 \|K_2^* f, a_2, \cdots, a_n\|,$$

for all $f \in H_F$. Then $\{f_i\}_{i=1}^{\infty}$ is a K_1-frame associated to $(a_2, \cdots, a_n)$ for H_1 if and only if it is a K_2-frame associated to $(a_2, \cdots, a_n)$ for H_1.

Proof. Proof of this Corollary is directly follows from the proof of the Theorem 3.11 by interchanging the role of K_1 and K_2. $\square$

Corollary 3.13. *Let $K \in \mathcal{B}(H_F)$ and $\{f_i\}_{i=1}^{\infty}$ be a K-frame associated to $(a_2, \cdots, a_n)$ for H_1. Suppose $0 \leq \lambda_1, \lambda_2 < 1$ such that*

$$\|K^* f - f, a_2, \cdots, a_n\| \leq \lambda_1 \|K^* f, a_2, \cdots, a_n\| + \lambda_2 \|f, a_2, \cdots, a_n\|,$$

for all $f \in H_F$. Then $\{f_i\}_{i=1}^{\infty}$ is a frame associated to $(a_2, \cdots, a_n)$ for H_1.

Proof. Proof of this Theorem is directly follows from the Theorem 3.11 by putting $K_1 = K$ and $K_2 = I_F$. $\square$

4. Tight K-frame and its properties in n-Hilbert space

Definition 4.1. A sequence $\{f_i\}_{i=1}^{\infty}$ in X is said to be a tight K-frame associated to $(a_2, \cdots, a_n)$ for X if there exist constants $0 < A \leq B < \infty$ such that

$$(4.1) \qquad \sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 = A \|K^* f, a_2, \cdots, a_n\|^2 \ \forall f \in X_F.$$

If $A = 1$, then $\{f_i\}_{i=1}^{\infty}$ is called Parseval K-frame associated to $(a_2, \cdots, a_n)$ for X.

Remark 4.2. From (4.1), we can write

$$\sum_{i=1}^{\infty} \left|\left\langle \frac{1}{\sqrt{A}} f, f_i | a_2, \cdots, a_n \right\rangle\right|^2 = \|K^* f, a_2, \cdots, a_n\|^2.$$

Therefore, if $\{f_i\}_{i=1}^{\infty}$ is a tight K-frame associated to $(a_2, \cdots, a_n)$ with bound A then the family $\left\{\dfrac{1}{\sqrt{A}}f_i\right\}_{i=1}^{\infty}$ is a Parseval K-frame associated to $(a_2, \cdots, a_n)$ for X.

Proposition 4.3. *Let $\{f_i\}_{i=1}^{\infty}$ be a tight frame associated to $(a_2, \cdots, a_n)$ for X with bound A and $K \in \mathcal{B}(X_F)$, then $\{Kf_i\}_{i=1}^{\infty}$ is a tight K-frame associated to $(a_2, \cdots, a_n)$ for X with bound A.*

Proof. Since $\{f_i\}_{i=1}^{\infty}$ is a tight frame associated to $(a_2, \cdots, a_n)$ for X with bound A, for any $f \in X_F$, we have

$$\sum_{i=1}^{\infty} |\langle f, Kf_i | a_2, \cdots, a_n \rangle|^2 = \sum_{i=1}^{\infty} |\langle K^*f, f_i | a_2, \cdots, a_n \rangle|^2$$

$$= A\,\|K^*f, a_2, \cdots, a_n\|^2$$

and hence $\{Kf_i\}_{i=1}^{\infty}$ is a tight K-frame associated to $(a_2, \cdots, a_n)$ for X with bound A. $\qquad\square$

Proposition 4.4. *Let $K, T \in \mathcal{B}(X_F)$ and $\{f_i\}_{i=1}^{\infty}$ be a tight K-frame associated to $(a_2, \cdots, a_n)$ for X with bound A. Then $\{Tf_i\}_{i=1}^{\infty}$ is a tight TK-frame associated to $(a_2, \cdots, a_n)$ for X with bound A.*

Proof. Since $\{f_i\}_{i=1}^{\infty}$ is a tight K-frame associated to $(a_2, \cdots, a_n)$ for X with bound A, for any $f \in X_F$, we have

$$\sum_{i=1}^{\infty} |\langle f, Tf_i | a_2, \cdots, a_n \rangle|^2 = \sum_{i=1}^{\infty} |\langle T^*f, f_i | a_2, \cdots, a_n \rangle|^2$$

$$= A\,\|K^*(T^*f), a_2, \cdots, a_n\|^2 = A\,\|(TK)^*f, a_2, \cdots, a_n\|^2$$

and hence $\{Tf_i\}_{i=1}^{\infty}$ is a tight TK-frame associated to $(a_2, \cdots, a_n)$ for X with bound A. $\qquad\square$

Theorem 4.5. *Let $\{f_i\}_{i=1}^{\infty}$ be a tight K-frame associated to $(a_2, \cdots, a_n)$ for X with bound A. Then there exists a Bessel sequence $\{g_i\}_{i=1}^{\infty}$ associated to $(a_2, \cdots, a_n)$ with bound B such that for all $f \in X_F$,*

$$K(f) = \sum_{i=1}^{\infty} \langle f, g_i | a_2, \cdots, a_n \rangle\, f_i \ \text{ and } \ AB \geq 1.$$

Proof. Let $\{f_i\}_{i=1}^{\infty}$ be a tight K-frame associated to $(a_2, \cdots, a_n)$ for X with bound A. Then by Theorem (2.4) of [3], there exists a Bessel sequence $\{g_i\}_{i=1}^{\infty}$ associated to $(a_2, \cdots, a_n)$ with bound B such that

$$K(f) = \sum_{i=1}^{\infty} \langle f, g_i | a_2, \cdots, a_n \rangle\, f_i \ \text{ and }$$

$$K^*(f) = \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n \rangle\, g_i \ \forall\, f \in X_F.$$

Since $\{f_i\}_{i=1}^{\infty}$ is a tight K-frame associated to $(a_2, \cdots, a_n)$ for X, we have

$$\sum_{i=1}^{\infty} |\langle f, f_i | a_2, \cdots, a_n \rangle|^2 = A\,\|K^*f, a_2, \cdots, a_n\|^2$$

$$= A \left\| \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n \rangle g_i, a_2, \cdots, a_n \right\|^2$$

$$= A \sup_{\| g, a_2, \cdots, a_n \| = 1} \left\{ \left\| \sum_{i=1}^{\infty} \langle \langle f, f_i | a_2, \cdots, a_n \rangle g_i, g | a_2, \cdots, a_n \rangle \right\|^2 \right\}$$

$$= A \sup_{\| g, a_2, \cdots, a_n \| = 1} \left\{ \left\| \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n \rangle \langle g_i, g | a_2, \cdots, a_n \rangle \right\|^2 \right\}$$

$$\leq A \sup_{\| g, a_2, \cdots, a_n \| = 1} \left\{ \sum_{i=1}^{\infty} | \langle f, f_i | a_2, \cdots, a_n \rangle |^2 \sum_{i=1}^{\infty} | \langle g_i, g | a_2, \cdots, a_n \rangle |^2 \right\}$$

$$\leq A \sup_{\| g, a_2, \cdots, a_n \| = 1} \left\{ \sum_{i=1}^{\infty} | \langle f, f_i | a_2, \cdots, a_n \rangle |^2 \, B \, \| g, a_2, \cdots, a_n \|^2 \right\}$$

[since $\{ g_i \}_{i=1}^{\infty}$ is a Bessel sequence associated to $(a_2, \cdots, a_n)$ with bound B]

$$= A B \sum_{i=1}^{\infty} | \langle f, f_i | a_2, \cdots, a_n \rangle |^2.$$

The above calculation shows that $AB \geq 1$. $\qquad\square$

Theorem 4.6. *Let $\{ f_i \}_{i=1}^{\infty}$ and $\{ g_i \}_{i=1}^{\infty}$ be two Parseval K-frame associated to $(a_2, \cdots, a_n)$ for X with the corresponding synthesis operators T and L, respectively. If $T L^* = \theta$, where θ is the null operator on X_F then $\{ f_i + g_i \}_{i=1}^{\infty}$ is a tight K-frame associated to $(a_2, \cdots, a_n)$ with frame bound 2.*

Proof. Let $\{ f_i \}_{i=1}^{\infty}$ and $\{ g_i \}_{i=1}^{\infty}$ be Parseval K-frames associated to $(a_2, \cdots, a_n)$ for X. Then by Theorem (3.6), there exist synthesis operators T and L such that $T e_i = f_i$, $L e_i = g_i$ with $\mathcal{R}(K) \subset \mathcal{R}(T)$, $\mathcal{R}(K) \subset \mathcal{R}(L)$ respectively, where $\{ e_i \}_{i=1}^{\infty}$ is a F-orthonormal basis for $l^2(\mathbb{N})$. Now, for $f \in X_F$,

$$T^* : X_F \to l^2(\mathbb{N}), \ T^*(f) = \sum_{i=1}^{\infty} \langle f, f_i | a_2, \cdots, a_n \rangle e_i, \ \text{and}$$

$$L^* : X_F \to l^2(\mathbb{N}), \ L^*(f) = \sum_{i=1}^{\infty} \langle f, g_i | a_2, \cdots, a_n \rangle e_i.$$

Now from the definition of Parseval K-frame associated to $(a_2, \cdots, a_n)$,

$$(4.2) \qquad \| K^* f, a_2, \cdots, a_n \|^2 = \sum_{i=1}^{\infty} | \langle f, f_i | a_2, \cdots, a_n \rangle |^2 = \| T^* f, a_2, \cdots, a_n \|^2$$

$$(4.3) \qquad \| K^* f, a_2, \cdots, a_n \|^2 = \sum_{i=1}^{\infty} | \langle f, f_i | a_2, \cdots, a_n \rangle |^2 = \| L^* f, a_2, \cdots, a_n \|^2.$$

Following the proof of the Theorem (3.7), it can be shown that for each $f \in X_F$,

$$\sum_{i=1}^{\infty} | \langle f, f_i + g_i | a_2, \cdots, a_n \rangle |^2 = \sum_{i=1}^{\infty} | \langle f, (T + L) e_i | a_2, \cdots, a_n \rangle |^2$$

$$= \| \, (\, T \, + \, L \,)^{\, *} \, f, \, a_2, \, \cdots, \, a_n \, \|^{\, 2}$$
$$= \langle \, T \, T^{\, *} f, \, f \, | \, a_2, \, \cdots, \, a_n \, \rangle + \langle \, T \, L^{\, *} f, \, f \, | \, a_2, \, \cdots, \, a_n \, \rangle + \langle \, L \, T^{\, *} f, \, f \, | \, a_2, \, \cdots, \, a_n \, \rangle$$
$$+ \, \langle \, L \, L^{\, *} f, \, f \, | \, a_2, \, \cdots, \, a_n \, \rangle$$
$$= \langle \, T \, T^{\, *} f, \, f \, | \, a_2, \, \cdots, \, a_n \, \rangle + \langle \, L \, L^{\, *} f, \, f \, | \, a_2, \, \cdots, \, a_n \, \rangle \; [\text{ since } T \, L^{\, *} \, = \, \theta \, = \, L \, T^{\, *} \,]$$
$$= \langle \, T^{\, *} f, \, T^{\, *} f \, | \, a_2, \, \cdots, \, a_n \, \rangle + \langle \, L^{\, *} f, \, L^{\, *} f \, | \, a_2, \, \cdots, \, a_n \, \rangle$$
$$= \| \, T^{\, *} f, \, a_2, \, \cdots, \, a_n \, \|^{\, 2} + \| \, L^{\, *} f, \, a_2, \, \cdots, \, a_n \, \|^{\, 2}$$
$$= \| \, K^{\, *} f, \, a_2, \, \cdots, \, a_n \, \|^{\, 2} + \| \, K^{\, *} f, \, a_2, \, \cdots, \, a_n \, \|^{\, 2} \; [\text{ using } (4.2) \text{ and } (4.3) \,]$$
$$= 2 \, \| \, K^{\, *} f, \, a_2, \, \cdots, \, a_n \, \|^{\, 2}.$$

Hence, $\{ \, f_i \, + \, g_i \, \}_{i=1}^{\infty}$ is a tight K-frame associated to $(\, a_2, \, \cdots, \, a_n \,)$ for X with bound 2. $\qquad \square$

Conclusion

In this paper, we have studied the notion of a K-frame associated to $(\, a_2, \, \cdots, \, a_n \,)$ for n-Hilbert space X and presented some of their characterizations. We have established stability condition of K-frame associated to $(\, a_2, \, \cdots, \, a_n \,)$ for X under some perturbations. We have verified that sum of two K-frames associated to $(\, a_2, \, \cdots, \, a_n \,)$ for X is also a K-frame associated to $(\, a_2, \, \cdots, \, a_n \,)$ for X. Also, the concept of tight K-frame associated to $(\, a_2, \, \cdots, \, a_n \,)$ for X is described and some properties of its are going to be established. Here we point out a few open problems for future work.

- $\{ \, U_1 \, f_i \, + \, U_2 \, f_i \, \}_{i=1}^{\infty}$ is K-frame associated to $(\, a_2, \, \cdots, \, a_n \,)$ for X, whether the operators U_1 and U_2 are positive or not?
- Let $\{ \, f_i \, \}_{i=1}^{\infty}$, $\{ \, g_i \, \}_{i=1}^{\infty}$ and $\{ \, h_i \, \}_{i=1}^{\infty}$ be K-frames associated to $(\, a_2, \, \cdots, \, a_n \,)$ for X whether $\{ \, f_i \, + \, g_i \, + \, h_i \, \}_{i=1}^{\infty}$ is K-frame associated to $(\, a_2, \, \cdots, \, a_n \,)$ for X or not?

Acknowledgments

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