Mathematical Analysis
& Convex Optimization

Vol. 4 (2023), No. 2, 1–6
https:\\ maco.lu.ac.ir
DOI: 10.22034/maco.4.2.1

Research Paper

DUALITY AND α-DUALITY OF G-FRAMES AND FUSION FRAMES

MORTEZA MIRZAEE AZANDARYANI AND MAHMOOD POURGHOLAMHOSSEIN*

Abstract. In this paper, we get some results about α-duals of g-frames and fusion frames in Hilbert spaces. Especially, the direct sums and tensor products for α-duals of g-frames and fusion frames are considered and some of the obtained results for duals are generalized to α-duals.

MSC(2010): 42C15.
Keywords: Hilbert space, g-frame, fusion frame, direct sum, tensor product, α-dual.

1. Introduction and preliminaries

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [5] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [4]. Many generalizations of frames have been introduced that one of the most important of them is g-frame introduced in [10].

Let H be a separable Hilbert space and let I be a finite or countable index set. A family $\mathcal{F} = \{f_i\}_{i\in I} \subseteq H$ is a *frame* for H, if there exist two positive numbers A and B such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2,$$

for each $f \in H$. A and B are the *lower* and *upper* frame bounds, respectively.

For each $i \in I$, let H_i be a Hilbert space. In this paper, $L(H, H_i)$ is the set of all bounded operators from H into H_i and $L(H, H)$ is denoted by $L(H)$.

Definition 1.1. We call $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ a *g-frame* for H with respect to $\{H_i : i \in I\}$ if there exist two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2,$$

for each $f \in H$. If only the second inequality is required, we call it a *g-Bessel sequence* with upper bound B. If $A = B$, Λ is called an $A-$tight g-frame.

Note that

$$\oplus_{i\in I} H_i = \left\{ \{f_i\}_{i\in I} | f_i \in H_i, \|\{f_i\}_{i\in I}\|_2{}^2 = \sum_{i\in I} \|f_i\|^2 < \infty \right\}$$

Date: Received: October 17, 2023 , Accepted: December 10, 2023.
*Corresponding author.

with pointwise operations and the inner product defined by

$$\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle$$

is a Hilbert space. If $H_i = H$ for each $i \in I$, we denote $\oplus_{i\in I}H_i$ by $\ell^2(I, H)$.
For a g-Bessel sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ the *synthesis operator* is $T_\Lambda : \oplus_{i\in I}H_i \longrightarrow H$, $T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I}\Lambda_i^* f_i$ and its adjoint operator which is $T_\Lambda^*(f) = \{\Lambda_i f\}_{i\in I}$ is called the *analysis operator* of Λ. The operator S_Λ is defined by $S_\Lambda = T_\Lambda T_\Lambda^*$. If Λ is a g-frame, then S_Λ is invertible. The *canonical g-dual* for Λ is defined by $\widetilde{\Lambda} = \{\tilde{\Lambda}_i\}_{i\in I}$ where $\tilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$ which is a g-frame and for each $f \in H$, we have

$$f = \sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i f = \sum_{i\in I}\tilde{\Lambda}_i^{\,*}\Lambda_i f.$$

Also a g-Bessel sequence $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ is called an *alternate g-dual* or a *g-dual* for a g-Bessel sequence Λ if

$$f = \sum_{i\in I}\Gamma_i^*\Lambda_i f = \sum_{i\in I}\Lambda_i^*\Gamma_i f,$$

for each $f \in H$.

Another important generalization of frames is the fusion frame introduced in [3].

Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of a Hilbert space H, and $\{\omega_i\}_{i\in I}$ be a family of weights, i.e., $\omega_i > 0$ for each $i \in I$. Then $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$ is a *fusion frame*, if there are two positive numbers A and B such that for each $f \in H$,

$$A\|f\|^2 \le \sum_{i\in I}\omega_i^2\|\pi_{W_i}(f)\|^2 \le B\|f\|^2,$$

where π_{W_i} is the orthogonal projection onto the subspace W_i. If only the right-hand inequality is required, then $\mathcal{W}$ is called a *Bessel fusion sequence*. If $A = B$, then $\mathcal{W}$ is called a *tight fusion frame*.

It is easy to see that if $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$ is a Bessel fusion sequence, then the operator $S_\mathcal{W}$ defined on H by $S_\mathcal{W} f = \sum_{i\in I}\omega_i^2\pi_{W_i} f$ is well-defined, bounded and positive. Also, if $\mathcal{W}$ is a fusion frame, then $S_\mathcal{W}$ is invertible.

Let $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$ and $\mathcal{V} = \{(V_i, \upsilon_i)\}_{i\in I}$ be two Bessel fusion sequences. Then, $\mathcal{V}$ is called a dual of $\mathcal{W}$ if $\sum_{i\in I}\upsilon_i\omega_i\pi_{W_i}\pi_{V_i} f = f$, for each $f \in H$, see [6].

Note that $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame if and only if $\Lambda_\mathcal{W} := \{\omega_i\pi_{W_i}\}_{i\in I}$ is a g-frame.

Direct sums and tensor products of g-frames have been studied recently (see [1, 8, 9] and the references stated in these papers). Also, direct sums and tensor products of fusion frames in Hilbert spaces have been considered by some authors (for more information, see [7, 8] and the references stated therein).

In this note, we obtain some results for the tensor product and direct sum of α-duals for g-frames and fusion frames, mostly, we generalize the obtained results for duals in [8, 9] to α-duals.

2. Main Results

In this paper I, J and I_k, for each $1 \le k \le n$, are finite or countable index sets. H, H_j, H_k, H_{kj}, $H_{i(k)}$ and $H_{i(k)j}$ are separable Hilbert spaces for each $j \in J$, $k \in \{1, \dots, n\}$ and

$i(k) \in I_k$. $\Phi_j = \{\Lambda_{ij} \in L(H_j, H_{ij}) : i \in I\}$, $\Psi_j = \{\Gamma_{ij} \in L(H_j, H_{ij}) : i \in I\}$, $\Phi^{(k)} = \{\Lambda_{i(k)} \in L(H_k, H_{i(k)})\}_{i(k)\in I_k}$, $\Psi^{(k)} = \{\Gamma_{i(k)} \in L(H_k, H_{i(k)})\}_{i(k)\in I_k}$, $\otimes_{k=1}^n \Phi^{(k)}$ is

$$\{\Lambda_{i(1)} \otimes \ldots \otimes \Lambda_{i(n)} \in L(\otimes_{k=1}^n H_k, H_{i(1)} \otimes \ldots \otimes H_{i(n)})\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)},$$

and $\Phi_j^{(k)} = \{\Lambda_{i(k)j} \in (H_{kj}, H_{i(k)j})\}_{i(k)\in I_k}$.

Recall that if H_k is a Hilbert space for each $1 \le k \le n$, then the (Hilbert) tensor product $\otimes_{k=1}^n H_k = H_1 \otimes \ldots \otimes H_n$ is a Hilbert space. The inner product for simple tensors is defined by $\langle \otimes_{k=1}^n f_k, \otimes_{k=1}^n g_k\rangle = \Pi_{k=1}^n \langle f_k, g_k\rangle$, where $f_k, g_k \in H_k$. If U_k is a bounded linear operator on H_k, then the tensor product $\otimes_{k=1}^n U_k$ is a bounded linear operator on $\otimes_{k=1}^n H_k$. Also $(\otimes_{k=1}^n U_k)^* = \otimes_{k=1}^n U_k^*$ and $\| \otimes_{k=1}^n U_k\| = \Pi_{k=1}^n \|U_k\|$.

Tensor products have important applications, for example tensor products are useful in the approximation of multi-variate functions of combinations of univariate ones.

We recall the following definition from [2].

Definition 2.1. Let $\alpha \in \mathbb{Z}$ and let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ be a g-frame. A g-frame $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ is called an α-dual of $\{\Lambda_i\}_{i\in I}$ if $\sum_{i\in I} \Lambda_i^* \Gamma_i f = S_\Lambda^\alpha f$, for each $f \in H$.

Example 2.2. (i) Since $\sum_{i\in I} \Lambda_i^* \Lambda_i S_\Lambda^{\alpha-1} f = S_\Lambda^\alpha f$, $\{\Lambda_i S_\Lambda^{\alpha-1}\}_{i\in I}$ is an α-dual of Λ.

(ii) If $\alpha = 0$, then $S_\Lambda^{\alpha-1} = S_\Lambda^{-1}$, so the canonical dual $(\{\Lambda_i S_\Lambda^{-1}\}_{i\in I})$ is a $0-$dual of Λ.

Now we get the following result for α-duals of g-frames.

Theorem 2.3. *Suppose that $\Phi^{(k)}$'s and $\Psi^{(k)}$'s are g-frames. If $\Psi^{(k)}$ is an α-dual of $\Phi^{(k)}$, for each $k \in \{1, \ldots, n\}$, then $\otimes_{k=1}^n \Psi^{(k)}$ is an α-dual of $\otimes_{k=1}^n \Phi^{(k)}$.*

Proof. Let A_k and B_k be bounds of $\Phi^{(k)}$. For each $1 \le k \le n$, we have

$$A_k.Id_{H_k} \le S_{\Phi^{(k)}} \le B_k.Id_{H_k},$$

so

$$(\Pi_{k=1}^n A_k).Id_{(\otimes_{k=1}^n H_k)} \le \otimes_{k=1}^n S_{\Phi^{(k)}} \le (\Pi_{k=1}^n B_k).Id_{(\otimes_{k=1}^n H_k)}.$$

Therefore, for each $z \in \otimes_{k=1}^n H_k$, we get

$$(\Pi_{k=1}^n A_k)\langle z, z\rangle = \langle \otimes_{k=1}^n S_{\Phi^{(k)}} z, z\rangle \le (\Pi_{k=1}^n B_k)\langle z, z\rangle$$

and since

$$\langle \otimes_{k=1}^n S_{\Phi^{(k)}} z, z\rangle = \sum_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)} \|(\Lambda_{i(1)} \otimes \ldots \otimes \Lambda_{i(n)})z\|^2, \tag{2.1}$$

we get $\otimes_{k=1}^n \Phi^{(k)}$ is a g-frame. Similarly, we obtain that $\otimes_{k=1}^n \Psi^{(k)}$ is a g-frame.
It is also obtained from (2.1) that $\otimes_{k=1}^n S_{\Phi^{(k)}} = S_{\otimes_{k=1}^n \Phi^{(k)}}$. Thus, for each $m \in \mathbb{N}$, we have

$$\otimes_{k=1}^n S_{\Phi^{(k)}}^m = (\otimes_{k=1}^n S_{\Phi^{(k)}})^m = S_{\otimes_{k=1}^n \Phi^{(k)}}^m,$$

and

$$\otimes_{k=1}^n S_{\Phi^{(k)}}^{-1} = (\otimes_{k=1}^n S_{\Phi^{(k)}})^{-1} = S_{\otimes_{k=1}^n \Phi^{(k)}}^{-1},$$

so for each $\alpha \in \mathbb{Z}$, we have

$$\otimes_{k=1}^n S_{\Phi^{(k)}}^\alpha = (\otimes_{k=1}^n S_{\Phi^{(k)}})^\alpha = S_{\otimes_{k=1}^n \Phi^{(k)}}^\alpha.$$

Hence, for each $\otimes_{k=1}^{n} f_{i(k)} \in \otimes_{k=1}^{n} H_k$, we have

$$\begin{aligned}
&\sum_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)} (\Lambda_{i(1)}\otimes\ldots\otimes\Lambda_{i(n)})^*(\Gamma_{i(1)}\otimes\ldots\otimes\Gamma_{i(n)})(\otimes_{k=1}^{n} f_{i(k)})\\
=\ & \otimes_{k=1}^{n} S^{\alpha}_{\Phi^{(k)}}(\otimes_{k=1}^{n} f_{i(k)}) = (\otimes_{k=1}^{n} S_{\Phi^{(k)}})^{\alpha}(\otimes_{k=1}^{n} f_{i(k)})\\
=\ & S^{\alpha}_{\otimes_{k=1}^{n}\Phi^{(k)}}(\otimes_{k=1}^{n} f_{i(k)}).
\end{aligned}$$

This implies that $\otimes_{k=1}^{n}\Psi^{(k)}$ is an α-dual of $\otimes_{k=1}^{n}\Phi^{(k)}$. □

Corollary 2.4. *Suppose that $\Phi^{(k)}$'s are A_k-tight g-frames. If $\Psi^{(k)}$ is an α-dual of $\Phi^{(k)}$, for each $k \in \{1,\ldots,n\}$, then* $\{\frac{\Lambda_{i(1)}\otimes\ldots\otimes\Lambda_{i(n)}}{A_1^{\alpha}}\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)}$ *and* $\{\frac{\Gamma_{i(1)}\otimes\ldots\otimes\Gamma_{i(n)}}{(\Pi_{k=2}^{n}A_k)^{\alpha}}\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)}$ *are g-duals.*

In the rest of this note, $\mathcal{W}$ and $\mathcal{V}$ are supposed to be $\{(W_i,\omega_i)\}_{i\in I}$ and $\{(V_i,\upsilon_i)\}_{i\in I}$, respectively. Also, here, I, J and I_k, for each $1 \le k \le n$, are finite or countable index sets. H, H_j, $H_{i(k)}$ and $H_{i(k)j}$ are separable Hilbert spaces for each $j \in J$, $k \in \{1,\ldots,n\}$ and $i(k) \in I_k$. $\mathcal{W}_j = \{(W_{ij},\omega_i) : i \in I\}$, $\mathcal{V}_j = \{(V_{ij},\upsilon_i) : i \in I\}$, $\mathcal{W}^{(k)} = \{(W_{i(k)},\omega_{i(k)})\}_{i(k)\in I_k}$, $\mathcal{V}^{(k)} = \{(V_{i(k)},\upsilon_{i(k)})\}_{i(k)\in I_k}$, $\otimes_{k=1}^{n}\mathcal{W}^{(k)}$ is

$$\{(W_{i(1)}\otimes\ldots\otimes W_{i(n)}, \omega_{i(1)}\ldots\omega_{i(n)})\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)},$$

and $\mathcal{W}_j^{(k)} = \{(W_{i(k)j},\omega_{i(k)})\}_{i(k)\in I_k}$, where W_{ij}, V_{ij} are closed subspaces of H_j, $W_{i(k)}$ is a closed subspace of $H_{i(k)}$ and $W_{i(k)j}$ is a closed subspace of $H_{i(k)j}$. Note that if M_k is a closed subspace of H_k, for each $1 \le k \le n$, then it is easy to see that $\pi_{\otimes_{k=1}^{n}M_K} = \otimes_{k=1}^{n}\pi_{M_k}$.

The concept of α-duality can also be defined for fusion frames similar to g-frames.

Definition 2.5. Let $\alpha \in \mathbb{Z}$ and $\mathcal{W}$ and $\mathcal{V}$ be two fusion frames for H. Then, $\mathcal{V}$ is called an α-dual of $\mathcal{W}$ if $\sum_{i\in I}\upsilon_i\omega_i\pi_{W_i}\pi_{V_i}f = S^{\alpha}_{\mathcal{W}}f$, for each $f \in H$.

Example 2.6. (i) Since $\sum_{i\in I}\omega_i\omega_i\pi_{W_i}\pi_{W_i}f = S_{\mathcal{W}}f$, $\mathcal{W}$ is a 1-dual of itself.
(ii) If $\mathcal{V}$ is a dual of $\mathcal{W}$, then $\mathcal{V}$ is a 0-dual of $\mathcal{W}$.

Now, we get the following result for α-duals of fusion frames.

Proposition 2.7. *Suppose that $\mathcal{W}^{(k)}$'s and $\mathcal{V}^{(k)}$'s are fusion frames. If $\mathcal{V}^{(k)}$ is an α-dual of $\mathcal{W}^{(k)}$, for each $k \in \{1,\ldots,n\}$, then $\otimes_{k=1}^{n}\mathcal{V}^{(k)}$ is an α-dual of $\otimes_{k=1}^{n}\mathcal{W}^{(k)}$.*

Proof. The result follows from Theorem 2.3 and using the fact that $\Phi^{(k)} := \{\omega_{i(k)}\pi_{W_{i(k)}}\}_{i(k)\in I_k}$ is a g-frame for each $1 \le k \le n$ if and only if $\otimes_{k=1}^{n}\Phi^{(k)} = \{\omega_{i(1)}\ldots\omega_{i(n)}\pi_{(W_{i(1)}\otimes\ldots\otimes W_{i(n)})}\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)}$ is a g-frame. □

Corollary 2.8. *Suppose that $\mathcal{W}^{(k)}$'s are A_k-tight fusion frames. If $\mathcal{V}^{(k)}$ is an α-dual of $\mathcal{W}^{(k)}$, for each $k \in \{1,\ldots,n\}$, then*

$$\left\{\left(W_{i(1)}\otimes\ldots\otimes W_{i(n)}, \frac{\omega_{i(1)}\ldots\omega_{i(n)}}{A_1^{\alpha}}\right)\right\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)}$$

and

$$\left\{\left(V_{i(1)}\otimes\ldots\otimes V_{i(n)}, \frac{\upsilon_{i(1)}\ldots\upsilon_{i(n)}}{(\Pi_{k=2}^{n}A_k)^{\alpha}}\right)\right\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)}$$

are duals.

Let $\Phi_j = \{\Lambda_{ij} \in L(H_j, H_{ij}) : i \in I\}$ be a g-Bessel sequence for H_j, $j \in J$, with upper bound B_j such that $B := sup\{B_j : j \in J\} < \infty$. Then $\{\Phi_j\}_{j\in J}$ is called a *B-Bounded family of g-Bessel sequences* or shortly *B-BFGBS*.

Let $\Phi_j = \{\Lambda_{ij} \in L(H_j, H_{ij}) : i \in I\}$ be an (A_j, B_j) g-frame for H_j, $j \in J$, such that $A := inf\{A_j : j \in J\} > 0$ and $B := sup\{B_j : j \in J\} < \infty$. Then we say that $\{\Phi_j\}_{j\in J}$ is an *(A, B)-bounded family of g-frames* or shortly *(A, B)-BFGF*.

Theorem 2.9. *Let $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ be BFGF. If Ψ_j is an α-dual of Φ_j, for each $j \in J$, then $\oplus_{j\in J}\Psi_j := \{\oplus_{j\in J}\Gamma_{ij} : i \in I\}$ is an α-dual for $\oplus_{j\in J}\Phi_j := \{\oplus_{j\in J}\Lambda_{ij} : i \in I\}$.*

Proof. Suppose that $\{\Phi_j\}_{j\in J}$ is an (A, B)-BFGF. Then,

$$A.Id_{H_j} \leq S_{\Phi_j} \leq B.Id_{H_j},$$

for each $j \in J$, so

$$A.Id_{\oplus_{j\in J}H_j} \leq \oplus_{j\in J}S_{\Phi_j} \leq B.Id_{\oplus_{j\in J}H_j}.$$

Consequently, for every $f_J = \{f_j\}_{j\in J} \in \oplus_{j\in J}H_j$, we get

$$\begin{aligned} A\langle\{f_j\}_{j\in J}, \{f_j\}_{j\in J}\rangle &\leq \langle S_{\oplus_{j\in J}\Phi_j}(f_J), f_J\rangle \\ &= \sum_{i\in I}\sum_{j\in J}\|\Lambda_{ij}(f_j)\|^2 \leq B\langle\{f_j\}_{j\in J}, \{f_j\}_{j\in J}\rangle. \end{aligned}$$

Since

$$\sum_{i\in I}\|(\oplus_{j\in J}\Lambda_{ij})f_J\|^2 = \sum_{i\in I}\sum_{j\in J}\|\Lambda_{ij}(f_j)\|^2,$$

we obtain that $\oplus_{j\in J}\Phi_j$ is a g-frame for $\oplus_{j\in J}H_j$. Similarly, we can see that $\oplus_{j\in J}\Psi_j = \{\oplus_{j\in J}\Gamma_{ij} : i \in I\}$ is a g-frame. Also, we have

$$\begin{aligned} \langle S_{\oplus_{j\in J}\Phi_j}(f_J), f_J\rangle &= \sum_{i\in I}\sum_{j\in J}\|\Lambda_{ij}(f_j)\|^2 \\ &= \sum_{j\in J}\sum_{i\in I}\|\Lambda_{ij}(f_j)\|^2 = \langle(\oplus_{j\in J}S_{\Phi_j})f_J, f_J\rangle, \end{aligned}$$

therefore $S_{\oplus_{j\in J}\Phi_j} = \oplus_{j\in J}S_{\Phi_j}$. Now, it is easy to see that $S^n_{\oplus_{j\in J}\Phi_j} = \oplus_{j\in J}S^n_{\Phi_j}$, for each $n \in \mathbb{N}$ and $S^{-1}_{\oplus_{j\in J}\Phi_j} = \oplus_{j\in J}S^{-1}_{\Phi_j}$, so for each $\alpha \in \mathbb{Z}$, we have $S^\alpha_{\oplus_{j\in J}\Phi_j} = \oplus_{j\in J}S^\alpha_{\Phi_j}$. Also, for each $\{f_j\}_{j\in J} \in \oplus_{j\in J}H_j$, it is easy to see that

$$\begin{aligned} \sum_{i\in I}(\oplus_{j\in J}\Lambda_{ij})^*(\oplus_{j\in J}\Gamma_{ij})\{f_j\}_{j\in J} &= \left\{\sum_{i\in I}\Lambda^*_{ij}\Gamma_{ij}(f_j)\right\}_{j\in J} \\ &= \{S^\alpha_{\Phi_j}(f_j)\}_{j\in J} = (\oplus_{j\in J}S^\alpha_{\Phi_j})(\{f_j\}_{j\in J}) \\ &= S^\alpha_{\oplus_{j\in J}\Phi_j}(\{f_j\}_{j\in J}). \end{aligned}$$

This means that $\oplus_{j\in J}\Psi_j = \{\oplus_{j\in J}\Gamma_{ij} : i \in I\}$ is an α-dual for $\oplus_{j\in J}\Phi_j = \{\oplus_{j\in J}\Lambda_{ij} : i \in I\}$. $\square$

Corollary 2.10. *Let $\{\Phi^{(k)}_j\}_{j\in J}$ and $\{\Psi^{(k)}_j\}_{j\in J}$ be BFGF, for each $1 \leq k \leq n$ and let $\Phi^{(k)}_j$ be an α-dual of $\Psi^{(k)}_j$, for each $j \in J$ and $k \in \{1, \dots, n\}$. Then $\otimes^n_{k=1}(\oplus_{j\in J}\Phi^{(k)}_j)$ is an α-dual of $\otimes^n_{k=1}(\oplus_{j\in J}\Psi^{(k)}_j)$.*

Proof. The result follows from Theorems 2.9 and 2.3. $\square$

Let $\mathcal{W}_j = \{(W_{ij}, \omega_i) : i \in I\}$ be a Bessel fusion sequence for H_j, $j \in J$, with upper bound B_j such that $B := sup\{B_j : j \in J\} < \infty$. Then $\{\mathcal{W}_j\}_{j\in J}$ is called a *B-Bounded family of Bessel fusion sequences* or shortly *B-BFBFS*.

Let $\mathcal{W}_j = \{(W_{ij}, \omega_i) : i \in I\}$ be an (A_j, B_j) fusion frame for H_j, $j \in J$, such that $A := inf\{A_j : j \in J\} > 0$ and $B := sup\{B_j : j \in J\} < \infty$. Then, we say that $\{\mathcal{W}_j\}_{j\in J}$ is an (A, B)-*bounded family of fusion frames* or shortly (A, B)-*BFFF*.

The next theorem and corollary are immediate consequences of the results obtained for g-frames in Theorem 2.9 and Corollary 2.10, respectively.

Theorem 2.11. *Let $\{\mathcal{W}_j\}_{j\in J}$ and $\{\mathcal{V}_j\}_{j\in J}$ be BFFF. If $\mathcal{V}_j$ is an α-dual for $\mathcal{W}_j$, for each $j \in J$, then $\oplus_{j\in J}\mathcal{V}_j := \{(\oplus_{j\in J}V_{ij}, \upsilon_i) : i \in I\}$ is an α-dual for $\oplus_{j\in J}\mathcal{W}_j := \{(\oplus_{j\in J}W_{ij}, \omega_i) : i \in I\}$.*

Corollary 2.12. *Let $\{\mathcal{W}_j^{(k)}\}_{j\in J}$ and $\{\mathcal{V}_j^{(k)}\}_{j\in J}$ be BFFF, for each $1 \le k \le n$ and let $\mathcal{V}_j^{(k)}$ be an α-dual of $\mathcal{W}_j^{(k)}$, for each $j \in J$ and $k \in \{1, \ldots, n\}$. Then $\otimes_{k=1}^{n}(\oplus_{j\in J}\mathcal{V}_j^{(k)})$ is an α-dual of $\otimes_{k=1}^{n}(\oplus_{j\in J}\mathcal{W}_j^{(k)})$.*

References

[1] A. Abdollahi and E. Rahimi, *Generalized frames on super Hilbert spaces*, Bull. Malays. Math. Sci. Soc., **35** (2012) 807–818.

[2] M. R. Abdollahpour and A. Najati, *G-frames and Hilbert-Schmidt operators*, Bull. Iranian Math. Soc., **4** (2011) 141–155.

[3] P. Casazza and G. Kutyniok, *Frames of subspaces*, Contemp. Math. Amer. Math. Soc., **345** (2004) 87–113.

[4] I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., **27** (1986) 1271–1283.

[5] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., **72** (1952) 341–366.

[6] P. Gavruta, *On the duality of fusion frames*, J. Math. Anal. Appl., **333** (2007) 871–879.

[7] A. Khosravi and M. S. Asgari, *Frames of subspaces and approximation of the inverse frame operator*, Houston J. Math., **33** (2007) 907–920.

[8] A. Khosravi and M. Mirzaee Azandaryani, *Fusion frames and g-frames in tensor product and direct sum of Hilbert spaces*, Appl. Anal. Discrete Math., **6** (2012) 287–303.

[9] A. Khosravi and M. Mirzaee Azandaryani, *G-frames and direct sums*, Bull. Malays. Math. Sci. Soc., **36** (2013) 313–323.

[10] W. Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl., **322** (2006) 437–452.

(Morteza Mirzaee Azandaryani) Department of Mathematics, University of Qom, Qom, Iran.
Email address: morteza_ma62@yahoo.com; m.mirzaee@qom.ac.ir

(Mahmood Pourgholamhossein) Department of Mathematics, University of Qom, Qom, Iran
Email address: purgol@yahoo.com