Mathematical Analysis
& Convex Optimization

Vol. 4 (2023), No. 1, 119–130
https:\\ maco.lu.ac.ir
DOI: 10.22034/maco.4.1.10

Research Paper

SOME NEW REFINEMENTS OF HERMITE-HADAMARD INEQUALITY VIA A SEQUENCE OF MAPPINGS

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Abstract. In this paper we introduce a new sequence of mappings in connection to Hermite-Hadamard type inequality. Some bounds and refinements of Hermite-Hadamard inequality for convex functions via this sequence are given.

MSC(2010): 26D15 ; 53C21.
Keywords: Hermite-Hadamard inequality, Jensen inequality, convex functions.

1. Introduction

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a convex function and $a, b \in I, a < b$. We consider the well-known Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Several refinements and generalizations of the inequality (1.1) have been found in [1-15] and references therein. In order to provide various refinements of this result, S.S. Dragomir introduced two mappings $H, F : [0,1] \to \mathbb{R}$, in [5] and [6] respectively as follows and established several results in connection to Hermite-Hadamard inequality;

$$H(t) := \frac{1}{b-a}\int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)dx,$$

$$F(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b f\big(tx + (1-t)y\big)dxdy.$$

Since then numerous articles have appeared in the literature reflecting further applications and properties of these mappings (see [3-11]) and references therein. On the other hand the sequence of mappings $H_n : [0,1] \to \mathbb{R}$ associated to mapping H defined by;

$$H_n(t) := \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\Big(t\frac{x_1 + \ldots + x_n}{n} + (1-t)\frac{a+b}{2}\Big)dx_1 \ldots dx_n,$$

is introduced by S.S. Dragomir in [9]. We recall some of the main properties of H_n:

Date: Received: February 1, 2023, Accepted: June 20, 2023.

Theorem 1.1. *Let $f : I \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then, we have*
(i) H_n is convex on $[0, 1]$.
(ii) One has the following bounds;

$$\inf_{t\in[0,1]} H_n(t) = H_n(0) = f\big(\frac{a+b}{2}\big) \tag{1.2}$$

and

$$\sup_{t\in[0,1]} H_n(t) = H_n(1). \tag{1.3}$$

(iii) H_n increases monotonically on $[0, 1]$.
(iv) For every $n \geq 1$ and $t \in [0, 1]$ one has

$$f\big(\frac{a+b}{2}\big) \leq ... \leq H_{n+1}(t) \leq H_n(t) \leq ... \leq H_1(t) = H(t). \tag{1.4}$$

(v) If $a, b \in I^\circ$ with $a < b$ then, for every $n \geq 1$ and $t \in [0, 1]$ we have

$$0 \leq H_n(t) - f\big(\frac{a+b}{2}\big) \leq \frac{t(b-a)M}{2\sqrt{3}\sqrt{n}}, \tag{1.5}$$

where $M := \sup_{x\in[a,b]} |f'_+(x)|$ and $f'_+(x)$ is the right derivative of f at x. In particular

$$\lim_{n\to\infty} H_n(t) = f\big(\frac{a+b}{2}\big). \tag{1.6}$$

In this paper we introduce a new sequence of mappings associated to the mapping F and establish new inequalities in connection to Hermite-Hadamard inequality.

2. Main Results

Motivated by [9] we define the sequence of mappings $F_n : [0, 1] \to \mathbb{R}$, associated to mapping F as follows,

$$F_n(t) := \frac{1}{(b-a)^{n+1}} \int_a^b ... \int_a^b f\Big(t\frac{x_1 + ... + x_n}{n} + (1-t)x_{n+1}\Big)dx_1...dx_{n+1},$$

where, $f : I \to \mathbb{R}$ is a real valued function, $I \subseteq \mathbb{R}$ is an interval and $a, b \in I$ with $a < b$. Note that for every $n \geq 1$,

$$F_n(1) = H_n(1), \; F_n(0) = \frac{1}{b-a}\int_a^b f(x)dx. \tag{2.1}$$

In this section we study the properties of this sequence and introduce some results in connection to Hermite-Hadamard inequality. We start with the the following theorem.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then;*

(i) The mapping F_n is convex on $[0, 1]$, for every $n \geq 1$.

(ii) For every $n \geq 1$ and $t \in [0,1]$ one has

$$f\big(\frac{a+b}{2}\big) \leq \int_0^1 F_n(t)dt \leq \frac{2(b-a)H_n(1) + \int_a^b f(x)dx}{2(b-a)}. \tag{2.2}$$

(iii) If $J_n(t) := \frac{F_n(t)+F_n(1-t)}{2}$ then, for every $n \geq 1$, J_n is convex on $[0,1]$.

(iv) For every $n \geq 1$ the following inequalities hold,

$$\inf_{t\in[0,1]} J_n(t) = J_n(\frac{1}{2}),$$

$$F_n(t) \leq F_n(0) = \frac{1}{b-a}\int_a^b f(x)dx, \text{ for all } t \in [0,1]. \tag{2.3}$$

(v) For every $n \geq 1$ and $t \in [0,1]$ we have

$$H_n(t) \leq F_n(t). \tag{2.4}$$

(vi) For every $n \geq 1$, $J_n(t)$ decreases monotonically on $[0, \frac{1}{2}]$ and increases monotonically on $[\frac{1}{2}, 1]$.

Proof. (i) Using the definition of F_n and the convexity of f, the proof is obvious.

(ii) By simple computation and using Jensen's integral type inequality we have

$$\begin{aligned}
&f\big(\frac{a+b}{2}\big)\\
&= f\left(\frac{1}{(b-a)^{n+1}}\int_a^b \dots \int_a^b \Big(t\frac{x_1+\dots+x_n}{n} + (1-t)x_{n+1}\Big)dx_1\dots dx_{n+1}\right)\\
&\leq \frac{1}{(b-a)^{n+1}}\int_a^b \dots \int_a^b f\Big(t\frac{x_1+\dots+x_n}{n} + (1-t)x_{n+1}\Big)dx_1\dots dx_{n+1}\\
&= F_n(t).
\end{aligned} \tag{2.5}$$

Since F_n is convex, by integrating in (2.5) and using Hermite-Hadamard inequality we obtain the required result in (2.2).

(iii) Using the convexity of F_n, the result is obvious.

(iv) By convexity of f for every $t \in [0,1]$ we have

$$\begin{aligned}
&\frac{1}{2}\left[f\big(t\frac{x_1+\dots+x_n}{n} + (1-t)x_{n+1}\big) + f\big((1-t)\frac{x_1+\dots+x_n}{n} + tx_{n+1}\big)\right]\\
&\geq f\Big(\frac{1}{2}\big(\frac{x_1+\dots+x_n}{n} + x_{n+1}\big)\Big).
\end{aligned}$$

Hence by integrating on $[a,b]^{n+1}$ we get

$$\begin{aligned}J_n(t) &= \frac{1}{2}\big(F_n(t)+F_n(1-t)\big)\\
&= \frac{1}{2(b-a)^{n+1}}\Big[\int_a^b ... \int_a^b \Big(f\big(t\frac{x_1+...+x_n}{n}+(1-t)x_{n+1}\big)\\
&+f\big((1-t)\frac{x_1+...+x_n}{n}+tx_{n+1}\big)\Big)dx_1...dx_{n+1}\Big]\\
&\geq \frac{1}{(b-a)^{n+1}}\Big[\int_a^b ... \int_a^b f\Big(\frac{1}{2}\big(\frac{x_1+...+x_n}{n}+x_{n+1}\big)\Big] = F_n(\frac{1}{2})\\
&= J_n(\frac{1}{2}).\end{aligned}$$

For second inequality in (iv) we note that,

$$\begin{aligned}&f\big(t\frac{x_1+...+x_n}{n}+(1-t)x_{n+1}\big)\\
&\leq tf\big(\frac{x_1+...+x_n}{n}\big)+(1-t)f(x_{n+1})\\
&\leq t\frac{f(x_1)+...+f(x_n)}{n}+(1-t)f(x_{n+1}),\end{aligned}$$

by discrete Jense's inequality. So integrating on $[a,b]^{n+1}$ implies that

$$\begin{aligned}F_n(t) &\leq \frac{1}{2(b-a)^{n+1}}\\
&\int_a^b ... \int_a^b \Big(t\frac{f(x_1)+...+f(x_n)}{n}+(1-t)f(x_{n+1})\big)\Big)dx_1...dx_{n+1}\\
&= t\frac{1}{b-a}\int_a^b f(x)dx+(1-t)\frac{1}{b-a}\int_a^b f(x)dx\\
&= \frac{1}{b-a}\int_a^b f(x)dx = F_n(0).\end{aligned}$$

(v) For every $n\geq 1$ and $t\in[0,1)$ applying Jensen's integral type inequality on $[a,b]$ give us

$$\begin{aligned}&\frac{1}{(b-a)}\int_a^b f\Big(t\frac{x_1+...+x_n}{n}+(1-t)x_{n+1}\Big)dx_{n+1}\\
&\geq f\Big[\frac{1}{(b-a)}\int_a^b \Big(t\frac{x_1+...+x_n}{n}+(1-t)x_{n+1}\Big)dx_{n+1}\Big]\\
&= f\Big(t\frac{x_1+...+x_n}{n}+(1-t)\frac{a+b}{2}\Big).\end{aligned}$$

Taking integral on $[a,b]^n$ give us the inequality in (2.4).

(vi) By statement (iv) for every $t\in[0,1]$, $J_n(t)\geq J_n(\frac{1}{2})$ so, by convexity of J_n, for every $1\geq s>t>\frac{1}{2}$ we have

$$\frac{J_n(s)-J_n(t)}{s-t}\geq\frac{J_n(t)-J_n(\frac{1}{2})}{t-\frac{1}{2}}\geq 0,$$

hence $J_n(s) \geq J_n(t)$. The fact that J_n decreases monotonically on $[0, \frac{1}{2}]$ is similar. □

Now, we give the following result on monotonicity of the sequence F_n which completes the above theorem.

Theorem 2.2. *Let $f : I \to \mathbb{R}$ be a convex function and $a, b \in I$, $a < b$. Then for every $t \in [0, 1]$ one has*

$$f\big(\frac{a+b}{2}\big) \leq ... \leq F_{n+1}(t) \leq F_n(t) \leq ... \leq F_1(t) = F(t). \tag{2.6}$$

Proof. If $t = 1$ then by (2.1) the inequality (2.6) is trivially holds. Suppose that $t \in [0, 1)$. Then, for every $x_1, ..., x_{n+2} \in [a, b]$ we define the real numbers $y_1, ..., y_{n+1}$ as follows

$$\begin{aligned} y_1 &:= t\frac{x_1 + ... + x_n}{n} + (1-t)x_{n+2}, \\ y_2 &:= t\frac{x_2 + x_1 + ... + x_{n-1}}{n} + (1-t)x_{n+2}, \\ &. \\ &. \\ &. \\ y_{n+1} &:= t\frac{x_{n+1} + x_1 + ... + x_{n-1}}{n} + (1-t)x_{n+2}. \end{aligned}$$

Note that

$$\frac{y_1 + ... + y_{n+1}}{n+1} = t\frac{x_1 + ... + x_{n+1}}{n+1} + (1-t)x_{n+2}.$$

Hence, by using Jensen's type inequality we get

$$\begin{aligned} &f\big(t\frac{x_1 + ... + x_{n+1}}{n+1} + (1-t)x_{n+2}\big) = f\big(\frac{y_1 + ... + y_{n+1}}{n+1}\big) \\ &\leq \frac{f(y_1) + ... + f(y_{n+1})}{n+1} \\ &= \frac{1}{n+1}\Big[f\big(t\frac{x_1 + ... + x_n}{n} + (1-t)x_{n+2}\big) + ... \\ &+ f\big(t\frac{x_{n+1} + x_1 + ... + x_{n-1}}{n} + (1-t)x_{n+2}\big)\Big]. \end{aligned}$$

Taking integral on $[a,b]^{n+2}$ implies that

$$F_{n+1}(t) \leq \frac{1}{n+1}\Bigg[\frac{1}{(b-a)^{n+2}}$$
$$\times \int_a^b \dots \int_a^b f\big(t\frac{x_1+\dots+x_n}{n} + (1-t)x_{n+2}\big)dx_1\dots dx_{n+2}$$
$$+\dots+ \frac{1}{(b-a)^{n+2}}\int_a^b \dots \int_a^b f\big(t\frac{x_{n+1}+x_1+\dots+x_{n-1}}{n}$$
$$+(1-t)x_{n+2}\big)dx_1\dots dx_{n+2}\Bigg] = \frac{1}{n+1}\Bigg[(n+1)\frac{b-a}{(b-a)^{n+1}}$$
$$\int_a^b \dots \int_a^b f\big(t\frac{x_1+\dots+x_n}{n} + (1-t)x_{n+2}\big)dx_1\dots dx_n dx_{n+2}\Bigg]$$
$$= F_n(t).$$

This completes the proof. □

Remark 2.3. From (1), (2.3), (2.6) for every $n \geq 1$ and $t \in [0,1]$ we have

$$\begin{aligned} f\big(\frac{a+b}{2}\big) &\leq \dots \leq F_{n+1}(t) \leq F_n(t) \leq \dots \leq F_1(t) \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{2.7}$$

Now, it is natural to ask what happens with the difference $\frac{1}{b-a}\int_a^b f(x)dx - F_n(t)$ for all $t \in [0,1)$. The following theorem give us an upper bound for this difference for $t \in [0,1)$.

Theorem 2.4. *Let $f : I \to \mathbb{R}$ be a convex function and $a, b \in I^\circ$, $a < b$. Then for every $t \in [0,1)$ we have the following inequality*

$$\begin{aligned} 0 &\leq \frac{1}{b-a}\int_a^b f(x)dx - F_n(t) \\ &\leq \frac{t\sqrt{2}(n+1)^{1/4}}{\sqrt{n}}\left[\int_a^b \Big(f'_+(x)\Big)^2 dx\right]^{1/2}. \end{aligned}$$

Proof. By convexity of f we have

$$\begin{aligned} &f\big(t\frac{x_1+\dots+x_n}{n} + (1-t)x_{n+1}\big) - f(x_{n+1}) \\ &\geq t f'_+(x_{n+1})\big(\frac{x_1+\dots+x_n}{n} - x_{n+1}\big). \end{aligned}$$

Integrating on $[a,b]^{n+1}$ and using Hölder's inequality deduce that

$$
\begin{aligned}
0 &\leq \frac{1}{b-a}\int_a^b f(x)dx - F_n(t) \\
&\leq \frac{t}{(b-a)^{n+1}}\int_a^b \dots \int_a^b f'_+(x_{n+1})\big(x_{n+1} - \frac{x_1+\dots+x_n}{n}\big)dx_1\dots dx_{n+1} \\
&\leq \frac{t}{(b-a)^{n+1}}\left[\int_a^b \dots \int_a^b \big(f'_+(x_{n+1})\big)^2 dx_1\dots dx_{n+1}\right]^{1/2} \\
&\times \left[\int_a^b \dots \int_a^b \Big(x_{n+1} - \frac{x_1+\dots+x_n}{n}\Big)^2 dx_1\dots dx_{n+1}\right]^{1/2} \\
&= \frac{t}{(b-a)^{n+2/2}}\left[\int_a^b \big(f'_+(x)\big)^2 dx\right]^{1/2} \\
&\times \left[\int_a^b \dots \int_a^b \Big(x_{n+1} - \frac{x_1+\dots+x_n}{n}\Big)^2 dx_1\dots dx_{n+1}\right]^{1/2}.
\end{aligned}
\tag{2.8}
$$

Let $g(x) := \big(x_{n+1} - \frac{x_1+\dots+x_n}{n}\big)^2 = \frac{1}{n^2}\Big(\sum_{i=1}^n (x_{n+1}-x_i)\Big)^2$ then,

$$\nabla g(x) = \frac{2}{n^2}\sum_{i=1}^n (x_{n+1}-x_i)(-1,\dots,-1,1).$$

Hence,

$$
\begin{aligned}
||\nabla g(x)|| &= \frac{2}{n^2}\left|\sum_{i=1}^n (x_i - x_{n+1})\right|(n+1)^{1/2} \\
&\leq \frac{2(n+1)^{1/2}}{n^2}\sum_{i=1}^n \left|x_i - x_{n+1}\right| \leq \frac{2(n+1)^{1/2}}{n}(b-a).
\end{aligned}
\tag{2.9}
$$

By combining (2.8) and (2.9) we obtain

$$
\begin{aligned}
0 &\leq \frac{1}{b-a}\int_a^b f(x)dx - F_n(t) \\
&\leq \frac{t\sqrt{2}(n+1)^{1/4}}{\sqrt{n}}\left[\int_a^b \Big(f'_+(x)\Big)^2 dx\right]^{1/2},
\end{aligned}
$$

and proof is completed. □

The following corollaries are immediate consequence of Theorem 2.4.

Corollary 2.5. *Under the assumptions of theorem 2.4 if $M := \sup_{x\in[a,b]}|f'_+(x)|$, then for all $t \in [0,1)$ and $n \geq 1$ we have the inequality*

$$0 \leq \frac{1}{b-a}\int_a^b f(x)dx - F_n(t) \leq \frac{\sqrt{2}(n+1)^{1/4}M\sqrt{b-a}}{\sqrt{n}}.$$

In particular we obtain

$$\lim_{n\to\infty} F_n(t) = \frac{1}{b-a}\int_a^b f(x)dx, \text{ for all } t \in [0,1).$$

Corollary 2.6. *Under the assumptions of theorem 2.4 one has the following inequality*

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b f(x)dx - J_n(t) \\ &\le \frac{(n+1)^{1/4}}{\sqrt{2n}}\left[\int_a^b \Big(f'_+(x)\Big)^2 dx\right]^{1/2}. \end{aligned}$$

The following result also holds;

Theorem 2.7. *Let $f : I \to \mathbb{R}$ be a convex function and $a, b \in I^\circ$ with $a < b$. Suppose that there exits a constant $K > 0$ such that*

$$\big|f'_+(x) - f'_+(y)\big| \le K|x-y|, \text{ for all } x, y \in [a,b].$$

Then we have the inequality

$$tF_n(1) + (1-t)F_n(0) - F_n(t) \le \frac{2t(1-t)(n+1)^{1/2}K}{n}(b-a),$$

for all $t \in [0,1]$ and $n \ge 1$.

Proof. By convexity of f for every $x_1, ..., x_{n+1} \in [a,b]$ and $t \in [0,1]$ we have

$$\begin{aligned} &f\Big(t\frac{x_1+...+x_n}{n} + (1-t)x_{n+1}\Big) - f\big(\frac{x_1+...+x_n}{n}\big) \\ &\ge (1-t)f'_+\big(\frac{x_1+...+x_n}{n}\big)\big(x_{n+1} - \frac{x_1+...+x_n}{n}\big), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &f\Big(t\frac{x_1+...+x_n}{n} + (1-t)x_{n+1}\Big) - f(x_{n+1}) \\ &\ge -tf'_+\big(x_{n+1}\big)\big(x_{n+1} - \frac{x_1+...+x_n}{n}\big). \end{aligned} \tag{2.11}$$

If we multiply the inequalities (2.10) and (2.11) by t and $1-t$, respectively and added the obtained results we obtain

$$\begin{aligned} &tf\big(\frac{x_1+...+x_n}{n}\big) + (1-t)f(x_{n+1}) \\ &- f\Big(t\frac{x_1+...+x_n}{n} + (1-t)x_{n+1}\Big) \le t(1-t) \\ &\Big[f'_+\big(\frac{x_1+...+x_n}{n}\big) - f'_+\big(x_{n+1}\big)\Big]\big(\frac{x_1+...+x_n}{n} - x_{n+1}\big). \end{aligned}$$

Integrating on $[a,b]^{n+1}$ and using (9) implies that

$$
\begin{aligned}
& tF_n(1)+(1-t)F_n(0)-F_n(t) \\
& \leq t(1-t)\frac{1}{(b-a)^{n+1}}\int_a^b ... \int_a^b \Big[f'_+\big(\frac{x_1+...+x_n}{n}\big)-f'_+\big(x_{n+1}\big)\Big] \\
& \times \big(\frac{x_1+...+x_n}{n}-x_{n+1}\big)dx_1...dx_{n+1} \\
& \leq \frac{t(1-t)K}{(b-a)^{n+1}}\int_a^b ... \int_a^b \big(\frac{x_1+...+x_n}{n}-x_{n+1}\big)^2 dx_1...dx_{n+1} \\
& \leq \frac{2t(1-t)K(n+1)^{1/2}}{n}(b-a).
\end{aligned}
$$

This completes the proof. □

Finally an upper bound for the difference $F_n(t)-H_n(t), n \geq 1,\ t \in [0,1]$, is as follows.

Theorem 2.8. *Let $f : I \to \mathbb{R}$ be a convex function and $a, b \in I^\circ$ with $a < b$. Then, for all $t \in [0,1]$ and $n \geq 1$ we have the inequality*

$$
\begin{aligned}
& 0 \leq F_n(t)-H_n(t) \\
& \leq \frac{(1-t)(b-a)}{2}\Bigg[\frac{1}{(b-a)^{n+1}}\int_a^b ... \int_a^b \Big(f'_+\big(t\frac{x_1+...+x_n}{n} \\
& +(1-t)x_{n+1}\big)\Big)^2 dx_1...dx_{n+1}\Bigg]^{1/2}.
\end{aligned}
$$

Proof. By convexity of f for every $x_1, ..., x_{n+1} \in [a,b]$ and $t \in [0,1]$ we have

$$
\begin{aligned}
& f\Big(t\frac{x_1+...+x_n}{n}+(1-t)\frac{a+b}{2}\Big)-f\Big(t\frac{x_1+...+x_n}{n}+(1-t)x_{n+1}\Big) \\
& \geq (1-t)f'_+\Big(t\frac{x_1+...+x_n}{n}+(1-t)x_{n+1}\Big)\Big(\frac{a+b}{2}-x_{n+1}\Big).
\end{aligned}
$$

Integrating on $[a,b]^{n+1}$ and using Hölder's inequality implies that

$$\begin{aligned}
&0 \leq F_n(t) - H_n(t) \\
&\leq \frac{1-t}{(b-a)^{n+1}} \int_a^b \dots \int_a^b f'_+\Big(t\frac{x_1+\dots+x_n}{n} + (1-t)x_{n+1}\Big) \\
&\Big(x_{n+1} - \frac{a+b}{2}\Big)dx_1\dots dx_{n+1} \leq \frac{1-t}{(b-a)^{n+1}} \Big[\int_a^b \dots \int_a^b \\
&\Big(f'_+\big(t\frac{x_1+\dots+x_n}{n} + (1-t)x_{n+1}\big)\Big)^2 dx_1\dots dx_{n+1}\Big]^{1/2} \\
&\Big[\int_a^b \dots \int_a^b (x_{n+1} - \frac{a+b}{2})^2 dx_1\dots dx_{n+1}\Big]^{1/2} \\
&= \frac{(1-t)(b-a)}{2} \Big[\frac{1}{(b-a)^{n+1}} \int_a^b \dots \int_a^b \Big(f'_+\big(t\frac{x_1+\dots+x_n}{n} \\
&+(1-t)x_{n+1}\big)\Big)^2 dx_1\dots dx_{n+1}\Big]^{1/2}.
\end{aligned}$$

This completes the proof. □

Corollary 2.9. *Under the assumptions as in theorem (2.8) if* $K := \sup_{x\in[a,b]} |f'_+(x)|$ *one has*

$$0 \leq F_n(t) - H_n(t) \leq \frac{(1-t)K}{2}(b-a),$$

for every $n \geq 1$, $t \in [0,1]$.
In particular we have

$$0 \leq F(t) - H(t) \leq \frac{(1-t)K}{2}(b-a).$$

The following example gives a refinement and upper bound related to inequality (1.1).

Example 2.10. Consider the convex function $f : I \to \mathbb{R}$, $f(x) := e^x$, for $n = 1$ and for every $t \in [0,1]$, we have

$$F_1(t) = \frac{1}{(b-a)^2} \int_a^b \int_a^b e^{tx+(1-t)y} dxdy,$$

where $a, b \in I$, with $a < b$. If $t = 0$ or $t = 1$, we see that

$$F_1(0) = F_1(1) = \frac{1}{b-a} \int_a^b e^x dx = \frac{e^b - e^a}{b-a}.$$

Thus inequalities in (2.7) are valid. It is easy to see that for every $t \in (0,1)$ we have

$$F_1(t) = \frac{1}{(b-a)^2(1-t)t}\big(e^{tb} - e^{ta}\big)\big(e^{(1-t)b} - e^{(1-t)a}\big).$$

From equality

$$\frac{a+b}{2}=\frac{1}{(b-a)^2}\int_a^b\int_a^b(tx+(1-t)y)dxdy,$$

by jensen's type integral inequality we get

$$\begin{aligned}f\Big(\frac{a+b}{2}\Big)=&f\Big(\frac{1}{(b-a)^2}\int_a^b\int_a^b(tx+(1-t)y)dxdy\Big)\\ \leq&\frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy\\ =&F_1(t).\end{aligned}$$

Also from the inequalities (2.3), (1.1) we note that

$$F_1(t)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2},$$

therefore the inequality (2.7) holds. Now, simple computation gives an upper bound for the difference $\frac{1}{b-a}\int_a^b f(x)dx-F_1(t)$ for all $t\in[0,1]$. Using Theorem 2.4 implies that

$$\begin{aligned}0&\leq\frac{1}{b-a}\int_a^b e^x dx-F_1(t)\\ &\leq t\sqrt{2}(2)^{1/4}\left[\int_a^b e^{2x}dx\right]^{1/2}\\ &=t(2)^{1/4}\big(e^{2b}-e^{2a}\big)^{1/2}.\end{aligned}$$

Conclusion

In this paper, we have given a sequence of mappings associated to the mapping F. This sequence gives us some new refinements and bounds related to well known Hermite-Hadamard inequality.

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