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Research Paper

DIRECT PRODUCT OF $IFMSN(G)$

RASUL RASULI* AND AMIR SHOMALI

Abstract. In this paper we introduced direct product of intuitionistic fuzzy multigroups of G under norms($IFMSN(G)$) and we prove that it will be also $IFMSN(G)$. Next we shall study some important properties and theorems for them. On the other hand we shall give the definition of the identity element, strong upper- lower and weak upper- lowerof them and study the main theorem for this. We shall also give new results on this subject. Also we define the concepts of conjugate and commutative of $IFMSN(G)$ and investigate them under direct product. Finally, we organize them under group homomorphisms and we prove that the image and preimage of direct product of $IFMSN(G)$ will be also $IFMSN(G)$.

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1. Introduction and Background

A multiset (mset), which is a generalization of classical or standard (Cantorian) set, is a "set" where an element can occur more than once. The term multiset (mset in short) as Knuth [4] notes, was first suggested by De Bruijn [3] in a private communication to him. The concept of fuzzy sets was proposed by Zaded [28] to capture uncertainty in a collection, which was neglected in crisp set. Fuzzy set theory has grown stupendously over the years giving birth to fuzzy groups introduced in [24]. Recently, Shinoj et al. [25] introduced a non-classical group called fuzzy multigroup, which generalized fuzzy group. In 1983, Atanassov [1, 2] introduced the concept of intuitionistic fuzzy sets. The concepts of intuitionistic fuzzy multiset and intuitionistic fuzzy multigroup are introduced in [26, 27], which have applications in medical diagnosis and robotics. The First author by using norms, investigated some properties of fuzzy algebraic structures [5-23] specially in [5-9] initiated the study of fuzzy multigroups, anti fuzzy multigroupsand and intuitionistic fuzzy multigroups under norms and investigated some properties of them.In this study, we introduce the concept of direct product, conjugate and commutative of $IFMSN(G)$ and we obtain some results about them. Also we discussed few results of them under group homomorphisms.

2. Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel. For details we refer to [5-9].

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*Corresponding author.

Definition 2.1. Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into $[0,1]$. The set of all fuzzy subsets of G is called the $[0,1]$-power set of G and is denoted $[0,1]^G$.

Definition 2.2. Let X be a set. A fuzzy multiset A of X is characterized by a count membership function

$$CM_A : X \to [0,1]$$

of which the value is a multiset of the unit interval $I = [0,1]$. That is,

$$CM_A(x) = \{\mu^1, \mu^2, ..., \mu^n, ...\} \forall x \in X,$$

where $\mu^1, \mu^2, ..., \mu^n, ... \in [0,1]$ such that

$$(\mu^1 \geq \mu^2 \geq ... \geq \mu^n \geq ...).$$

Whenever the fuzzy multiset is finite, we write

$$CM_A(x) = \{\mu^1, \mu^2, ..., \mu^n\},$$

where $\mu^1, \mu^2, ..., \mu^n \in [0,1]$ such that

$$(\mu^1 \geq \mu^2 \geq ... \geq \mu^n),$$

or simply

$$CM_A(x) = \{\mu^i\},$$

for $\mu^i \in [0,1]$ and $i = 1, 2, ..., n$.
Now, a fuzzy multiset A is given as
$A = \{\dfrac{CM_A(x)}{x} : x \in X\}$ or $A = \{(CM_A(x), x) : x \in X\}$.
The set of all fuzzy multisets is depicted by $FMS(X)$.

Example 2.3. Assume that $X = \{a, b, c\}$ is a set. Then for $CM_A(a) = \{1, 0.5, 0.4\}$ and $CM_A(b) = \{0.9, 0.6\}$ and $CM_A(c) = \{0\}$ we get that A is a fuzzy multiset of X written as

$$A = \{\frac{1, 0.5, 0.4}{a}, \frac{0.9, 0.6}{b}\}.$$

Definition 2.4. Let $A, B \in FMS(X)$. Then A is called a fuzzy submultiset of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x)$ for all $x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 2.5. A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(T1) $T(x, 1) = x$ (neutral element),
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),
(T3) $T(x, y) = T(y, x)$ (commutativity),
(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
for all $x, y, z \in [0,1]$.
We say that T be idempotent if $T(x, x) = x$ for all $x \in [0,1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

Example 2.6. (1) Standard intersection t-norm $T_m(x,y) = \min\{x,y\}$.
(2) Bounded sum t-norm $T_b(x,y) = \max\{0, x+y-1\}$.
(3) algebraic product t-norm $T_p(x,y) = xy$.
(4) Drastic T-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum t-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product t-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \le T(x,y) \le T_{\min}(x,y)$ for all $x, y \in [0,1]$.

Definition 2.7. ([6]) A t-conorm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(C1) $C(x,0) = x$
(C2) $C(x,y) \le C(x,z)$ if $y \le z$
(C3) $C(x,y) = C(y,x)$
(C4) $C(x, C(y,z)) = C(C(x,y),z)$,
for all $x,y,z \in [0,1]$.

We say that C be idempotent if $C(x,x) = x$ for all $x \in [0,1]$.

Example 2.8. (1) Standard union t-conorm $C_m(x,y) = \max\{x,y\}$.
(2) Bounded sum t-conorm $C_b(x,y) = \min\{1, x+y\}$.
(3) Algebraic sum t-conorm $C_p(x,y) = x+y-xy$.
(4) Drastic T-conorm

$$C_D(x,y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise,} \end{cases}$$

dual to the drastic t-norm.
(5) Nilpotent maximum t-conorm , dual to the nilpotent minimum t-norm:

$$C_{nM}(x,y) = \begin{cases} \max\{x,y\} & \text{if } x+y < 1 \\ 1 & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity) $C_{H_2}(x,y) = \dfrac{x+y}{1+xy}$ is a dual to one of the Hamacher t-norms. Note that all t-conorms are bounded by the maximum and the drastic t-conorm: $C_{\max}(x,y) \le C(x,y) \le C_D(x,y)$ for any t-conorm C and all $x, y \in [0,1]$.

Recall that t-conorm C is idempotent if for all $x \in [0,1]$, we have that $C(x,x) = x$.

Lemma 2.9. *Let T be a t-norm and C be a t-conorm. Then*

$$T(T(x,y),T(w,z)) = T(T(x,w),T(y,z)),$$

and

$$C(C(x,y),C(w,z)) = C(C(x,w),C(y,z)),$$

for all $x,y,w,z \in [0,1]$.

Definition 2.10. Let $A = (CM_A, CN_A) \in IFMS(G)$. Then A is said to be an intuitionistic fuzzy multigroup of G under norms(t-norm T and t-conorm C) if it satisfies the following conditions:
(1) $CM_A(xy) \geq T(CM_A(x), CM_A(y))$,
(2) $CM_A(x^{-1}) \geq CM_A(x)$,
(3) $CN_A(xy) \leq C(CN_A(x), CN_A(y))$,
(4) $CN_A(x^{-1}) \leq CN_A(x)$,
for all $x, y \in G$.
The set of all intuitionistic fuzzy multigroups of G under norms(t-norm T and t-conorm C) is depicted by $IFMSN(G)$.

Theorem 2.11. *Let $A = (CM_A, CN_A) \in IFMSN(G)$ and T, C be idempotent. Then*
(1) $A(e) \supseteq A(x)$ for all $x \in G$.
(2) $A(x^n) \supseteq A(x)$ for all $x \in G$ and $n \geq 1$.
(3) $A(x) = A(x^{-1})$ for all $x \in G$.

3. Direct Product of $IFMSN(G)$

Definition 3.1. Let $A = (CM_A, CN_A) \in IFMSN(G)$ and $B = (CM_B, CN_B) \in IFMSN(H)$. The direct product of A and B, denoted by

$A \times B = (CM_A, CN_A) \times (CM_B, CN_B) = (CM_A \times CM_B, CN_A \times CN_B) = (CM_{A\times B}, CN_{A\times B})$,

is characterized by as functions

$$CM_{A\times B} : G \times H \to [0,1]$$

and

$$CN_{A\times B} : G \times H \to [0,1]$$

such that

$$CM_{A\times B}(x,y) = T(CM_A(x), CM_B(y))$$

and

$$CN_{A\times B}(x,y) = C(CM_A(x), CM_B(y))$$

for all $x \in G$ and $y \in H$.

Example 3.2. Let $G = \{1, x\}$ be a group, where $x^2 = 1$ and $H = \{e, a, b, c\}$ be a Klein 4-group, where $a^2 = b^2 = c^2 = e$. Let $CM_A = \{\frac{0.5, 0.4}{1}, \frac{0.6, 0.3}{x}\}$ and $CN_A = \{\frac{0.4, 0.2, 0.1}{1}, \frac{0.2, 0.1}{x}\}$ and

$$CM_B = \{\frac{0.6, 0.25}{e}, \frac{0.35, 0.25}{a}, \frac{0.50, 0.40}{b}, \frac{0.4, 0.3}{c}\}$$

and

$$CN_B = \{\frac{0.2, 0.15}{e}, \frac{0.5, 0.45, 0.25}{a}, \frac{0.20, 0.15}{b}, \frac{0.25, 0.15}{c}\}$$

be fuzzy multigroups of G and H. Let

$$G \times H = \{(1,e),(1,a),(1,b),(1,c),(x,e),(x,a),(x,b),(x,c)\}$$

be a a group from the classical sense. Define

$$A \times B = \{\frac{0.2, 0.1}{(1,e)}, \frac{0.55, 0.35}{(1,a)}, \frac{0.45, 0.35}{(1,b)}, \frac{0.6, 0.2}{(1,c)}, \frac{0.4, 0.3}{(x,e)}, \frac{0.25, 0.15}{(x,a)}, \frac{0.7, 0.3}{(x,b)}, \frac{0.7, 0.6}{(x,c)}\}.$$

Let $T(x,y) = T_p(x,y) = xy$ and $C(x,y) = C_p(x,y) = x + y - xy$ for all $x, y \in [0,1]$.
Then $A = (CM_A, CN_A) \in IFMSN(G)$ and $B = (CM_B, CN_B) \in IFMSN(H)$ thus $A \times B \in IFMSN(G \times H)$.

Proposition 3.3. *Let $A_i = (CM_{A_i}, CN_{A_i}) \in IFMSN(G_i)$ for $i = 1, 2$. Then $A_1 \times A_2 \in IFMSN(G_1 \times G_2)$.*

Proof. Let $(a_1, b_1), (a_2, b_2) \in G_1 \times G_2$. Then
(1)

$$\begin{aligned}(CM_{A_1 \times A_2})((a_1,b_1)(a_2,b_2)) &= (CM_{A_1 \times A_2})(a_1a_2, b_1b_2) \\ &= T(CM_{A_1}(a_1a_2), CM_{A_2}(b_1b_2)) \\ &\geq T(T(CM_{A_1}(a_1), CM_{A_1}(a_2)), T(CM_{A_2}(b_1), CM_{A_2}(b_2))) \\ &= T(T(CM_{A_1}(a_1), CM_{A_2}(b_1), T(CM_{A_1}(a_2), CM_{A_2}(b_2)) \ (\textit{Lemma 2.9}) \\ &= T((CM_{A_1 \times A_2})(a_1,b_1), (CM_{A_1 \times A_2})(a_2,b_2))\end{aligned}$$

thus

$$(CM_{A_1 \times A_2})((a_1,b_1)(a_2,b_2)) \geq T((CM_{A_1 \times A_2})(a_1,b_1), (CM_{A_1 \times A_2})(a_2,b_2)).$$

(2)

$$\begin{aligned}(CN_{A_1 \times A_2})((a_1,b_1)(a_2,b_2)) &= (CN_{A_1 \times A_2})(a_1a_2, b_1b_2) \\ &= C(CN_{A_1}(a_1a_2), CN_{A_2}(b_1b_2)) \\ &\leq C(C(CN_{A_1}(a_1), CN_{A_1}(a_2)), C(CN_{A_2}(b_1), CN_{A_2}(b_2))) \\ &= C(C(CN_{A_1}(a_1), CN_{A_2}(b_1), C(CN_{A_1}(a_2), CN_{A_2}(b_2)) \ (\textit{Lemma 2.9}) \\ &= C((CN_{A_1 \times A_2})(a_1,b_1), (CN_{A_1 \times A_2})(a_2,b_2))\end{aligned}$$

so

$$(CN_{A_1 \times A_2})((a_1,b_1)(a_2,b_2)) \leq C((CN_{A_1 \times A_2})(a_1,b_1), (CN_{A_1 \times A_2})(a_2,b_2)).$$

Let $(a, b) \in G_1 \times G_2$. Then
(3)

$$\begin{aligned}(CM_{A_1 \times A_2})(a,b)^{-1} &= (CM_{A_1 \times A_2})(a^{-1}, b^{-1}) \\ &= T(CM_{A_1}(a^{-1}), CM_{A_2}(b^{-1})) \\ &\geq T(CM_{A_1}(a), CM_{A_2}(b))\end{aligned}$$

then

$$(CM_{A_1 \times A_2})(a,b)^{-1} \geq T(CM_{A_1}(a), CM_{A_2}(b)).$$

(4)

$$\begin{aligned}(CN_{A_1\times A_2})(a,b)^{-1} &= (CN_{A_1\times A_2})(a^{-1},b^{-1})\\ &= C(CN_{A_1}(a^{-1}),CN_{A_2}(b^{-1}))\\ &\leq C(CN_{A_1}(a),CN_{A_2}(b))\end{aligned}$$

thus

$$(CN_{A_1\times A_2})(a,b)^{-1} \leq C(CN_{A_1}(a),CN_{A_2}(b)).$$

Therefore (1)-(4) give us that $A_1 \times A_2 = (CM_{A_1\times A_2}, CN_{A_1\times A_2}) \in IFMSN(G_1 \times G_2)$. □

Corollary 3.4. *Let $A_i = (CM_{A_i}, CN_{A_i}) \in IFMSN(G_i)$ for $i = 1, 2, ..., n$. Then*

$$A_1 \times A_2 \times ... \times A_n \in IFMSN(G_1 \times G_2 \times ... \times G_n).$$

Definition 3.5. Let $A = (CM_A, CN_A) \in IFMSN(G)$ and $\alpha, \beta \in [0, 1]$. Then we define
(1) $A^\star = \{x \in G \mid A(x) = A(e_G)\}$ where e_G is the identity element of G.
(2) $A^{[\beta]}_{[\alpha]} = \{x \in G \mid A(x) \supseteq (\alpha, \beta)\}$ is called strong upper- lower (α, β)-cut of A.
(3) $A^{(\beta)}_{(\alpha)} = \{x \in G \mid A(x) \supset (\alpha, \beta)\}$ is called weak upper- lower (α, β)-cut of A.

Proposition 3.6. *Let $A = (CM_A, CN_A) \in IFMSN(G)$ and $B = (CM_B, CN_B) \in IFMSN(H)$ such that T, C be idempotent norms. Then for all $\alpha, \beta \in [0, 1]$ the following assertions hold.*
(1) $(A \times B)^\star = A^\star \times B^\star$.
(2) $(A \times B)^{[\beta]}_{[\alpha]} = A^{[\beta]}_{[\alpha]} \times B^{[\beta]}_{[\alpha]}$.
(3) $(A \times B)^{(\beta)}_{(\alpha)} = A^{(\beta)}_{(\alpha)} \times B^{(\beta)}_{(\alpha)}$.

Proof. (1) Let

$$\begin{aligned}(A\times B)^\star &= \{(x,y)\in G\times H \mid (A\times B)(x,y) = (A\times B)(e_G,e_H)\}\\ &= \{(x,y)\in G\times H \mid (CM_{A\times B},CN_{A\times B})(x,y) = (CM_{A\times B},CN_{A\times B})(e_G,e_H)\}\\ &= \{(x,y)\in G\times H \mid CM_{A\times B}(x,y) = CM_{A\times B}(e_G,e_H), CN_{A\times B}(x,y) = CN_{A\times B}(e_G,e_H)\}\end{aligned}$$

so

$$(x,y) \in (A \times B)^\star$$

if and only if

$$CM_{A\times B}(x,y) = CM_{A\times B}(e_G,e_H)$$

and

$$CN_{A\times B}(x,y) = CN_{A\times B}(e_G,e_H)$$

if and only if

$$T(CM_A(x),CM_B(y)) = T(CM_A(e_G),CM_B(e_H))$$

and

$$C(CN_A(x),CN_B(y)) = C(CN_A(e_G),CN_B(e_H))$$

if and only if

$$CM_A(x) = CM_A(e_G)$$

and

$$CM_B(y) = CM_B(e_H)$$

if and only if

$$x \in A^\star$$

and

$$y \in B^\star$$

if and only if

$$(x, y) \in A^\star \times B^\star$$

thus

$$(A \times B)^\star = A^\star \times B^\star.$$

(2) Let

$$\begin{aligned}(A \times B)^{[\beta]}_{[\alpha]} &= \{(x, y) \in G \times H \mid (A \times B)(x, y) \supseteq (\alpha, \beta)\}\\ &= \{(x, y) \in G \times H \mid (CM_{A\times B}, CN_{A\times B})(x, y) \supseteq (\alpha, \beta)\}\\ &= \{(x, y) \in G \times H \mid CM_{A\times B}(x, y) \geq \alpha, CN_{A\times B}(x, y) \leq \beta\}.\end{aligned}$$

Now

$$(x, y) \in (A \times B)^{[\beta]}_{[\alpha]} \Longleftrightarrow CM_{A\times B}(x, y) \geq \alpha \; and \; CN_{A\times B}(x, y) \leq \beta$$
$$\Longleftrightarrow T(CM_A(x), CM_B(y)) \geq \alpha = T(\alpha, \alpha) \; and \; C(CN_A(x), CN_B(y)) \leq \beta = C(\beta, \beta)$$
$$\Longleftrightarrow CM_A(x) \geq \alpha \; and \; CM_B(y) \geq \alpha \; and \; CN_A(x) \leq \beta \; and \; CN_B(y) \leq \beta$$
$$\Longleftrightarrow x \in A^{[\beta]}_{[\alpha]} \; and \; y \in B^{[\beta]}_{[\alpha]} \Longleftrightarrow (x, y) \in A^{[\beta]}_{[\alpha]} \times B^{[\beta]}_{[\alpha]}$$

thus

$$(A \times B)^{[\beta]}_{[\alpha]} = A^{[\beta]}_{[\alpha]} \times B^{[\beta]}_{[\alpha]}.$$

(3) As

$$\begin{aligned}(A \times B)^{(\beta)}_{(\alpha)} &= \{(x, y) \in G \times H \mid (A \times B)(x, y) \supset (\alpha, \beta)\}\\ &= \{(x, y) \in G \times H \mid (CM_{A\times B}, CN_{A\times B})(x, y) \supset (\alpha, \beta)\}\\ &= \{(x, y) \in G \times H \mid CM_{A\times B}(x, y) > \alpha, CN_{A\times B}(x, y) < \beta\}\end{aligned}$$

so

$$(x, y) \in (A \times B)^{(\beta)}_{(\alpha)} \Longleftrightarrow CM_{A\times B}(x, y) > \alpha \; and \; CN_{A\times B}(x, y) < \beta$$
$$\Longleftrightarrow T(CM_A(x), CM_B(y)) > \alpha = T(\alpha, \alpha) \; and \; C(CN_A(x), CN_B(y)) < \beta = C(\beta, \beta)$$
$$\Longleftrightarrow CM_A(x) > \alpha \; and \; CM_B(y) < \alpha \; and \; CN_A(x) < \beta \; and \; CN_B(y) < \beta$$
$$\Longleftrightarrow x \in A^{(\beta)}_{(\alpha)} \; and \; y \in B^{(\beta)}_{(\alpha)} \Longleftrightarrow (x, y) \in A^{(\beta)}_{(\alpha)} \times B^{(\beta)}_{(\alpha)}$$

thus

$$(A \times B)^{(\beta)}_{(\alpha)} = A^{(\beta)}_{(\alpha)} \times B^{(\beta)}_{(\alpha)}.$$

□

Proposition 3.7. *Let $A = (CM_A, CN_A) \in IFMSN(G)$ and $B = (CM_B, CN_B) \in IFMSN(H)$ such that T, C be idempotent norms. Then for all $(x, y) \in G \times H$ the following assertions hold.*
(1) $(A \times B)(e_G, e_H) \supseteq (A \times B)(x, y)$.
(2) $(A \times B)((x, y)^n) \supseteq (A \times B)(x, y)$.
(3) $(A \times B)(x, y) = (A \times B)(x^{-1}, y^{-1})$.

Proof. As Proposition 3.3 we get that $A \times B \in IFMSN(G \times H)$ so Theorem 2.11 gives us that assertions are hold. □

Proposition 3.8. *Let $A = (CM_A, CN_A) \in IFMSN(G)$ and $B = (CM_B, CN_B) \in IFMSN(H)$ such that T, C be idempotent norms. Then for all $\alpha, \beta \in [0, 1]$ the following assertions hold.*
(1) $(A \times B)^\star$ is a subgroup of $G \times H$.
(2) $(A \times B)^{[\beta]}_{[\alpha]}$ is a subgroup of $G \times H$.
(3) $(A \times B)^{(\beta)}_{(\alpha)}$ is a subgroup of $G \times H$.

Proof. (1) Let $(x_1, y_1), (x_2, y_2) \in (A \times B)^\star$ and we must prove that $(x_1, y_1)(x_2, y_2)^{-1} \in (A \times B)^\star$. Because $(x_1, y_1), (x_2, y_2) \in (A \times B)^\star$ then

$$CM_{A\times B}(x_1, y_1) = CM_{A\times B}(x_2, y_2) = CM_{A\times B}(e_G, e_H)$$

and

$$CN_{A\times B}(x_1, y_1) = CN_{A\times B}(x_2, y_2) = CN_{A\times B}(e_G, e_H)$$

which mean that

$$T(CM_A(x_1), CM_B(y_1)) = T(CM_A(x_2), CM_B(y_2)) = T(CM_A(e_G), CM_B(e_H))$$

and

$$C(CN_A(x_1), CN_B(y_1)) = C(CN_A(x_2), CN_B(y_2)) = C(CN_A(e_G), CN_B(e_H))$$

so $CM_A(x_1) = CM_A(x_2) = CM_A(e_G)$ and $CM_A(y_1) = CM_A(y_2) = CM_A(e_H)$. Then

$$\begin{aligned}
CM_{A\times B}((x_1, y_1)(x_2, y_2)^{-1}) &= CM_{A\times B}((x_1, y_1)(x_2^{-1}, y_2^{-1})) \\
&= CM_{A\times B}(x_1x_2^{-1}, y_1y_2^{-1}) \\
&= T(CM_A(x_1x_2^{-1}), CM_B(y_1y_2^{-1})) \\
&\geq T(T(CM_A(x_1), CM_A(x_2^{-1})), T(CM_B(y_1), CM_B(y_2^{-1}))) \\
&\geq T(T(CM_A(x_1), CM_A(x_2)), T(CM_B(y_1), CM_B(y_2))) \\
&= T(T(CM_A(e_G), CM_A(e_G)), T(CM_B(e_H), CM_B(e_H))) \\
&= T(CM_A(e_G), CM_B(e_H)) = CM_{A\times B}(e_G, e_H) \\
&\geq CM_{A\times B}((x_1, y_1)(x_2, y_2)^{-1}) \text{ (Proposition 3.7 part(1))}
\end{aligned}$$

thus

$$CM_{A\times B}((x_1, y_1)(x_2, y_2)^{-1}) = CM_{A\times B}(e_G, e_H).$$

Also

$$\begin{aligned}
CN_{A\times B}((x_1, y_1)(x_2, y_2)^{-1}) &= CN_{A\times B}((x_1, y_1)(x_2^{-1}, y_2^{-1})) \\
&= CN_{A\times B}(x_1x_2^{-1}, y_1y_2^{-1}) \\
&= C(CN_A(x_1x_2^{-1}), CN_B(y_1y_2^{-1})) \\
&\leq C(C(CN_A(x_1), CN_A(x_2^{-1})), C(CN_B(y_1), CN_B(y_2^{-1}))) \\
&\leq C(C(CN_A(x_1), CN_A(x_2)), C(CN_B(y_1), CN_B(y_2))) \\
&= C(C(CM_A(e_G), CN_A(e_G)), C(CN_B(e_H), CN_B(e_H))) \\
&= C(CN_A(e_G), CN_B(e_H)) = CN_{A\times B}(e_G, e_H) \\
&\leq CN_{A\times B}((x_1, y_1)(x_2, y_2)^{-1}) \text{ (Proposition 3.7 part(1))}
\end{aligned}$$

then

$$CN_{A\times B}((x_1, y_1)(x_2, y_2)^{-1}) = CN_{A\times B}(e_G, e_H).$$

Therefore

$$\begin{aligned}(A\times B)^{\star}((x_1,y_1)(x_2,y_2)^{-1}) &= (CM_{A\times B}((x_1,y_1)(x_2,y_2)^{-1}), CN_{A\times B}((x_1,y_1)(x_2,y_2)^{-1}))\\ &= (CM_{A\times B}(e_G,e_H), CN_{A\times B}(e_G,e_H))\\ &= (A\times B)^{\star}(e_G,e_H)\end{aligned}$$

so $(x_1,y_1)(x_2,y_2)^{-1}\in (A\times B)^{\star}$ thus $(A\times B)^{\star}$ is a subgroup of $G\times H$.

(2) Let $(x_1,y_1),(x_2,y_2)\in (A\times B)^{[\beta]}_{[\alpha]}$ and we show that $(x_1,y_1)(x_2,y_2)^{-1}\in (A\times B)^{[\beta]}_{[\alpha]}$. As $(x_1,y_1),(x_2,y_2)\in (A\times B)^{[\beta]}_{[\alpha]}$ so $CM_{A\times B}(x_1,y_1)\geq \alpha$ and $CM_{A\times B}(x_2,y_2)\geq\alpha$. Now

$$\begin{aligned}CM_{A\times B}((x_1,y_1)(x_2,y_2)^{-1}) &= CM_{A\times B}((x_1,y_1)(x_2^{-1},y_2^{-1}))\\ &= CM_{A\times B}(x_1x_2^{-1},y_1y_2^{-1})\\ &= T(CM_A(x_1x_2^{-1}),CM_B(y_1y_2^{-1}))\\ &\geq T(T(CM_A(x_1),CM_A(x_2^{-1})),T(CM_B(y_1),CM_B(y_2^{-1})))\\ &\geq T(T(CM_A(x_1),CM_A(x_2)),T(CM_B(y_1),CM_B(y_2)))\\ &= T(T(CM_A(x_1),CM_B(y_1)),T(CM_A(x_2),CM_B(y_2)))\ (\textit{Lemma 2.9})\\ &= T(CM_{A\times B}(x_1,y_1),CM_{A\times B}(x_2,y_2))\\ &\geq T(\alpha,\alpha)=\alpha\end{aligned}$$

thus

$$CM_{A\times B}((x_1,y_1)(x_2,y_2)^{-1})\geq\alpha.$$

Also since $CN_{A\times B}(x_1,y_1)\leq\beta$ and $CN_{A\times B}(x_2,y_2)\leq\beta$ so

$$\begin{aligned}CN_{A\times B}((x_1,y_1)(x_2,y_2)^{-1}) &= CN_{A\times B}((x_1,y_1)(x_2^{-1},y_2^{-1}))\\ &= CN_{A\times B}(x_1x_2^{-1},y_1y_2^{-1})\\ &= C(CN_A(x_1x_2^{-1}),CN_B(y_1y_2^{-1}))\\ &\leq C(C(CN_A(x_1),CN_A(x_2^{-1})),C(CN_B(y_1),CN_B(y_2^{-1})))\\ &\leq C(C(CN_A(x_1),CN_A(x_2)),C(CN_B(y_1),CN_B(y_2)))\\ &= C(C(CN_A(x_1),CN_B(y_1)),C(CN_A(x_2),CN_B(y_2)))\ (\textit{Lemma 2.9})\\ &= C(CN_{A\times B}(x_1,y_1),CN_{A\times B}(x_2,y_2))\\ &\leq C(\beta,\beta)=\beta\end{aligned}$$

then

$$CN_{A\times B}((x_1,y_1)(x_2,y_2)^{-1})\leq\beta.$$

Therefore

$$(A\times B)^{[\beta]}_{[\alpha]}((x_1,y_1)(x_2,y_2)^{-1}) = (CM_{A\times B}((x_1,y_1)(x_2,y_2)^{-1}),CN_{A\times B}((x_1,y_1)(x_2,y_2)^{-1}))\supseteq(\alpha,\beta)$$

thus $(x_1,y_1)(x_2,y_2)^{-1}\in (A\times B)^{[\beta]}_{[\alpha]}$. Then $(A\times B)^{[\beta]}_{[\alpha]}$ is a subgroup of $G\times H$.

(3) The proof is similar to (2). □

Proposition 3.9. *Let $A=(CM_A,CN_A)\in IFMSN(G)$ and $B=(CM_B,CN_B)\in IFMSN(H)$. If $A\times B\in IFMSN(G\times H)$, then at least one of the following statements hold.*

(1) $B(e_H) \supseteq A(x)$ *for all* $x \in G$.
(2) $A(e_G) \supseteq B(y)$ *for all* $y \in G$.

Proof. By contrapositive, suppose that none of the statements holds. Then suppose we can find $a \in G$ and $b \in H$ such that $A(a) \supset B(e_H)$ and $B(b) \supset A(e_G)$. Thus

$$A(a) = (CM_A(a), CN_A(a)) \supset B(e_H) = (CM_B(e_H), CN_B(e_H))$$

and

$$B(b) = (CM_B(b), CN_B(b)) \supset A(e_G) = (CM_A(e_G), CN_A(e_G)).$$

Now

$$\begin{aligned} CM_{A\times B}(a,b) &= T(CM_A(a), CM_B(b)) \\ &> T(CM_B(e_H), CM_A(e_G)) \\ &= T(CM_A(e_G), CM_B(e_H)) \\ &= CM_{A\times B}(e_G, e_H) \end{aligned}$$

and

$$\begin{aligned} CN_{A\times B}(a,b) &= C(CN_A(a), CN_B(b)) \\ &< C(CN_B(e_H), CN_A(e_G)) \\ &= C(CN_A(e_G), CN_B(e_H)) \\ &= CN_{A\times B}(e_G, e_H) \end{aligned}$$

thus $CM_{A\times B}(a,b) > CM_{A\times B}(e_G, e_H)$ and $CN_{A\times B}(a,b) < CN_{A\times B}(e_G, e_H)$. Therefore

$$\begin{aligned} (A\times B)(a,b) &= (CM_{A\times B}(a,b), CN_{A\times B}(a,b)) \\ &\supset (CM_{A\times B}(e_G, e_H), CN_{A\times B}(e_G, e_H)) \\ &= (A\times B)(e_G, e_H)) \end{aligned}$$

this is contradiction with Proposition 3.7 part (1). Then at least one of the statements hold. □

Proposition 3.10. *Let* $A = (CM_A, CN_A) \in IFMS(G)$ *and* $B = (CM_B, CN_B) \in IFMS(H)$. *Let* $A \times B \in IFMSN(G \times H)$ *and* $A(x) \subseteq B(e_H)$ *for all* $x \in G$. *Then* $A \in IFMSN(G)$.

Proof. Since $A(x) = (CM_A(x), CN_A(x)) \subseteq B(e_H) = (CM_B(e_H), CN_B(e_H))$ for all $x \in G$ then $A(y) = (CM_A(y), CN_A(y)) \subseteq B(e_H)$ and $A(xy) = (CM_A(xy), CN_A(xy)) \subseteq B(e_H) = B(e_He_H) = (CM_B(e_He_H), CN_B(e_He_H))$ for all $y \in G$. Now

$$\begin{aligned} CM_A(xy) &= T(CM_A(xy), CM_B(e_He_H)) \\ &= CM_{A\times B}(xy, e_He_H) \\ &= CM_{A\times B}((x, e_H)(y, e_H)) \\ &\geq T(CM_{A\times B}(x, e_H), CM_{A\times B}(y, e_H)) \\ &= T(T(CM_A(x), CM_B(e_H)), T(CM_A(y), CM_B(e_H))) \\ &= T(CM_A(x), CM_A(y)) \end{aligned}$$

and so

$$CM_A(xy) \geq T(CM_A(x), CM_A(y)). \qquad (1)$$

Also

$$\begin{aligned} CN_A(xy) &= C(CN_A(xy), CN_B(e_H e_H)) \\ &= CN_{A\times B}(xy, e_H e_H) \\ &= CN_{A\times B}((x, e_H)(y, e_H)) \\ &\leq C(CN_{A\times B}(x, e_H), CN_{A\times B}(y, e_H)) \\ &= C(C(CN_A(x), CN_B(e_H)), C(CN_A(y), CN_B(e_H))) \\ &= C(CN_A(x), CN_A(y)) \end{aligned}$$

then

$$CN_A(xy) \leq C(CN_A(x), CN_A(y)). \qquad (2)$$

Further since $A(x) \subseteq B(e_H)$ for all $x \in G$ so $A(x^{-1}) \subseteq B(e_H)$. Thus

$$\begin{aligned} CM_A(x^{-1}) &= T(CM_A(x^{-1}), CM_A(e_H)) \\ &= T(CM_A(x^{-1}), CM_A(e_H^{-1})) \\ &= CM_{A\times B}((x, e_H)^{-1}) \\ &\geq CM_{A\times B}(x, e_H) \\ &= T(CM_A(x), CM_A(e_H)) \\ &= CM_A(x) \end{aligned}$$

and then

$$CM_A(x^{-1}) \geq CM_A(x). \qquad (3)$$

And

$$\begin{aligned} CN_A(x^{-1}) &= C(CN_A(x^{-1}), CN_A(e_H)) \\ &= C(CN_A(x^{-1}), CN_A(e_H^{-1})) \\ &= CN_{A\times B}((x, e_H)^{-1}) \\ &\leq CN_{A\times B}(x, e_H) \\ &= C(CN_A(x), CN_A(e_H)) \\ &= CN_A(x) \end{aligned}$$

thus

$$CN_A(x^{-1}) \leq CN_A(x). \qquad (4)$$

Therefore (1)-(4) give us that $A = (CM_A, CN_A) \in IFMSN(G)$. □

Proposition 3.11. *Let $A = (CM_A, CN_A) \in IFMS(G)$ and $B = (CM_B, CN_B) \in IFMS(H)$. Let $A \times B \in IFMSN(G \times H)$ and $B(x) \subseteq A(e_G)$ for all $x \in H$. Then $B \in IFMSN(H)$.*

Proof. The proof is similar to Proposition 3.10. □

Corollary 3.12. *Let $A = (CM_A, CN_A) \in IFMS(G)$ and $B = (CM_B, CN_B) \in IFMS(H)$ such that $A \times B \in IFMSN(G \times H)$. Then either $A \in IFMSN(G)$ or $B \in IFMSN(H)$.*

Proof. Using Proposition 3.9 we get that $B(e_H) \supseteq A(x)$ for all $x \in G$ or $A(e_G) \supseteq B(y)$ for all $y \in G$. Then from Proposition 3.10 and Proposition 3.11 we will have that either $A \in IFMSN(G)$ or $B \in IFMSN(H)$. □

Definition 3.13. Let $A = (CM_A, CN_A) \in IFMS(X)$ and $C = (CM_C, CN_C) \in IFMS(X)$.
(1) We say A is conjugate to B if $A(x) = B(yxy^{-1})$ for all $x, y \in X$.
(2) We say A is commutative if $A(xy) = A(yx)$ for all $x, y \in X$.

Proposition 3.14. *Let $A = (CM_A, CN_A) \in IFMSN(G)$ and $C = (CM_C, CN_C) \in IFMSN(G)$ and $B = (CM_B, CN_B) \in IFMSN(H)$ and $D = (CM_D, CN_D) \in IFMSN(H)$. If A is conjugate to B and C is conjugate to D, then $A \times C$ is conjugate to $B \times D$.*

Proof. As A is conjugate to B so $A(x) = B(kxk^{-1})$ and as C is conjugate to D so $C(y) = D(hyh^{-1})$ for all $x, y \in G$ and $k, h \in H$. Thus

$$A(x) = (CM_A(x), CN_A(x)) = B(kxk^{-1}) = (CM_B(kxk^{-1}), CN_B(kxk^{-1}))$$

and

$$C(y) = (CM_C(y), CN_C(y)) = D(hyh^{-1}) = (CM_D(hyh^{-1}), CN_D(hyh^{-1})).$$

$$\begin{aligned} CM_{A\times C}(x, y) &= T(CM_A(x), CM_C(y)) \\ &= T(CM_B(kxk^{-1}), CM_D(hyh^{-1})) \\ &= CM_{B\times D}(kxk^{-1}, hyh^{-1}) \\ &= CM_{B\times D}((k, h)(x, y)(k^{-1}, h^{-1})) \\ &= CM_{B\times D}((k, h)(x, y)(k, h)^{-1}) \end{aligned}$$

and thus

$$CM_{A\times C}(x, y) = CM_{B\times D}((k, h)(x, y)(k, h)^{-1}).$$

Also

$$\begin{aligned} CN_{A\times C}(x, y) &= C(CN_A(x), CN_C(y)) \\ &= C(CN_B(kxk^{-1}), CN_D(hyh^{-1})) \\ &= CN_{B\times D}(kxk^{-1}, hyh^{-1}) \\ &= CN_{B\times D}((k, h)(x, y)(k^{-1}, h^{-1})) \\ &= CN_{B\times D}((k, h)(x, y)(k, h)^{-1}) \end{aligned}$$

hence

$$CN_{A\times C}(x, y) = CN_{B\times D}((k, h)(x, y)(k, h)^{-1}).$$

Therefore

$$\begin{aligned} (A \times C)(x, y) &= (CM_{A\times C}(x, y), CN_{A\times C}(x, y)) \\ &= (CM_{B\times D}((k, h)(x, y)(k, h)^{-1}), CN_{B\times D}((k, h)(x, y)(k, h)^{-1})) \\ &= (B \times D)((k, h)(x, y)(k, h)^{-1}) \end{aligned}$$

then $(A \times C)(x, y) = (B \times D)((k, h)(x, y)(k, h)^{-1})$ and thus $A \times C$ will be conjugate to $B \times D$. □

Proposition 3.15. *Let $A = (CM_A, CN_A) \in IFMSN(G)$ and $B = (CM_B, CN_B) \in IFMSN(H)$. Then A and B are commutative if and only if $A \times B$ is a commutative.*

Proof. Let $x_1, y_1 \in G$ and $x_2, y_2 \in H$ such that $x = (x_1, x_2) \in G \times H$ and $y = (y_1, y_2) \in G \times H$. Let A and B are commutative then $A(x_1y_1) = A(y_1x_1)$ and $B(x_2y_2) = B(y_2x_2)$. Thus

$$A(x_1y_1) = (CM_A(x_1y_1), CN_A(x_1y_1)) = A(y_1x_1) = (CM_A(y_1x_1), CN_A(y_1x_1))$$

and

$$B(x_1y_1) = (CM_B(x_1y_1), CN_B(x_1y_1)) = B(y_1x_1) = (CM_B(y_1x_1), CN_B(y_1x_1)).$$

Then

$$\begin{aligned} CM_{A\times B}(xy) &= CM_{A\times B}((x_1, x_2)(y_1, y_2)) \\ &= CM_{A\times B}(x_1y_1, x_2y_2) \\ &= T(CM_A(x_1y_1), CM_B(x_2y_2)) \\ &= T(CM_A(y_1x_1), CM_B(y_2x_2)) \\ &= CM_{A\times B}(y_1x_1, y_2x_2) \\ &= CM_{A\times B}((y_1, y_2)(x_1, x_2)) \\ &= CM_{A\times B}(yx) \end{aligned}$$

thus $CM_{A\times B}(xy) = CM_{A\times B}(yx)$. Also

$$\begin{aligned} CN_{A\times B}(xy) &= CN_{A\times B}((x_1, x_2)(y_1, y_2)) \\ &= CN_{A\times B}(x_1y_1, x_2y_2) \\ &= C(CN_A(x_1y_1), CN_B(x_2y_2)) \\ &= C(CN_A(y_1x_1), CN_B(y_2x_2)) \\ &= CN_{A\times B}(y_1x_1, y_2x_2) \\ &= CN_{A\times B}((y_1, y_2)(x_1, x_2)) \\ &= CN_{A\times B}(yx) \end{aligned}$$

then $CN_{A\times B}(xy) = CN_{A\times B}(yx)$. Therefore

$$(A \times B)(xy) = (CM_{A\times B}(xy), CN_{A\times B}(xy)) = (CM_{A\times B}(yx), CN_{A\times B}(yx)) = (A \times B)(yx)$$

and then $A \times B$ is a commutative.
Conversely, suppose that $A \times B$ is a commutative. Then

$$(A \times B)(xy) = (A \times B)(yx)$$

if and only if

$$CM_{A\times B}((x_1, x_2)(y_1, y_2)) = CM_{A\times B}((y_1, y_2)(x_1, x_2))$$

and

$$CN_{A\times B}((x_1, x_2)(y_1, y_2)) = CN_{A\times B}((y_1, y_2)(x_1, x_2))$$

if and only if

$$CM_{A\times B}(x_1y_1, x_2y_2) = CM_{A\times B}(y_1x_1, y_2x_2)$$

and

$$CN_{A\times B}(x_1y_1, x_2y_2) = CN_{A\times B}(y_1x_1, y_2x_2)$$

if and only if

$$T(CM_A(x_1y_1), CM_B(x_2y_2)) = T(CM_A(y_1x_1), CM_B(y_2x_2))$$

and

$$C(CN_A(x_1y_1), CN_B(x_2y_2)) = C(CN_A(y_1x_1), CN_B(y_2x_2))$$

if and only if

$$CM_A(x_1y_1) = CM_A(y_1x_1) \text{ and } CM_B(x_2y_2) = CM_B(y_2x_2)$$

and

$$CN_A(x_1y_1) = CN_A(y_1x_1) \text{ and } CN_B(x_2y_2) = CN_B(y_2x_2).$$

Thus

$$A(x_1y_1) = (CM_A(x_1y_1), CN_A(x_1y_1)) = (CM_A(y_1x_1), CN_A(y_1x_1)) = A(y_1x_1)$$

and

$$B(x_2y_2) = (CM_B(x_2y_2), CN_B(x_2y_2)) = (CM_B(y_2x_2), CN_B(y_2x_2)) = B(y_2x_2)$$

hence $A = (CM_A, CN_A)$ and $B = (CM_B, CN_B)$ will be commutatives. □

Definition 3.16. Let $G \times H$ and $I \times J$ be groups and $f : G \times H \to I \times J$ be a homomorphism. Let $A = (CM_A, CN_A) \in IFMS(G)$ and $B = (CM_B, CN_B) \in IFMS(H)$ and $C = (CM_C, CN_C) \in IFMS(I)$ and $D = (CM_D, CN_D) \in IFMS(J)$ such that $A \times B \in IFMS(G \times H)$ and $C \times D \in IFMS(I \times J)$. Define $f(A \times B) \in IFMS(I \times J)$ as

$$f(A \times B) = f(CM_{A\times B}, CN_{A\times B}) = (f(CM_{A\times B}), f(CN_{A\times B})) = (CM_{f(A\times B)}, CN_{f(A\times B)})$$

such that for all $(i, j) \in I \times J$

$$f(CM_{A\times B})(i, j) = (CM_{f(A\times B)})(i, j)$$

$$= \begin{cases} \sup\{CM_{A\times B}(g, h) \mid (g, h) \in G \times H, f(g, h) = (i, j)\} & \text{if } f^{-1}(i, j) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$f(CN_{A\times B})(i, j) = (CN_{f(A\times B)})(i, j)$$

$$= \begin{cases} \inf\{CN_{A\times B}(g, h) \mid (g, h) \in G \times H, f(g, h) = (i, j)\} & \text{if } f^{-1}(i, j) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Also we define $f^{-1}(C \times D) \in IFMS(G \times H)$ as

$$f^{-1}(C\times D) = f^{-1}(CM_{C\times D}, CN_{C\times D}) = (f^{-1}(CM_{C\times D}), f^{-1}(CN_{C\times D})) = (CM_{f^{-1}(C\times D)}, CN_{f^{-1}(C\times D)})$$

such that for all $(g, h) \in G \times H$

$$f^{-1}(C\times D)(g, h) = (CM_{f^{-1}(C\times D)}(g, h), CN_{f^{-1}(C\times D)}(g, h)) = (CM_{C\times D}(f(g, h)), CN_{C\times D}(f(g, h))).$$

Proposition 3.17. *Let $G\times H$ and $I \times J$ be groups and $f : G\times H \to I \times J$ be an epimorphism. If $A = (CM_A, CN_A) \in IFMSN(G)$ and $B = (CM_B, CN_B) \in IFMSN(H)$ and $A \times B \in IFMSN(G \times H)$, then $f(A \times B) \in IFMSN(I \times J)$.*

Proof. Let $X = (i_1, j_1) \in I \times J$ and $Y = (i_2, j_2) \in I \times J$ such that

$$f^{-1}(XY) = f^{-1}((i_1, j_1)(i_2, j_2)) = f^{-1}(i_1 i_2, j_1 j_2) \neq \emptyset.$$

Then

(1)

$$\begin{aligned}
CM_{f(A\times B)}(XY) &= CM_{f(A\times B)}((i_1, j_1)(i_2, j_2)) = CM_{f(A\times B)}(i_1 i_2, j_1 j_2) \\
&= \sup\{CM_{A\times B}(g_1g_2, h_1h_2) \mid g_1, g_2 \in G, h_1, h_2 \in H, f(g_1g_2, h_1h_2) = (i_1i_2, j_1j_2)\} \\
&= \sup\{CM_{A\times B}(g_1g_2, h_1h_2) \mid g_1, g_2 \in G, h_1, h_2 \in H, (f(g_1g_2), f(h_1h_2)) = (i_1i_2, j_1j_2)\} \\
&= \sup\{CM_{A\times B}(g_1g_2, h_1h_2) \mid g_1, g_2 \in G, h_1, h_2 \in H, f(g_1g_2) = i_1i_2, f(h_1h_2) = j_1j_2\} \\
&= \sup\{T(CM_A(g_1g_2), CM_B(h_1h_2)) \mid g_1, g_2 \in G, h_1, h_2 \in H, f(g_1g_2) = i_1i_2, f(h_1h_2) = j_1j_2\} \\
&\geq \sup\{T(T(CM_A(g_1), CM_A(g_2)), T(CM_B(h_1), CM_B(h_2))) \mid f(g_1g_2) = i_1i_2, f(h_1h_2) = j_1j_2\} \\
&= \sup\{T(T(CM_A(g_1), CM_B(h_1)), T(CM_A(g_2), CM_B(h_2))) \mid f(g_1g_2) = i_1i_2, f(h_1h_2) = j_1j_2\} \\
&= \sup\{T(T(CM_A(g_1), CM_B(h_1)), T(CM_A(g_2), CM_B(h_2))) \\
&\qquad \mid f(g_1) = i_1, f(g_2) = i_2, f(h_1) = j_1, f(h_2) = j_2\} \\
&= \sup\{T(CM_{A\times B}(g_1, h_1), CM_{A\times B}(g_2, h_2)) \mid f(g_1) = i_1, f(g_2) = i_2, f(h_1) = j_1, f(h_2) = j_2\} \\
&= T(\sup\{CM_{A\times B}(g_1, h_1) \mid f(g_1, h_1) = (i_1, j_1)\}, \sup\{CM_{A\times B}(g_2, h_2) \mid f(g_2, h_2) = (i_2, j_2)\}) \\
&= T(CM_{f(A\times B)}(i_1, j_1), CM_{f(A\times B)}(i_2, j_2)) = T(CM_{f(A\times B)}(X), CM_{f(A\times B)}(Y))
\end{aligned}$$

thus

$$CM_{f(A\times B)}(XY) \geq T(CM_{f(A\times B)}(X), f(A \times B)(Y)).$$

(2)

$$\begin{aligned}
CN_{f(A\times B)}(XY) &= CN_{f(A\times B)}((i_1, j_1)(i_2, j_2)) = CN_{f(A\times B)}(i_1 i_2, j_1 j_2) \\
&= \sup\{CN_{A\times B}(g_1g_2, h_1h_2) \mid g_1, g_2 \in G, h_1, h_2 \in H, f(g_1g_2, h_1h_2) = (i_1i_2, j_1j_2)\} \\
&= \inf\{CN_{A\times B}(g_1g_2, h_1h_2) \mid g_1, g_2 \in G, h_1, h_2 \in H, (f(g_1g_2), f(h_1h_2)) = (i_1i_2, j_1j_2)\} \\
&= \inf\{CN_{A\times B}(g_1g_2, h_1h_2) \mid g_1, g_2 \in G, h_1, h_2 \in H, f(g_1g_2) = i_1i_2, f(h_1h_2) = j_1j_2\} \\
&= \inf\{C(CN_A(g_1g_2), CM_B(h_1h_2)) \mid g_1, g_2 \in G, h_1, h_2 \in H, f(g_1g_2) = i_1i_2, f(h_1h_2) = j_1j_2\} \\
&\leq \inf\{C(C(CN_A(g_1), CN_A(g_2)), C(CN_B(h_1), CN_B(h_2))) \mid f(g_1g_2) = i_1i_2, f(h_1h_2) = j_1j_2\} \\
&= \inf\{C(C(CN_A(g_1), CN_B(h_1)), C(CN_A(g_2), CN_B(h_2))) \mid f(g_1g_2) = i_1i_2, f(h_1h_2) = j_1j_2\} \\
&= \inf\{C(C(CN_A(g_1), CN_B(h_1)), C(CN_A(g_2), CN_B(h_2))) \\
&\qquad \mid f(g_1) = i_1, f(g_2) = i_2, f(h_1) = j_1, f(h_2) = j_2\} \\
&= \inf\{C(CN_{A\times B}(g_1, h_1), CN_{A\times B}(g_2, h_2)) \mid f(g_1) = i_1, f(g_2) = i_2, f(h_1) = j_1, f(h_2) = j_2\} \\
&= C(\inf\{CN_{A\times B}(g_1, h_1) \mid f(g_1, h_1) = (i_1, j_1)\}, \inf\{CN_{A\times B}(g_2, h_2) \mid f(g_2, h_2) = (i_2, j_2)\}) \\
&= C(CN_{f(A\times B)}(i_1, j_1), CN_{f(A\times B)}(i_2, j_2)) = C(CN_{f(A\times B)}(X), CN_{f(A\times B)}(Y))
\end{aligned}$$

thus

$$CN_{f(A\times B)}(XY) \leq C(CN_{f(A\times B)}(X), f(A \times B)(Y)).$$

Let $X = (i, j) \in I \times J$ then

(3)

$$\begin{aligned}
CM_{f(A\times B)}(X^{-1}) &= CM_{f(A\times B)}((i, j)^{-1}) = CM_{f(A\times B)}(i^{-1}, j^{-1}) \\
&= \sup\{CM_{A\times B}(g^{-1}, h^{-1}) \mid g \in G, h \in H, f(g^{-1}, h^{-1}) = (i^{-1}, j^{-1})\} \\
&= \sup\{CM_{A\times B}(g^{-1}, h^{-1}) \mid g \in G, h \in H, (f(g^{-1}), f(h^{-1})) = (i^{-1}, j^{-1})\} \\
&= \sup\{CM_{A\times B}(g^{-1}, h^{-1}) \mid g \in G, h \in H, f(g^{-1}) = i^{-1}, f(h^{-1})) = j^{-1}\} \\
&= \sup\{T(CM_A(g^{-1}), CM_B(h^{-1})) \mid g \in G, h \in H, f(g^{-1}) = i^{-1}, f(h^{-1})) = j^{-1}\}
\end{aligned}$$

$$\geq \sup\{T(CM_A(g), CM_B(h)) \mid g \in G, h \in H, f^{-1}(g) = i^{-1}, f^{-1}(h) = j^{-1}\}$$
$$= \sup\{T(CM_A(g), CM_B(h)) \mid g \in G, h \in H, f(g) = i, f(h) = j\}$$
$$= \sup\{CM_{A\times B}(g, h) \mid (g, h) \in G \times H, f(g, h) = (i, j)\}$$
$$= CM_{f(A\times B)}(i, j) = CM_{f(A\times B)}(X)$$

and then

$$CM_{f(A\times B)}(X^{-1}) \geq CM_{f(A\times B)}(X).$$

(4)

$$CN_{f(A\times B)}(X^{-1}) = CN_{f(A\times B)}((i, j)^{-1}) = CN_{f(A\times B)}(i^{-1}, j^{-1})$$
$$= \inf\{CN_{A\times B}(g^{-1}, h^{-1}) \mid g \in G, h \in H, f(g^{-1}, h^{-1}) = (i^{-1}, j^{-1})\}$$
$$= \inf\{CN_{A\times B}(g^{-1}, h^{-1}) \mid g \in G, h \in H, (f(g^{-1}), f(h^{-1})) = (i^{-1}, j^{-1})\}$$
$$= \inf\{CN_{A\times B}(g^{-1}, h^{-1}) \mid g \in G, h \in H, f(g^{-1}) = i^{-1}, f(h^{-1})) = j^{-1}\}$$
$$= \inf\{C(CN_A(g^{-1}), CN_B(h^{-1})) \mid g \in G, h \in H, f(g^{-1}) = i^{-1}, f(h^{-1})) = j^{-1}\}$$
$$\leq \inf\{C(CN_A(g), CN_B(h)) \mid g \in G, h \in H, f^{-1}(g) = i^{-1}, f^{-1}(h) = j^{-1}\}$$
$$= \inf\{C(CN_A(g), CN_B(h)) \mid g \in G, h \in H, f(g) = i, f(h) = j\}$$
$$= \inf\{CN_{A\times B}(g, h) \mid (g, h) \in G \times H, f(g, h) = (i, j)\}$$
$$= CN_{f(A\times B)}(i, j) = CM_{f(A\times B)}(X)$$

and then

$$CN_{f(A\times B)}(X^{-1}) \leq CN_{f(A\times B)}(X).$$

Therefore $f(A \times B) = (CM_{f(A\times B)}, CN_{f(A\times B)}) \in IFMSN(I \times J)$. □

Proposition 3.18. *Let $G\times H$ and $I\times J$ be groups and $f : G\times H \to I\times J$ be a homomorphism. If $C = (CM_C, CN_C) \in IFMSN(I)$ and $D = (CM_D, CN_D) \in IFMSN(J)$ and $C \times D \in IFMSN(I \times J)$, then $f^{-1}(C \times D) \in IFMSN(G \times H)$.*

Proof. Let $X = (g_1, h_1) \in G \times H$ and $Y = (g_2, h_2) \in G \times H$. Then
(1)

$$CM_{f^{-1}(C\times D)}(XY) = CM_{f^{-1}(C\times D)}((g_1, h_1)(g_2, h_2))$$
$$= CM_{f^{-1}(C\times D)}(g_1g_2, h_1h_2)) = CM_{C\times D}(f(g_1g_2, h_1h_2))$$
$$= CM_{C\times D}(f(g_1g_2), f(h_1h_2)) = T(CM_C(f(g_1g_2)), CM_D(f(h_1h_2)))$$
$$= T(CM_C(f(g_1)f(g_2)), CM_D(f(h_1)f(h_2)))$$
$$\geq T(T(CM_C(f(g_1)), CM_C(f(g_2))), T(CM_D(f(h_1)), CM_D(f(h_2)))$$
$$= T(T(CM_C(f(g_1)), CM_D(f(h_1))), T(CM_C(f(g_2), CM_D(f(h_2))) \text{ (Lemma 2.10)}$$
$$= T(CM_{C\times D}(f(g_1), f(h_1)), CM_{C\times D}(f(g_2), f(h_2)))$$
$$= T(CM_{C\times D}(f(g_1, h_1)), CM_{C\times D}(f(g_2, h_2)))$$
$$= T(CM_{f^{-1}(C\times D)}(g_1, h_1), CM_{f^{-1}(C\times D)}(g_2, h_2))$$
$$= T(CM_{f^{-1}(C\times D)}(X), CM_{f^{-1}(C\times D)}(Y))$$

and then

$$CM_{f^{-1}(C\times D)}(XY) \geq T(CM_{f^{-1}(C\times D)}(X), CM_{f^{-1}(C\times D)}(Y)).$$

(2)

$$CN_{f^{-1}(C\times D)}(XY) = CN_{f^{-1}(C\times D)}((g_1, h_1)(g_2, h_2))$$
$$= CN_{f^{-1}(C\times D)}(g_1g_2, h_1h_2)) = CN_{C\times D}(f(g_1g_2, h_1h_2))$$
$$= CN_{C\times D}(f(g_1g_2), f(h_1h_2)) = C(CN_C(f(g_1g_2)), CN_D(f(h_1h_2)))$$

$$= C(CN_C(f(g_1)f(g_2)), CN_D(f(h_1)f(h_2)))$$

$$\leq C(C(CN_C(f(g_1)), CN_C(f(g_2))), C(CN_D(f(h_1)), CN_D(f(h_2)))$$

$$= C(C(CN_C(f(g_1)), CN_D(f(h_1))), C(CN_C(f(g_2), CN_D(f(h_2))) \text{ (Lemma 2.10)}$$

$$= C(CN_{C\times D}(f(g_1), f(h_1)), CN_{C\times D}(f(g_2), f(h_2)))$$

$$= C(CN_{C\times D}(f(g_1, h_1)), CN_{C\times D}(f(g_2, h_2)))$$

$$= C(CN_{f^{-1}(C\times D)}(g_1, h_1), CM_{f^{-1}(C\times D)}(g_2, h_2))$$

$$= C(CN_{f^{-1}(C\times D)}(X), CM_{f^{-1}(C\times D)}(Y))$$

hence

$$CN_{f^{-1}(C\times D)}(XY) \leq C(CM_{f^{-1}(C\times D)}(X), CN_{f^{-1}(C\times D)}(Y)).$$

Let $X = (g, h) \in G \times H$. Then

(3)

$$CM_{f^{-1}(C\times D)}(X^{-1}) = CM_{f^{-1}(C\times D)}((g_1, h_1)^{-1}) = CM_{C\times D}(f(g, h)^{-1})$$

$$= CM_{C\times D}(f(g^{-1}, h^{-1})) = CM_{C\times D}(f^{-1}(g), f^{-1}(h)) = T(CM_C(f^{-1}(g)), CM_D(f^{-1}(h)))$$

$$\geq T(CM_C(f(g)), CM_D(f(h))) = CM_{C\times D}(f(g), f(h)) = CM_{C\times D}(f(g, h))$$

$$= CM_{f^{-1}(C\times D)}(g, h) = CM_{f^{-1}(C\times D)}(X)$$

and then

$$CM_{f^{-1}(C\times D)}(X^{-1}) \geq CM_{f^{-1}(C\times D)}(X).$$

(4)

$$CN_{f^{-1}(C\times D)}(X^{-1}) = CN_{f^{-1}(C\times D)}((g_1, h_1)^{-1}) = CN_{C\times D}(f(g, h)^{-1})$$

$$= CN_{C\times D}(f(g^{-1}, h^{-1})) = CN_{C\times D}(f^{-1}(g), f^{-1}(h)) = C(CN_C(f^{-1}(g)), CN_D(f^{-1}(h)))$$

$$\leq C(CN_C(f(g)), CN_D(f(h))) = CN_{C\times D}(f(g), f(h)) = CN_{C\times D}(f(g, h))$$

$$= CN_{f^{-1}(C\times D)}(g, h) = CN_{f^{-1}(C\times D)}(X)$$

and then

$$CN_{f^{-1}(C\times D)}(X^{-1}) \leq CN_{f^{-1}(C\times D)}(X).$$

Hence $f^{-1}(C \times D) = (CM_{f^{-1}(C\times D)}, CN_{f^{-1}(C\times D)}) \in IFMSN(G \times H)$. □

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(Rasul Rasuli) Department of Mathematics, Payame Noor University(PNU), P. O. Box 19395-4697, Tehran, Iran.
Email address: Rasuli@pnu.ac.ir

(Amir Shomali) Department of Mathematics, Sanandaj Branch, Islamic Azad University, Sanandaj, Iran.
Email address: shomali.math@gmail.com