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Research Paper

# MAPS COMPLETELY PRESERVING ZERO TRIPLE JORDAN PRODUCT OF OPERATORS

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two standard operator algebras on Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. In this paper, we determine the forms of the surjective maps from $\mathcal{A}$ onto $\mathcal{B}$ such that completely preserve zero triple Jordan product in both directions.

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**Keywords:** Preserving problem, Completely preserving problem, Triple Jordan product, Operator algebra.

## 1. Introduction and Background

Mappings that preserve a certain property is a subject that has attracted the attention of many mathematicians and they seek to obtain other properties of these maps as well as their forms. In the field of preserving problems, the issue of maps that completely preserve a specific property has recently been taken into consideration. For example, you can see papers [1-5].

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $\mathcal{B}(\mathcal{X})$ denote the Banach algebra of all bounded linear operators on $\mathcal{X}$. Let $\mathcal{S} \subseteq B(X)$ and $\mathcal{T} \subseteq B(Y)$ be linear subspaces and $\phi : \mathcal{S} \longrightarrow \mathcal{T}$ be a map. Define for each $n \in \mathbb{N}$, a map $\phi_n : \mathcal{S} \otimes \mathcal{M}_n(\mathbb{F}) \longrightarrow \mathcal{T} \otimes \mathcal{M}_n(\mathbb{F})$ by

$$\phi_n((s_{ij})_{n\times n}) = (\phi(s_{ij}))_{n\times n} \qquad (\forall s_{ij} \in \mathcal{S}).$$

Let $(p)$ be a property. We say that $\phi$ preserves $n-(p)$, whenever $\phi_n$ preserves $(p)$ and $\phi$ completely preserves $(p)$, whenever $\phi_n$ preserves $(p)$ for each $n$.

Recall that a standard operator algebra on $\mathcal{X}$ is a norm closed subalgebra of $\mathcal{B}(\mathcal{X})$ which contains the identity and all finite rank operators. Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on Banach spaces $X$ and $Y$, respectively. Recently in [3] completely idempotent and completely square-zero preserving maps and in [4] completely commutativity (in fact completely Lie zero product) and completely Jordan zero product preserving maps are discussed.

Triple Jordan product of two operators $A, B \in \mathcal{A}$ is defined as $ABA$. In this paper, surjective maps from $\mathcal{A}$ onto $\mathcal{B}$ such that completely preserve zero triple Jordan product, are determined.

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Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a map. If for every $A_{ij}, B_{ij} \in \mathcal{A}$, $1 \le i, j \le n$ we have

$$\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \times \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & & \\ B_{n1} & \cdots & B_{nn} \end{bmatrix} \times \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} \times\phi(A_{11}) & \cdots & \phi(A_{1n}) \\ \vdots & & \\ \phi(A_{n1}) & \cdots & \phi(A_{nn}) \end{bmatrix} \times \begin{bmatrix} \phi(B_{11}) & \cdots & \phi(B_{1n}) \\ \vdots & & \\ \phi(B_{n1}) & \cdots & \phi(B_{nn}) \end{bmatrix} \times$$

$$\begin{bmatrix} \phi(A_{11}) & \cdots & \phi(A_{1n}) \\ \vdots & & \\ \phi(A_{n1}) & \cdots & \phi(A_{nn}) \end{bmatrix} = 0$$

for each $n$, then we say that $\phi$ completely preserves zero triple Jordan product of operators in both directions. Our main theorems of this paper are as following.

**Theorem 1.1.** *Let $\mathcal{X}$, $\mathcal{Y}$ be infinite dimensional Banach spaces and $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective map. Then the following statements are equivalent:*

(1) *$\phi$ is completely preserving zero triple Jordan product in both directions.*

(2) *$\phi$ is 2-zero triple Jordan product preserving in both directions.*

(3) *There exist a bounded invertible linear or (in the complex case) conjugate- linear operator $A : \mathcal{X} \to \mathcal{Y}$ and a scalar $\lambda$ such that*

$$\phi(T) = \lambda A T A^{-1},$$

*for all $T \in \mathcal{A}$.*

**Theorem 1.2.** *Let $\phi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ $(n \ge 3)$ be a surjective map, where $\mathbb{F}$ is the real or complex field. Then the following statements are equivalent:*

(1) *$\phi$ is completely preserving zero triple Jordan product in both directions.*

(2) *$\phi$ is 2-zero triple Jordan product preserving in both directions.*

(3) *There exist an invertible matrix $A \in \mathcal{M}_n$, a scalar $\lambda$ and an automorphism $\tau : \mathbb{F} \to \mathbb{F}$ such that*

$$\phi(T) = \lambda A T_\tau A^{-1},$$

*for all $T \in \mathcal{M}_n(\mathbb{F})$. Here $T_\tau = (\tau(t_{ij}))$ for $T = (t_{ij})$.*

## 2. **Main Results**

Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a map. If for every $A_{ij} \in \mathcal{A}$, $1 \le i, j \le n$ we have

$$\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}^2 = 0 \Leftrightarrow \begin{bmatrix} \times\phi(A_{11}) & \cdots & \phi(A_{1n}) \\ \vdots & & \\ \phi(A_{n1}) & \cdots & \phi(A_{nn}) \end{bmatrix}^2 = 0$$

for each $n$, then we say that $\phi$ completely preserves square-zero operators in both directions. If for every $A_{ij} \in \mathcal{A}$, $1 \le i, j \le 2$ we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^2 = 0 \Leftrightarrow \begin{bmatrix} \phi(A_{11}) & \phi(A_{12}) \\ \phi(A_{21}) & \phi(A_{22}) \end{bmatrix}^2 = 0,$$

then we say that $\phi$ preserves 2-square-zero operators in both directions. See the following result from [3]. We use from the following theorems in the proof of our main results.

**Theorem 2.1.** [3] *Let $\mathcal{X}$, $\mathcal{Y}$ be infinite dimensional Banach spaces and $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective map. Then the following statements are equivalent:*

(1) *$\phi$ is completely square-zero preserving in both directions.*

(2) *$\phi$ is 2-square-zero preserving operators in both directions.*

(3) *There exist a bounded invertible linear or (in the complex case) conjugate- linear operator $A : \mathcal{X} \to \mathcal{Y}$ and a scalar $c$ such that*

$$\phi(T) = cATA^{-1},$$

*for all $T \in \mathcal{A}$.*

**Proposition 2.2.** [3] *Let $\phi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ $(n \geq 3)$ be a surjective map, where $\mathbb{F}$ is the real or complex field. Then the following statements are equivalent:*

(1) *$\phi$ is completely square-zero preserving in both directions.*

(2) *$\phi$ is 2-square-zero preserving in both directions.*

(3) *There exist an invertible matrix $A \in \mathcal{M}_n$, a scalar $c$ and an automorphism $\tau : \mathbb{F} \to \mathbb{F}$ such that*

$$\phi(T) = cAT_\tau A^{-1},$$

*for all $T \in \mathcal{M}_n(\mathbb{F})$. Here $T_\tau = (\tau(t_{ij}))$ for $T = (t_{ij})$.*

In order to prove the main theorems, we need some auxiliary lemmas. In the following lemmas suppose that $\mathcal{A}$ and $\mathcal{B}$ are two standard operator algebra on Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ with dimensions greater than 2, respectively. Also assume that $\phi : \mathcal{A} \to \mathcal{B}$ is a surjection preserving 2-zero triple Jordan product in both directions.

We denote by $\mathcal{X}^*$ the dual space of $\mathcal{X}$. For every nonzero $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, the symbol $x \otimes f$ stands for the rank one linear operator on $\mathcal{X}$ defined by $(x \otimes f)y = f(y)x$ for any $y \in \mathcal{X}$. Note that every rank one operator in $\mathcal{B}(\mathcal{X})$ can be written in this way.

**Lemma 2.3.** $\phi(0) = 0$.

*Proof.* For any $T \in \mathcal{A}$,

$$\begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} = 0_{\mathcal{A} \otimes \mathcal{M}_2}$$

which implies

$$\begin{bmatrix} \phi(T) & \phi(0) \\ \phi(0) & \phi(0) \end{bmatrix} \begin{bmatrix} \phi(0) & \phi(0) \\ \phi(0) & \phi(0) \end{bmatrix} \begin{bmatrix} \phi(T) & \phi(0) \\ \phi(0) & \phi(0) \end{bmatrix} = 0_{\mathcal{B} \otimes \mathcal{M}_2}.$$

So we obtain

$$\phi(T)\phi(0)\phi(T) + \phi(0)^2\phi(T) + \phi(T)\phi(0)^2 + \phi(0)^3 = 0 \tag{2.1}$$

$$2\phi(0)^2\phi(T) + 2\phi(0)^3 = 0. \tag{2.2}$$

Since $\phi$ is surjective, then there exist $T_0, T_1 \in \mathcal{A}$ such that $\phi(T_0) = 0$ and $\phi(T_1) = I$. Setting $T = T_0$ in (2.1) yields that $\phi(0)^3 = 0$. This together with setting $T = T_1$ in (2.1) yields that

$$\phi(0) + 2\phi(0)^2 = 0 \tag{2.3}$$

and also from (2.2) we have $2\phi(0)^2\phi(T) = 0$ which by setting $T = T_1$, $\phi(0)^2 = 0$. This and (2.3) follows that $\phi(0) = 0$. ∎

**Lemma 2.4.** *$\phi(-S) = -\phi(-S)$ for every $S \in \mathcal{A}$.*

*Proof.* For any $T, S \in \mathcal{A}$,

$$\begin{bmatrix} T & T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S & 0 \\ -S & 0 \end{bmatrix} \begin{bmatrix} T & T \\ 0 & 0 \end{bmatrix} = 0_{\mathcal{A}\otimes\mathcal{M}_2}$$

which implies

$$\begin{bmatrix} \phi(T) & \phi(T) \\ \phi(0) & \phi(0) \end{bmatrix} \begin{bmatrix} \phi(S) & \phi(0) \\ \phi(-S) & \phi(0) \end{bmatrix} \begin{bmatrix} \phi(T) & \phi(T) \\ \phi(0) & \phi(0) \end{bmatrix} = 0_{\mathcal{B}\otimes\mathcal{M}_2}.$$

So we obtain

$$\phi(T)\phi(S)\phi(T) + \phi(T)\phi(-S)\phi(T) = 0. \tag{2.4}$$

From surjectivity of $\phi$ there exists a $T_1 \in \mathcal{A}$ such that $\phi(T_1) = I$. Setting $T = T_1$ in (2.4) yields that $\phi(-S) = -\phi(S)$. □

**Lemma 2.5.** *$\phi(I) = \lambda I$ for some constant $\lambda$.*

*Proof.* For any $T \in \mathcal{A}$,

$$\begin{bmatrix} T & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -T & 0 \end{bmatrix} \begin{bmatrix} T & I \\ 0 & 0 \end{bmatrix} = 0_{\mathcal{A}\otimes\mathcal{M}_2}$$

which implies

$$\begin{bmatrix} \phi(T) & \phi(I) \\ \phi(0) & \phi(0) \end{bmatrix} \begin{bmatrix} \phi(I) & \phi(0) \\ \phi(-T) & \phi(0) \end{bmatrix} \begin{bmatrix} \phi(T) & \phi(I) \\ \phi(0) & \phi(0) \end{bmatrix} = 0_{\mathcal{B}\otimes\mathcal{M}_2}.$$

So we obtain

$$\phi(T)\phi(I)^2 = \phi(I)\phi(T)\phi(I). \tag{2.5}$$

Let $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$ be arbitrary non-zero elements. From surjectivity of $\phi$ there exists a $S \in \mathcal{A}$ such that $\phi(S) = x \otimes f$. Setting $T = S$ in (2.5) yields that $x \otimes f\phi(I)^2 = \phi(I)x \otimes f\phi(I)$ which implies that $\phi(I)x$ and $x$ are linearly dependent and so $\phi(I)$ is a mutiple of $I$. □

**Lemma 2.6.** *$\phi$ is 2-square-zero preserving operators in both directions.*

*Proof.* If for every $A_{ij} \in \mathcal{A}$, $1 \le i, j \le 2$ we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^2 = 0$$

then

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = 0$$

which by lemmas 2.3, 2.5 and assumption

$$\begin{bmatrix} \phi(A_{11}) & \phi(A_{12}) \\ \phi(A_{21}) & \phi(A_{22}) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \phi(A_{11}) & \phi(A_{12}) \\ \phi(A_{21}) & \phi(A_{22}) \end{bmatrix} = 0.$$

Therefore

$$\begin{bmatrix} \phi(A_{11}) & \phi(A_{12}) \\ \phi(A_{21}) & \phi(A_{22}) \end{bmatrix}^2 = 0$$

and this completes the proof. □

*Proof of Theorems 1.1 and 1.2:* Clearly, we have (3) ⇒ (1) ⇒ (2). We only need to show (2) ⇒ (3). By Lemma 2.6 $\phi$ is 2-square-zero preserving operators in both directions and so using Theorem 2.1 and Proposition 2.2 follows the assertions.

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