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Research Paper

INFINITELY MANY SOLUTIONS FOR A CLASS OF NONLINEAR FRACTIONAL EQUATIONS WITH IMPULSIVE EFFECTS

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Abstract. The existence of infinitely many solutions for a class of impulsive fractional boundary value problems is established. Our approach is based on recent variational methods for smooth functionals defined on reflexive Banach spaces. Some recent results are extended and improved. One example is given in this paper to illustrate the main results.

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1. Introduction and Background

The aim of this paper is to investigate the existence of infinitely many classical solutions for the following nonlinear impulsive fractional boundary value problem (BVP, for short):

$$(P_\lambda^f)\qquad \begin{cases} {}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right)+a(t)u(t)=\lambda f(t,u(t)), \quad t\neq t_j,\ \text{a.e. } t\in[0,T],\\ \Delta\left({}_tD_T^{\alpha-1}\left({}_0^cD_t^{\alpha}u\right)\right)(t_j)=I_j(u(t_j)),\quad j=1,\dots m,\\ u(0)=u(T)=0 \end{cases}$$

where $\alpha\in(\frac{1}{2},1]$, $a\in C([0,T])$ such that there are $a_0,\ a_1>0$ such that $0<a_0\leq a(t)\leq a_1$, $\lambda>0$, $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is an L^1-Carathéodory function, $0=t_0<t_1<\cdots<t_n<t_{n+1}=T$, $\Delta({}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right))(t_j)={}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right)(t_j^+)-{}_tD_T^{\alpha}\left({}_0^cD_t^{\alpha}u(t)\right)(t_j^-)$ and $I_j:\mathbb{R}\to\mathbb{R}$, $j=1,\dots,m$ are continuous.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium or polymer rheology. On this kind of equations the derivatives of fractional order [18, 19, 20] are involved. The interest of the study of fractional-order differential equations lies in the fact that fractional-order models are more accurate than integer-order models, that is, there are more degrees of freedom in the fractional-order models. Furthermore, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes due to the existence of a “memory” term in a model. This memory term insures the history and its impact to the present and future, see [26]. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [4, 6, 8, 10] and the references therein.

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In recent years, a great deal of work has been done in the study of the existence of solutions for impulsive boundary value problems (IBVPs for short), by which a number of chemotherapy, population dynamics, optimal control, ecology, industrial robotics and physics phenomena are described. For the general aspects of impulsive differential equations, we refer the reader to the classical monograph [22]. For some general and recent works on impulsive differential equations studied via variational methods and critical point theory, we refer the reader to [1, 2, 13, 14, 15, 16] and the references therein.

Due to the great development in the theory of fractional calculus and impulsive differential equations as well as having wide applications in several fields. On the other hand, in recent years, some researchers have used variational methods to study the existence of solutions for fractional differential equations with impulses, see for instance, [5, 7, 11] and the references therein for detailed discussions.

Motivated by the above works, in this paper, by using some critical theorems obtained in [23, Theorem 9.12] which we recall in the next section (see Theorem 2.1), under Ambrosetti-Rabinowitz condition (AR) on the nonlinear term and impulsive functions we discuss the existence of infinitely many classical solutions for the problem (P_λ^f) (see Theorem 3.1). We present Example 3.5, in which the hypotheses of Theorem 3.1 are fulfilled. In Theorem 3.6 we discuss the existence of infinitely many solutions for the problem (P_λ^f) when the nonlinear term is superlinear.

2. Preliminaries

In this section, we formulate our main results on the existence infinitely many weak solutions for the problem (P_λ^f). Our main tool to ensure the results is Theorem 9.12 of [23] that we now recall here.

Theorem 2.1. [23, Theorem 9.12] *Let X be an infinite dimensional real Banach space. Let $\varphi \in C^1(X,\mathbb{R})$ be an even functional which satisfies the PS condition, and $\varphi(0) = 0$. Suppose that $X = V \oplus E$, where V is infinite dimensional, and φ satisfies that*

- (i) *there exist $\alpha > 0$ and $\rho > 0$ such that $\varphi(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$,*
- (ii) *for any finite dimensional subspace $W \subset X$, there is $R = R(W)$ such that $\varphi(u) \leq 0$ on $W \setminus B_{R(W)}$.*

Then φ possesses an unbounded sequence of critical values.

Theorem 2.1 has been successfully used to ensure the existence of infinitely many solutions for boundary value problems in the papers [3, 12, 24].

To create suitable function spaces and apply critical point theory to explore the existence of solutions for the problem (P_λ^f), we require the following essential notations and findings which will be used in establishing our main results.

Definition 2.2. [18] For a function f defined on $[0,T]$ and $\alpha > 0$, the left and right Riemann-Liouville fractional integrals of order α for the function f are defined by

$$_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad t \in [0,T]$$

and

$$_tD_T^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s)\mathrm{d}s, \quad t \in [0,T]$$

while the right-hand sides are point-wise defined on $[0,T]$ where $\Gamma(\alpha)$ is the gamma function.

Definition 2.3. [18] Let $a, T \in \mathbb{R}$ and $\mathrm{AC}([0,T])$ be the space of absolutely continuous functions on $[0,T]$. For $0 < \alpha \leq 1$, $f \in \mathrm{AC}([0,T])$ left and right Riemann-Liouville and Caputo fractional derivatives are defined by:

$$ {}_0D_t^{\alpha} f(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t} {}_0D_t^{\alpha-1} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\alpha} f(s)\mathrm{d}s, $$

$$ {}_tD_T^{\alpha} f(t) \equiv -\frac{\mathrm{d}}{\mathrm{d}t} {}_tD_T^{\alpha-1} f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} f(s)\mathrm{d}s, $$

$$ {}_0^cD_t^{\alpha} f(t) \equiv {}^cD_{0+}^{\alpha} f(t) := {}_0D_t^{\alpha-1} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\mathrm{d}s $$

and

$$ {}_t^cD_T^{\alpha} f(t) \equiv {}^cD_{T-}^{\alpha} f(t) := -{}_tD_T^{\alpha-1} f'(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} f'(s)\mathrm{d}s $$

where $\Gamma(\alpha)$ is the gamma function. Note that when $\alpha = 1$, ${}_0^cD_t^1 f(t) = f'(t)$ and ${}_t^cD_T^1 f(t) = -f'(t)$.

We have the following property of fractional integration.

Proposition 2.4. [18, 21] *We have the following property of fractional integration*

$$ \int_0^T [{}_0D_t^{-\gamma} f(t)] g(t)\mathrm{d}t = \int_0^T [{}_tD_T^{-\gamma} g(t)] f(t)\mathrm{d}t, \quad \gamma > 0, $$

provided that $f \in \mathrm{L}^p([0,T], \mathbb{R}^N)$, $g \in \mathrm{L}^q([0,b], \mathbb{R}^N)$ *and* $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1 + \gamma$ *or* $p \neq 1$, $q \neq 1$, $1/p + 1/q = 1 + \gamma$.

Let $0 < \alpha \leq 1$, $1 < p < \infty$ and $\mathrm{E}_0^{\alpha,p}(0,T)$ be the Banach space, which is closure of $\mathrm{C}_0^{\infty}([0,T])$ with respect to the norm

$$ \|u\|^p_{\mathrm{E}_0^{\alpha,p}(0,T)} = \|{}_a^cD_t^{\alpha} u(t)\|^p_{\mathrm{L}^p(0,T)} + \|u\|^p_{\mathrm{L}^p(0,T)}. $$

It is an established fact that $\mathrm{E}_0^{\alpha,p}(0,T)$ is a reflexive and separable Banach space (see [17, Proposition 3.1]). In short $\mathrm{E}_{0,T}^{\alpha,2} = \mathrm{E}^{\alpha}$, and by $\|.\|$ and $\|.\|_{\infty}$ the norms in $\mathrm{L}^2(0,T)$ and $\mathrm{C}([0,T])$:

$$ \|u\|^2 = \int_0^T |u(t)|^2 \,\mathrm{d}t, \quad u \in \mathrm{L}^2[0,T] $$

and

$$ \|u\|_{\infty} = \max_{t \in [0,T]} |u(t)|, \quad u \in \mathrm{C}([0,T]). $$

E^{α} is a Hilbert space with inner product

$$ (u,v)_{\alpha} = \int_0^T ({}_0^cD_t^{\alpha} u(t)\, {}_0^cD_t^{\alpha} v(t) + u(t)v(t))\,\mathrm{d}t $$

and the norm

$$ \|u\|^2_{\alpha} = \int_0^T \left(|{}_0^cD_t^{\alpha} u(t)|^2 + |u(t)|^2\right) \mathrm{d}t. $$

Pay attention that if $a \in \mathrm{C}([0,T])$ and there are two positive constants a_1 and a_2, so that $0 < a_1 \le a(t) \le a_2$, an equivalent norm in E^α is

$$\|u\|_{a,\alpha}^2 = \int_0^T \left(|{}_0^c D_t^\alpha u(t)|^2 + a(t)|u(t)|^2 \right) \mathrm{d}t.$$

Proposition 2.5. [17] *Let $0 < \alpha \le 1$. For every $u \in \mathrm{E}^\alpha$, we have*

$$\|u\| \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^c D_t^\alpha u\|. \tag{2.1}$$

In addition, for $\frac{1}{2} < \alpha \le 1$,

$$\|u\|_\infty \le \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}} \|{}_0^c D_t^\alpha u\|.$$

By (2.1), we can take E^α with the norm

$$\|u\|_{0,\alpha} = \left(\int_0^T |{}_0^c D_t^\alpha u(t)|^2 \mathrm{d}t \right)^{1/2} = \|{}_0^c D_t^\alpha u\|, \quad \forall u \in \mathrm{E}^\alpha$$

in the following literature.

By Proposition 2.5, when $\alpha > 1/2$, for every $u \in \mathrm{E}^\alpha$, we have

$$\|u\|_\infty \le k \left(\int_0^T |{}_0^c D_t^\alpha u(t)|^2 \mathrm{d}t \right)^{1/2} = k\|u\|_{0,\alpha} < k\|u\|_{a,\alpha} \tag{2.2}$$

where

$$k = \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}.$$

Here we give the definition of weak and classical solutions for the problem (P_λ^f) as below:

Definition 2.6. A function $u \in \mathrm{E}^\alpha$ is said to be a weak solution of the problem (P_λ^f), if for every $v \in \mathrm{E}^\alpha$,

$$\begin{aligned} \int_0^T \left[({}_0^c D_t^\alpha u(t))({}_0^c D_t^\alpha v(t)) + a(t)u(t)v(t) \right] \mathrm{d}t + \sum_{j=1}^m I_j(u(t_j))v(t_j) \\ - \lambda \int_0^T f(t,u(t))\, v(t) \mathrm{d}t = 0. \end{aligned}$$

Definition 2.7. A function

$$u \in \left\{ u \in \mathrm{AC}([0,T]) \,:\, \int_{t_j}^{t_{j+1}} \left(|{}_0^c D_t^\alpha u(t)|^2 + |u(t)|^2 \right) \mathrm{d}t < \infty,\ j = 0, \ldots, m \right\}$$

is called to be a classical solution of problem (P_λ^f) if

$${}_t D_T^\alpha \left({}_0^c D_t^\alpha u(t) \right) + a(t)u(t) = \lambda f(t,u(t)) + h(u(t)), \quad \text{a.e. } t \in [0,T] \backslash \{t_1, \ldots, t_m\},$$

the limits ${}_t D_T^{\alpha-1} \left({}_0^c D_t^\alpha u \right)(t_j^+)$ and ${}_t D_T^{\alpha-1} \left({}_0^c D_t^\alpha u \right)(t_j^-)$ exist, $\Delta \left({}_t D_T^{\alpha-1} \left({}_0^c D_t^\alpha u \right) \right)(t_j) = I_j(u(t_j))$ and $u(0) = u(T) = 0$.

Lemma 2.8. [9, Lemma 2.1] *The function $u \in \mathrm{E}^\alpha$ is a weak solution of (P_λ^f) if and only if u is a classical solution of (P_λ^f).*

For each $u \in E^\alpha$, consider the functional φ defined on E^α by

$$\varphi(u) = \frac{1}{2}\|u\|_{a,\alpha}^2 + \sum_{j=1}^m \int_0^u I_j(\xi)\mathrm{d}\xi - \lambda \int_0^T F(t,u(t))\mathrm{d}t. \tag{2.3}$$

It is clear that φ is differentiable at any $u \in E^\alpha$ and

$$\begin{aligned}\varphi'(u)(v) = &\int_0^T \left[({}_0^cD_t^\alpha u(t))({}_0^cD_t^\alpha v(t)) + a(t)u(t)v(t)\right]\mathrm{d}t + \sum_{j=1}^m I_j(u(t_j))v(t_j)\\ &-\lambda \int_0^T f(t,u(t))v(t)\mathrm{d}t\end{aligned} \tag{2.4}$$

for any $v \in E^\alpha$. Obviously, φ' is continuous.

Corresponding to the functions f and I_j, $j = 1 \ldots, m$, we introduce the functions $F : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $J_j : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, $j = 1, \ldots, m$, respectively, as follows

$$F(t,\xi) = \int_0^\xi f(t,x)\mathrm{d}x, \quad \text{for all } (t,\xi) \in [0,T] \times \mathbb{R}$$

and

$$J_j(x) = \int_0^x I_j(\xi)\mathrm{d}\xi, \quad j = 1, \ldots, m \quad \text{for every } x \in \mathbb{R}.$$

3. Main result

In this section, we formulate our main results on the existence infinitely many classical solutions for the problem (P_λ^f).

Theorem 3.1. *Suppose that the following conditions are satisfied:*

(A_1) *there exists a constant $\nu > 2$ such that*

$$0 < \nu F(t,u) \le uf(t,u) \quad \textit{for all } (t,u) \in [0,T] \times \mathbb{R}\backslash\{0\},$$

(A_2)

$$0 < uI_j(u) \le \nu \int_0^\xi I_j(s)\mathrm{d}s \textit{ for all } \xi \in \mathbb{R}\backslash\{0\},\ j = 1, \ldots, m.$$

Moreover, if $f(t,u)$ and I_j are odd about u, then the impulsive problem (P_λ^f) has infinitely many classical solutions for $\lambda > 0$.

Lemma 3.2. *Assume that $(A_1) - (A_2)$ hold and $\lambda > 0$. Then $\varphi(u)$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\}$ be a sequence in E^α such that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to +\infty$. First, we prove that $\{u_n\}$ is bounded. By (2.3) and (2.4), one has

$$\begin{aligned}\nu\varphi(u_n) - \varphi'(u_n)(u_n) = &\left(\frac{\nu}{2} - 1\right)\|u_n\|_{a,\alpha}^2\\ &+\nu \sum_{j=1}^m \int_0^{u_n(t_j)} I_j(s)\mathrm{d}s - \sum_{j=1}^m I_j(u_n(t_j))u_n(t_j)\\ &-\lambda \int_0^T \left(\nu F(t,u_n(t)) - f(t,u_n(t))u_n(t)\right)\mathrm{d}t.\end{aligned}$$

By (A_1) and (A_2), one has

$$\nu \sum_{j=1}^{m} \int_0^{u_n(t_j)} I_j(s)\mathrm{d}s - \sum_{j=1}^{m} I_j(u_n(t_j))u_n(t_j) \geq 0 \tag{3.1}$$

and

$$\lambda \left(\int_0^T f(t, u_n(t))u_n(t)\mathrm{d}t - \nu \int_0^T F(t, u_n(t))\mathrm{d}t \right) \geq 0. \tag{3.2}$$

By considering (3.1)-(3.2), we conclude that

$$\nu\varphi(u_n) - \varphi'(u_n)(u_n) \geq \left(\frac{\nu}{2} - 1\right) \|u_n\|^2_{a,\alpha}.$$

Since $\nu > 2$ this implies that $\{u_n\}$ is bounded. Consequently, as E^α is a reflexive Banach space, we have, up to a subsequence,

$$u_n \rightharpoonup u \qquad \text{in } \mathrm{E}^\alpha,$$

$$u_n \to u \qquad \text{in } \ C[0, T]$$

and

$$u_n \to u \qquad \text{a.e. on } \ [0, T].$$

By $\varphi'(u_n) \to 0$ and $u_n \to u$, we obtain that

$$\left(\varphi'(u_n) - \varphi'(u)\right)(u_n - u) \to 0.$$

From the continuity of f and I_j $(j = 1, \ldots, m)$, we know that

$$\int_0^T \left(f(t, u_n(t)) - f(t, u(t))\right)(u_n(t) - u(t))\mathrm{d}t \to 0, \qquad \text{as } n \to \infty$$

and

$$\sum_{j=1}^{m} \left(I_j(u_n(t_j)) - I_j(u(t_j))\right)(u_n(t_j) - u(t_j)) \to 0, \qquad \text{as } n \to \infty.$$

Moreover, an easy computation shows that

$$\begin{aligned}
&\left(\varphi'(u_n) - \varphi'(u)\right)(u_n - u) \\
=& \int_0^T \left[({}_0^cD_t^\alpha u_n(t) - {}_0^cD_t^\alpha u(t))({}_0^cD_t^\alpha u_n(t) - {}_0^cD_t^\alpha u(t)) + a(t)(u_n(t) - u(t))(u_n(t) - u(t)) \right] \mathrm{d}t \\
&+ \sum_{j=1}^{m} \left(I_j(u_n(t_j) - I_j(u(t_j))\right)(u_n(t_j) - u(t_j)) \\
&- \lambda \int_0^T (f(t, u_n(t) - f(t, u(t))(u_n(t) - u(t))\mathrm{d}t \\
\geq& \|u_n - u\|^2_{a,\alpha}.
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \left(\varphi'(u_n) - \varphi'(u)\right)(u_n - u) \geq \lim_{n\to\infty} \|u_n - u\|^2_{a,\alpha}.$$

So $\|u_n - u\|_{a,\alpha} \to 0$ as $n \to \infty$, which implies that $\{u_n\}$ converges strongly to u in E^α. Therefore, φ satisfies the Palais-Smale condition. □

Lemma 3.3. [25, Lemma 2.2] *Assume that* (A_1) *holds. Then, for every* $t \in [0,T]$, *the following inequalities hold:*

$$F(t,x) \leq F\left(t, \frac{x}{|x|}\right)|x|^{\nu} \quad \text{if} \;\; 0<|x|\leq 1$$

and

$$F(t,x) \geq F\left(t, \frac{x}{|x|}\right)|x|^{\nu} \qquad \text{if} \;\; |x|\geq 1.$$

In view of Lemma 3.3, (A_1) implies that, for every $t \in [0,T]$

$$F(t,x) \leq M|x|^{\nu} \text{ if } 0<|x|\leq 1 \tag{3.3}$$

and

$$F(t,x) \geq m|x|^{\nu} \qquad \text{if } |x|\geq 1 \tag{3.4}$$

where $M = \max_{t\in[0,T],|x|=1} F(t,x)$, $m = \min_{t\in[0,T],|x|=1} F(t,x)$. Thanks to (A_1), one has $M>0$ and $m>0$. Since $F(t,x)-m|x|^{\beta}$ is continuous on $[0,T]\times[-1,1]$, there exists a constant C_2 such that

$$F(t,x) \geq m|x|^{\nu} - C_2 \;\; \text{for every} \;\; (t,x)\in[0,T]\times[-1,1]. \tag{3.5}$$

So it follows from (3.4) and (3.5) that

$$F(t,x) \geq m|x|^{\nu} - C_2 \;\; \text{for every} \;\; (t,x)\in[0,T]\times\mathbb{R}. \tag{3.6}$$

Remark 3.4. From (A_2), we can obtain that there exist constants $a,b>0$ such that

$$\int_0^{\xi} I_j(s)\mathrm{d}s < a|\xi|^{\nu} + b \tag{3.7}$$

for all $\xi \in \mathbb{R}$.

Proof of Theorem 3.1. Using the continuity of f and I_j, $j=1,\dots,m$, we obtain that $\varphi(u)$ is continuously and differentiable. In view of (2.3), it is obvious that $\varphi(u)$ is even and $\varphi(0)=0$. First, we show that φ satisfies condition (i) in Theorem 2.1. For any $u\in \mathrm{E}^{\alpha}$, by (2.3) and (A_2), one has

$$\begin{aligned}\varphi(u) =&\frac{1}{2}\|u\|^2_{a,\alpha} + \sum_{j=1}^{m} J_j(u(t_j)) - \lambda\int_0^T F(t,u(t))\mathrm{d}t\\ \geq&\frac{1}{2}\|u\|^2_{a,\alpha} - \lambda\int_0^T F(t,u(t))\mathrm{d}t.\end{aligned}$$

It is clear that $\|u\|_{a,\alpha}\leq \dfrac{1}{k}$ implies $\|u\|_{\infty}\leq 1$. Thanks to (3.3), one has

$$\int_0^T F(t,u(t))\mathrm{d}t \leq M\int_0^T |u|^{\nu}\mathrm{d}t \leq MTk^{\nu}\|u\|^{\nu}_{a,\alpha},\;\; \|u\|_{a,\alpha}\leq\frac{1}{k}. \tag{3.8}$$

Combining (3.8) and (A_1), one has

$$\varphi(u) \geq \frac{1}{2}\|u\|_{a,\alpha} - \lambda MTk^{\nu}\|u\|^{\nu}_{a,\alpha}.$$

Which implies that we can choose $\rho>0$ small enough such that $\varphi(u)\geq\alpha>0$ with $\|u\|=\rho$. Second, we show that φ satisfies condition (ii) in Theorem 2.1. Let $W\subset \mathrm{E}^{\alpha}$ is a finite dimensional subspace. For every $r \in \mathbb{R} \setminus \{0\}$ and $u \in W \setminus \{0\}$ with $\|u\|_{a,\alpha} = 1$, by (3.6)-(3.7), we can imply that

$$\begin{aligned}\varphi(ru) &= \frac{1}{2}\|ru\|_{a,\alpha}^2 + \sum_{j=1}^{m}\int_0^{ru(t)} I_j(s)\mathrm{d}s - \lambda\int_0^T F(t, ru(t))\mathrm{d}t \\ &\leq \frac{r^2}{2}\|u\|_{a,\alpha}^2 + \sum_{j=1}^{m}(a|r|^\nu|u(t)|^\nu + b) - \lambda|r|^\nu m\int_0^T |u(t)|^\nu \mathrm{d}t + \lambda T C_2.\end{aligned}$$

Noting that $\nu > 2$, the above inequality implies that there exists r_0 such that $\|ru\| > \rho$ and $\varphi(ru) < 0$ for every $r \geq r_0 > 0$. Since W is a finite dimensional subspace, there exists $R(W) > 0$ such that $\varphi(u) \leq 0$ on $W \setminus B_{R(W)}$. According to Theorem 2.1, the functional $\varphi(u)$ possesses infinitely many critical points, i.e. the impulsive problem (P_λ^f) has infinitely many classical solutions. □

Now, we illustrate Theorem 3.1 by presenting the following example.

Example 3.5. Consider the problem

$$\begin{aligned} {}_tD_1^\alpha\left({}_0^cD_t^\alpha u(t)\right) + u(t) = \lambda f(t,u), \quad t \neq \frac{1}{2}, \text{ a.e. } t \in [0,1], \\ \Delta\left({}_tD_1^{\alpha-1}\left({}_0^cD_t^\alpha u\right)\right)(\frac{1}{2}) = I_1(u(\frac{1}{2})), \\ u(0) = u(1) = 0 \end{aligned} \tag{3.9}$$

where

$$f(t,u) = t^2\left(u^3 + u^3e^{u^4} + u^7e^{u^4}\right)$$

for every $(t,u) \in [0,1] \times \mathbb{R}$ and $I_1(\zeta) = \dfrac{1}{5}\zeta$ for each $\zeta \in \mathbb{R}$. By the expression of f, we have

$$F(t,u) = t^2\left(\frac{u^4}{4} + \frac{u^4e^{u^4}}{4}\right)$$

for every $(t,u) \in [0,1]\times\mathbb{R}$. By choosing $\nu = 4 > 2$, the assumptions (A_1) and (A_2) are fulfilled. Since $f(t,u)$ and I_1 are odd about u, we clearly see that all assumptions of Theorem 3.1 are fulfilled. Therefore, the impulsive problem (3.9) has infinitely many classical solutions.

Theorem 3.6. *Suppose that the following conditions are satisfied:*

(A_3) *there exist constants $R > 0$ and $0 < \lambda L_1 < \dfrac{1}{2Tk^2}$ such that*

$$F(t,u) \leq L_1|u|^2 \text{ for all } (t,u) \in [0,T] \times \mathbb{R}, \ |u| \leq R,$$

(A_4) *$F(t,u) \geq 0$ for all $(t,u) \in [0,T] \times \mathbb{R}$ and there exist constants $R_1 > 0$, $\delta_1 > 0$ and $\alpha_1 > \nu$ such that*

$$F(t,u) \geq \delta_1|u|^{\alpha_1} \text{ for all } (t,u) \in [0,T] \times \mathbb{R}, \ |u| \geq R,$$

(A_5) *there exists a constant $\nu > 2$, $\delta_1 \geq 0$ and $0 < \alpha_2 < 2$ such that*

$$\nu F(t,\xi) - \xi f(t,\xi) \leq \delta_2|u|^{\alpha_2},$$

(A_6)

$$0 < uI_j(u) \leq \nu\int_0^\xi I_j(s)\mathrm{d}s, \quad j = 1,\ldots,m.$$

Moreover, if $f(t,u)$ and I_j for $j=1,\dots,m$ are odd about u, then the impulsive problem (P_λ^f) has infinitely many classical solutions for $\lambda>0$.

Lemma 3.7. *Under the assumptions of Theorem 3.6, $\varphi(u)$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\}$ be a sequence in E^α such that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n)\to 0$ as $n\to+\infty$. By (2.3), (2.4) and (A_5), one has

$$\begin{aligned}\nu\varphi(u_n)-\varphi'(u_n)(u_n) &= \left(\frac{\nu}{2}-1\right)\|u_n\|^2_{a,\alpha}\\ &\quad+\nu\sum_{j=1}^m\int_0^{u_n(t_j)}I_j(s)\mathrm{d}s-\sum_{j=1}^m I_j(u_n(t_j))u_n(t_j)\\ &\quad-\lambda\int_0^T\left(\nu F(t,u_n(t))-f(t,u_n(t))u_n(t)\right)\mathrm{d}t\\ &\ge\left(\frac{\nu}{2}-1\right)\|u_n\|^2_{a,\alpha}-\lambda T\delta_2C^{\alpha_2}\|u_n\|^{\alpha_2}_{a,\alpha}.\end{aligned}$$

Since $0<\alpha_2<2$ it follows that $\{u_n\}$ is bounded on E^α. The proof of the PS condition is similar to that in Lemma 3.3. We omit it here. □

Proof. Using the continuity of f and I_j for $j=1,2,\dots,m$, we obtain that $\varphi(u)$ is continuously and differentiable. In view of (2.3), it is obvious that $\varphi(u)$ is even and $\varphi(0)=0$. By Lemma 3.7, $\varphi(u)$ satisfies the PS condition. First, we show that φ satisfies condition (i) in Theorem 2.1. It is clear that $\|u\|_{a,\alpha}\le\dfrac{\Gamma(\alpha)\sqrt{2\alpha-1}}{T^{\alpha-\frac12}}R=\dfrac{1}{k}R$ implies $\|u\|_\infty\le R$. Thanks to (A_3), one has

$$\lambda\int_0^T F(t,u(t))\mathrm{d}t\le\lambda L_1\int_0^T|u|^2\mathrm{d}t\le\lambda L_1Tk^2\|u\|^2_{a,\alpha},\ \ \|u\|_{a,\alpha}\le\frac{R}{k}. \tag{3.10}$$

Combining (3.10) and (A_6), one has

$$\varphi(u)\ge\frac12\|u\|^2_{a,\alpha}-\lambda L_1Tk^2\|u\|^2_{a,\alpha}=\left(\frac12-\lambda L_1Tk^2\right)\|u\|^2_{a,\alpha}.$$

Which implies that we can choose $\rho>0$ small enough such that $\varphi(u)\ge\alpha>0$ with $\|u\|=\rho$. Second, we show that φ satisfies condition (ii) in Theorem 2.1. Let $W\subset\mathrm{E}^\alpha$ is a finite dimensional subspace. According to Remark 3.4, it follows from (A_6) that there exist constants $a,b,c,d>0$ such that

$$\int_0^\xi I_j(s)\mathrm{d}s<a|u|^\nu+b \tag{3.11}$$

for all $u\in\mathbb{R}$. By (A_4), we can imply that there exists a constant $C_3>0$ such that

$$F(t,x)\ge\delta_1|x|^{\alpha_1}-C_3\ \ \text{for every}\ \ (t,x)\in[0,T]\times\mathbb{R}. \tag{3.12}$$

For every $r \in \mathbb{R} \setminus \{0\}$ and $u \in W \setminus \{0\}$ with $\|u\|_{a,\alpha} = 1$, by (3.11)-(3.12), we can imply that

$$\begin{aligned} \varphi(ru) &= \frac{1}{2}\|ru\|^2_{a,\alpha} + \sum_{j=1}^{m} \int_0^{ru(t_j)} I_j(s)\mathrm{d}s \\ &- \lambda \int_0^T F(t, ru(t))\mathrm{d}t \\ &\leq \frac{r^2}{2}\|u\|^2_{a,\alpha} + \sum_{j=1}^{m} c|r|^{\nu}|u(t)|^{\nu} + b \\ &- \lambda |r|^{\alpha_1}\delta_1 \int_0^T |u(t)|^{\alpha_1}\mathrm{d}t + \lambda T C_3. \end{aligned}$$

Noting that $\alpha_1 > \nu$, the above inequality implies that there exists r_0 such that $\|ru\|_{a,\alpha} > \rho$ and $\varphi(ru) < 0$ for every $r \geq r_0 > 0$. Since W is a finite dimensional subspace, there exists $R(W) > 0$ such that $\varphi(u) \leq 0$ on $W \setminus B_{R(W)}$. □

Conclusion

We studied a class of impulsive fractional boundary value problems. We discussed the existence of infinitely many solutions for the problem employing a recent variational methods for smooth functionals defined on reflexive Banach spaces under Ambrosetti-Rabinowitz condition (AR) on the nonlinear term and impulsive functions. We discussed the existence of infinitely many solutions for the problem when the nonlinear term is superlinear.

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