Mathematical Analysis
& Convex Optimization

Vol. 4 (2023), No. 2, 109–114
https:\\ maco.lu.ac.ir
DOI: 10.22034/maco.4.2.10

Research Paper

REMARKS ON BANACH SPACES RELATED TO UNITARY REPRESENTATIONS

SEYEDEH SOMAYEH JAFARI

Abstract. Let (π, H) be a unitary representation of G. We study some Banach spaces related to π. In particular, we investigate the subject by subrepresentations and finite direct sum of given representations.

MSC(2010): 22D10; 43A60; 43A65.
Keywords: Locally compact group, unitary representation, uniformly G-continuous operator, weakly almost G-periodic operators.

1. Introduction

Throughout this note, G is a locally compact group with a fixed left Haar measure dx. A unitary representation of G will always mean a pair (π, H) where π is a homomorphism of G into the group unitary operators on the Hilbert space H that is continuous with respect to the strong operator topology on $B(H)$, consisting of all bounded linear operators on H; see for example [4]. In the attractive works, Bekka [1] and Xu [10] have introduced some spaces of operators associated with a unitary representation, corresponding to $LUC(G)$ and the space of $WAP(G)$. Let now us recall these notions as follow.

Given any unitary representation (π, H) of G, note that $B(H)$ is a right G-module under the following action

$$T \cdot_\pi x = \pi(x^{-1})T\pi(x) \quad (T \in B(H),\, x \in G).$$

In general $B(H)$ is not Banach G-module in terms of Johnson's notion, [9]. In fact, for $T \in B(H)$, the map $x \mapsto T \cdot_\pi x$, $G \longrightarrow B(H)$ is not norm continuous, necessarily. We say that T is *uniformly G-continuous operator* if the mapping $x \mapsto T \cdot_\pi x$ are norm continuous. Suppose that the notation $UCB(\pi)$ refers to the collection of such operators. Then $UCB(\pi)$ is a C^*-subalgebra of $B(H)$, and also it is a right Banach G-module. We also say that T is *weakly almost G-periodic operator* if the set of all $T \cdot_\pi x$, where $x \in G$ is relatively weakly compact. The collection of such operators that denotes $WAP(\pi)$ is a closed subspace of $B(H)$. Note that [3, Proposition 4.16] ensouras that $K(H) \subseteq WAP(\pi)$, where $K(H)$ is the set of all compact operators on H. As might be expected, there exist the same style of G-versions of the above spaces; i.e., $WAP(\pi) \subseteq UCB(\pi)$. Moreover, $WAP(\pi)$ is a right Banach G-submodule of $UCB(\pi)$; see [10], for more details. The reader can also refer to recent works of the author, [5]-[8].

Our interest to us here is some properties and applications of these spaces.

Date: Received: February 10, 2023 , Accepted: December 1, 2023.

2. The results

For any unitary representation (π, H) of G, let $M \in B(H)^*$ and $T \in B(H)$. Then define the complex-valued function MT on G by

$$MT(x) = \langle M, T \cdot_\pi x\rangle \quad (x \in G).$$

Obviously, MT is bounded by $\|M\|\|T\|$. Also, compatible results exist between locally compact groups and their unitary representations. For instance, $T \in UCB(\pi)$ if and only if $MT \in LUC(G)$ for all $M \in B(H)^*$. Moreover, if $T \in WAP(\pi)$, then $MT \in WAP(G)$ for all $M \in B(H)^*$. But we have been unable to confirm the converse. We refer the reader to our recent work [5] for more details. Now, suppose that $\mathcal{L}_\pi$ and $\mathcal{W}_\pi$ are respectively the closure of the linear span of the sets

$$\{MT \,|\, M \in B(H)^*, \, T \in UCB(\pi)\}$$

in $LUC(G)$, and

$$\{MT \,|\, M \in B(H)^*, \, T \in WAP(\pi)\}$$

in $WAP(G)$. If G is non-compact and the set

$$N = \{x \in G \,|\, T \cdot_\pi x = T \,\text{for all T} \in \text{UCB}(\pi)\}$$

is non-trivial, then as seen in the proof of [2, Proposition 4.9] $\mathcal{L}_\pi$ is properly contained in $LUC(G)$. A similar result holds when replacing the notions of uniformly continuous bounded by weakly almost periodic. We have the following example for the left regular representation of G.

Example 2.1. Let $\big(\lambda, L^2(G)\big)$ be the left regular representation of G. We recall that $\lambda : G \longrightarrow B\big(L^2(G)\big)$ is given by $x \mapsto l_x$, and $l_x(f)(y) = f(xy)$ for all $f \in L^2(G)$, $x, y \in G$. As mentioned [5, Remark 3.11], $f \in LUC(G)$ if and only if $T_f \in UCB(\lambda)$, and also $f \in WAP(G)$ if and only if $T_f \in WAP(\lambda)$, where T_f is the multiplication operator on $L^2(G)$ for each $f \in L^\infty(G)$; i.e., $T_f(g) = fg$ for all $g \in L^2(G)$. Note that the proof of [2, Corollary 4.10] states $LUC(G) = \mathcal{L}_\lambda$. Also, one can verify that $WAP(G) = \mathcal{W}_\lambda$.

It is clear that $\mathcal{W}_\pi = \mathcal{L}_\pi$ for all unitary representations (π, H) of compact groups. Note that there exist some unitary representations (π, H) of a non-compact group such that $\mathcal{W}_\pi = \mathcal{L}_\pi$; for instance, see [10, Example 5.4.1 and Remark 5.4.2]. In fact, we have the following result.

Proposition 2.2. *Let G be a locally compact group. Then G is compact if and only if $\mathcal{W}_\pi = \mathcal{L}_\pi$ for each unitary representation (π, H) of G.*

Proof. One implication is trivial. Suppose that $\mathcal{W}_\pi = \mathcal{L}_\pi$ for each unitary representation (π, H) of G. So, as seen in Example 2.1, we have

$$WAP(G) = \mathcal{W}_\lambda = \mathcal{L}_\lambda = LUC(G).$$

It follows that G is compact. □

Let us recall that $L^1(G)$ is the group algebra equipped with the convolution product $*$ and the norm $\|.\|_1$ as defined in [4]. Also, let $L^\infty(G)$ refers to the Lebesgue space equipped with the essential supremum norm $\|.\|_\infty$ as defined in [4]. Then $L^\infty(G)$ is the dual of $L^1(G)$ for the pairing

$$\langle f, \phi\rangle = \int_G f(x)\, \phi(x)\, dx$$

for all $\phi \in L^1(G)$ and $f \in L^\infty(G)$.

Note that $UCB(\pi)$ is a unital Banach $L^1(G)$-module by [9, Proposition 2.1] by the following action

$$T \cdot_\pi \phi = \int_G T \cdot_\pi x \, \phi(x) \, dx \quad (T \in UCB(\pi), \, \phi \in L^1(G)).$$

In fact,

$$UCB(\pi) \cdot_\pi L^1(G) = B(H) \cdot_\pi L^1(G) = UCB(\pi).$$

Let (π_0, H_0) and (π, H) be unitary representations of G such that π_0 is a subrepresentation of π. Let also $P : H \longrightarrow H_0$ be the canonical projection. Then there exists a surjective map from $UCB(\pi)$ onto $UCB(\pi_0)$; see [2, Lemma 7.1] for details. Our next theorem reveals that the above statement holds also for weakly almost G-periodic operators. Before stating, however, we need the following lemma that was proven in [5, Lemma 3.3].

Lemma 2.3. *Let (π, H) be a unitary representation of G and $T \in B(H)$. Then $T \in WAP(\pi)$ if and only if $T \in UCB(\pi)$ and γ_T is a weakly compact operator, where $\gamma_T : L^1(G) \longrightarrow B(H)$ is given by*

$$\phi \mapsto T \cdot_\pi \phi \quad (\phi \in L^1(G)).$$

Theorem 2.4. *Let (π_0, H_0) and (π, H) be unitary representations of G such that π_0 is a subrepresentation of π. Then the following assertions hold.*

(a) $PT|_{H_0} \in WAP(\pi_0)$ for all $T \in WAP(\pi)$.
(b) There exists a surjective map from $WAP(\pi)$ onto $WAP(\pi_0)$.

Proof. Suppose that $T \in B(H)$. Then $PT|_{H_0} \in B(H_0)$. Also, for each $x \in G$, we have

$$(PT|_{H_0}) \cdot_{\pi_0} x = (P)(T \cdot_\pi x)|_{H_0}.$$

For each $M_0 \in UCB(\pi_0)^*$, the linear bounded functional M on $UCB(\pi)$ is defined by

$$\langle M, \, T \rangle = \langle M_0, \, PT|_{H_0} \rangle \quad (T \in UCB(\pi)).$$

Let now $T \in WAP(\pi) \subseteq UCB(\pi)$ and $T_0 = PT|_{H_0}$. Then $T_0 \in UCB(\pi_0)$. We claim that the mapping $\gamma_{T_0} : L^1(G) \longrightarrow UCB(\pi_0)$ is weakly compact. Note that for each $\phi \in L^1(G)$, we have

$$\begin{aligned} T_0 \cdot_{\pi_0} \phi &= \int_G T_0 \cdot_{\pi_0} x \, \phi(x) \, dx \\ &= \int_G (P)(T \cdot_\pi x)|_{H_0} \, \phi(x) \, dx \\ &= (P)(T \cdot_\pi \phi)|_{H_0}. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \gamma_T^*(M), \, \phi \rangle &= \langle M, \, T \cdot_\pi \phi \rangle \\ &= \langle M_0, \, (P)(T \cdot_\pi \phi)|_{H_0} \rangle \\ &= \langle M_0, \, T_0 \cdot_{\pi_0} \phi \rangle \\ &= \langle \gamma_{T_0}^*(M_0), \, \phi \rangle. \end{aligned}$$

So, $\gamma_T^*(M) = \gamma_{T_0}^*(M_0)$. On the other hand, since $UCB(\pi)$ is the neo-unital $L^1(G)$-module, $T = S \cdot_\pi \phi$ for some $S \in UCB(\pi)$ and $\phi \in L^1(G)$. Suppose now that $M_0^\alpha \xrightarrow{w^*} M_0$ in $UCB(\pi_0)^*$. Then

$$\begin{aligned}\langle M^\alpha, T\rangle &= \langle M^\alpha, S \cdot_\pi \phi\rangle \\ &= \langle M_0^\alpha, S_0 \cdot_{\pi_0} \phi\rangle \\ &\longrightarrow \langle M_0, S_0 \cdot_{\pi_0} \phi\rangle \\ &= \langle M_0, T_0\rangle \\ &= \langle M, T\rangle,\end{aligned}$$

where, $S_0 = PS|_{H_0}$. So, $M^\alpha \xrightarrow{w^*} M$ in $UCB(\pi)^*$. On the other hand, since $T \in WAP(\pi)$, we have

$$\gamma_{T_0}^*(M_0^\alpha) = \gamma_T^*(M^\alpha) \xrightarrow{w} \gamma_T^*(M) = \gamma_{T_0}^*(M_0).$$

So, γ_{T_0} is weakly compact. It follows that T_0 lies in $WAP(\pi_0)$ by Lemma 2.3.

For the second one, we show that the mapping $T \mapsto PT|_{H_0}$ from $WAP(\pi)$ into $WAP(\pi_0)$ is surjective. For each $M \in UCB(\pi)^*$, we consider the linear bounded functional M_0 on $UCB(\pi_0)$ as defined by

$$\langle M_0, T_0\rangle = \langle M, T_0P\rangle \quad (T_0 \in UCB(\pi_0)).$$

One shows that

$$\begin{aligned}(M)(T_0P)(x) &= \langle M, T_0P \cdot_\pi x\rangle = \langle M, (T_0 \cdot_{\pi_0} x)P\rangle \\ &= \langle M_0, T \cdot_{\pi_0} x\rangle = M_0T_0(x);\end{aligned}$$

that is,

$$(M)(T_0P) = M_0T_0.$$

So, $\gamma_{T_0P}^*(M) = \gamma_{T_0}^*(M_0)$. On the other hand, the mapping $T \mapsto PT|_{H_0}$ from $WAP(\pi)$ into $WAP(\pi_0)$ is well-defined by part (a). If $T_0 \in WAP(\pi_0) \subseteq UCB(\pi_0)$, then $T_0P \in UCB(\pi)$. Now, we show that $T_0P \in WAP(\pi)$. For this aim, let $M^\alpha \xrightarrow{w^*} M$ in $UCB(\pi)^*$. Then $M_0^\alpha \xrightarrow{w^*} M_0$ in $UCB(\pi_0)^*$. Since $T_0 \in WAP(\pi_0)$, we have

$$\gamma_{T_0P}^*(M^*) = \gamma_{T_0}^*(M_0^*) \xrightarrow{w} \gamma_{T_0}^*(M_0) = \gamma_{T_0P}^*(M).$$

It follows that $T_0P \in WAP(\pi)$. □

We have the following consequence as an immediate result of the above theorem together with [2, Lemma 7.1].

Corollary 2.5. *Let (π_0, H_0) and (π, H) be unitary representations of G such that π_0 is a subrepresentation of π. Then $\mathcal{L}_{\pi_0} \subseteq \mathcal{L}_\pi$ and $\mathcal{W}_{\pi_0} \subseteq \mathcal{W}_\pi$.*

Now, we study the finite direct sum of π on the subject. Suppose that (π, H_π) be a unitary representation of G. We recall some usual notations as follows. Let $H'_\pi = \oplus_n H_\pi$ and $\pi' = \oplus_n \pi$, the direct sum of n copies of π. Let $H_i = H_\pi$ for each $i = 1, ..., n$ and write $H'_\pi = \oplus_{i=1}^n H_i$, in order to avoid confusion. Let also, for each $i = 1, ..., n$ consider following maps

$$P_i : H'_\pi \longrightarrow H_i \text{ and } I_i : H_i \longrightarrow H'_\pi,$$

where P_i and I_i are the canonical projection and injection, respectively. For each $T \in B(H'_\pi)$, define a component of T as follows.

$$\{T_{ij} : H_j \longrightarrow H_i \,|\, i, j = 1, ..., n\},$$

where $T_{ij} = P_i T I_j$. As is pointed out [2], if $M \in B(H'_\pi)^*$, then for $i, j = 1, ..., n$ the elements M_{ij} in $B(H_\pi)^*$ are a components of M which are given via the formula

$$\langle M_{ij}, T\rangle = \langle M, I_i T P_j\rangle \quad (T \in B(H_\pi)).$$

According to [2], we have $T \in UCB(\pi')$ if and only if $T_{ij} \in UCB(\pi)$ for each $i, j = 1, ..., n$. Our next theorem shows that the above statement is valid also for weakly almost G-periodic operators and operators that vanish at infinity.

Theorem 2.6. *Let (π, H_π) be a unitary representation of a locally compact group G, and let $\pi' = \oplus_n \pi$ be the direct sum of n copies of π. Then $T \in WAP(\pi')$ if and only if $T_{ij} \in WAP(\pi)$ for each $i, j = 1, ..., n$.*

Proof. Let $T \in B(H'_\pi)$ and $x \in G$. Then [2, Lemma 7.4] states that

$$(T \cdot_{\pi'} x)_{ij} = T_{ij} \cdot_\pi x \quad (1 \leq i, j \leq n).$$

Suppose now that $T \in WAP(\pi')$. Then

$$\begin{aligned} \overline{\{T_{ij} \cdot_\pi x \,| x \in G\}}^{\sigma(B(H_\pi),\, B(H_\pi)^*)} &= \overline{\{(T \cdot_{\pi'} x)_{ij} \,| x \in G\}}^{\sigma(B(H_\pi),\, B(H_\pi)^*)} \\ &\subseteq \overline{\{\Sigma_{i,j}(T \cdot_{\pi'} x)_{ij} \,| x \in G\}}^{\sigma(B(H'_\pi),\, B(H'_\pi)^*)} \\ &= \overline{\{T \cdot_{\pi'} x \,| x \in G\}}^{\sigma(B(H'_\pi),\, B(H'_\pi)^*)}, \end{aligned}$$

and so $T_{ij} \in WAP(\pi)$ for each $i, j = 1, ..., n$, where the notation σ denotes the weak topology.

For the converse, let $T_{ij} \in WAP(\pi)$ for each $i, j = 1, ..., n$, and let $M^\alpha \xrightarrow{w^*} M$ in $UCB(\pi)^*$. Then $M^\alpha_{ij} \xrightarrow{w^*} M_{ij}$ in $UCB(\pi)^*$ by [2, Lemma 7.6]. Therefore,

$$\begin{aligned} \gamma^*_T(M^\alpha) &= M^\alpha T = \sum_{i,j} M^\alpha_{ij} T_{ij} \\ &= \sum_{i,j} \gamma^*_{T_{ij}}(M^\alpha_{ij}) \xrightarrow{w} \sum_{i,j} \gamma^*_{T_{ij}}(M_{ij}) \\ &= \sum_{i,j} M_{ij} T_{ij} = \gamma^*_T(M). \end{aligned}$$

It means that $T \in WAP(\pi)$. □

Corollary 2.7. *Let (π, H_π) be a unitary representation of a locally compact group G, and let $\pi' = \oplus_n \pi$ be the direct sum of n copies of π. Then $\mathcal{W}_\pi = \mathcal{W}_{\pi'}$.*

Proof. Noting Corollary 2.5, we need only prove that $\mathcal{W}_{\pi'} \subseteq \mathcal{W}_\pi$. Suppose that $M \in B(H')^*$ and $T \in WAP(\pi')$. Then $T_{ij} \in WAP(\pi)$ for each $i, j = 1, ..., n$. On the other hand,

$$\begin{aligned} MT(x) = \langle M,\, T \cdot_{\pi'} x\rangle &= \sum_{i,\,j} \langle M_{ij},\, (T \cdot_{\pi'} x)_{ij}\rangle \\ &= \sum_{i,\,j} \langle M_{ij},\, T_{ij} \cdot_\pi x\rangle = \sum_{i,\,j} M_{ij} T_{ij}(x). \end{aligned}$$

It follows that $MT = \sum_{i,\,j} M_{ij} T_{ij} \in \mathcal{W}_\pi$. □

Conclusion

Regarding every unitary representation of (π, H) of G, we studied some special closed subspaces of $B(H)$ and $LUC(G)$. On the base of these notions, we stated a characterization of compact groups. Moreover, we explored the relations between these spaces for sub-representations and finite direct sums.

Acknowledgments

The author would like to thank the editor and referees for giving their valuable time to the paper.

References

[1] M. E. B. Bekka, Amenable unitary representations of locally compact groups. *Invent. Math.*, **100**:383-401, 1990.

[2] P. K. Chan, Topological centers of module actions induced by unitary representations. *J. Funct. Anal.*, **259**:2193-2214, 2010.

[3] C. Chou and A. T. Lau, *Vector-valued invariant means on spaces of bounded operators associated to a locally compact group*, Illinois J. Math. **45** : 581–602, 2001.

[4] G. B. Folland, *A course in abstract harmonic analysis*, CRC Press, Boca Raton, 1995.

[5] S. S. Jafari, On topological centers induced by unitary representations. *Arch. Math.*, **117**(3):323–333, 2021.

[6] S. S. Jafari, Operators commuting with certain module actions. *Int. J. Nonlinear Anal. Appl.* **14** (12): 53–58, 2023.

[7] S. S. Jafari, On the generalized notion of amenable locally compact groups. *Khayyam J. Math.*, Accepted, 2024.

[8] S. S. Jafari and Y. Zohrevand, Topological centers induced by $L^1(G)^{**}$-module actions. *Contemp. Math.* **5** (3), 2933–2939, 2024.

[9] B. E. Johnson, *Cohomology in Banach algebras*, Memoirs of the American Mathematical Society, **127**, 1972.

[10] Q. Xu, *Representations of locally compact groups, amenability and compactifications.* Ph.D., University of Alberta, Edmonton, Canada, 1993.

(Seyedeh Somayeh Jafari) Department of Mathematics, Payame Noor University, Tehran, Iran.
Email address: ss.jafari@pnu.ac.ir