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Research Paper

MODULE CONNES AMENABILITY FOR PROJECTIVE TENSOR PRODUCT AND Θ-LAU PRODUT OF BANACH ALGEBRAS

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Abstract. Let $\mathbb{A}$ and $\mathbb{B}$ be Banach algebras with preduals $\mathbb{A}_*$ and $\mathbb{B}_*$ respectively, and $\Theta : \mathbb{B} \to \mathbb{A}$ be an algebraic homomorphism. In this paper, we derive some specific results concerning the characterizations of module Connes amenability of certain Banach algebras. Indeed, we investigate and give necessary and sufficient conditions for module Connes amenability of projective tensor product $\mathbb{A}\widehat{\otimes}\mathbb{B}$. Moreover, we characterize the module (ψ,θ)-Connes amenability of Θ-Lau product $\mathbb{A}\times_\Theta\mathbb{B}$, which ψ and θ are homomorphisms in $\mathbb{A}_*$ and $\mathbb{B}_*$, respectively.

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1. Introduction

The concept of amenability of a Banach algebra $\mathbb{A}$ to the case that there exists an additional $\mathbb{U}$-module structure on $\mathbb{A}$, where $\mathbb{U}$ is a Banach algebra, is extended by Amini [1]. The definition of Connes amenability makes sense for a larger class of Banach algebras (called dual Banach algebras in [13]). Runde in [14], introduced the concept of Connes amenability of dual Banach algebras and this type of Banach algebras is interested by many researchers. Connes amenability for certain product of Banach algebras such as projective tensor product, Lau product and Connes amenability of l^1-Munn algebras were studied by Ghaffari et al. in [7, 8]. The concept of module Connes amenability for dual Banach algebras which are also Banach modules with respect to the compatible action, first defined by Amini [2]. It is shown that for a Banach algebra, there exists a relation between its module Connes amenability and the existance of its normal module virtual diagonal. Moreover, module Connes amenability of semigroup algebras is studied in [16]. The θ-Lau product was introduced by Lau for certain class of Banach algebras in [10], and was followed by Monfared for the general case [12]. Also, characterization of the projective tensor product Banach algebras have been studied widely by many researchers and mathematicians in recent decades (see [7, 12]).

In this paper, we are going to investigate the module Connes amenability and character module Connes amenability for projective tensor product $\mathbb{A}\widehat{\otimes}\mathbb{B}$ and Θ-Lau product $\mathbb{A}\times_\Theta\mathbb{B}$, where $\mathbb{A}$ and $\mathbb{B}$ are dual Banach algebras and $\Theta : \mathbb{B} \to \mathbb{A}$ is an algebraic homomorphism.

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2. Preliminaries

Let $\mathbb{A}$ be a dual Banach algebra and $\mathbb{E}$ be a Banach $\mathbb{A}$-bimodule. The collection of all elements of $\mathbb{E}$ that module maps from $\mathbb{A}$ onto $\mathbb{E}$ are w^*-weakly continuous and is denoted by $\sigma wc(\mathbb{E})$. Suppose that $\Delta : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ is the multiplication operator. We can develop Δ to an $\mathbb{A}$-bimodule homomorphism $\Delta_{\sigma wc}$ from $\sigma wc((\mathbb{A}\hat{\otimes}\mathbb{A})^*)^*$ to $\mathbb{A}$.

Definition 2.1. A σwc-virtual diagonal for Banach algebra $\mathbb{A}$ is an element $\mu \in \sigma wc((\mathbb{A}\hat{\otimes}\mathbb{A})^*)^*$ such that $a.\mu = \mu.a$ and $a.\Delta_{\sigma wc}\mu = a$, for all $a \in \mathbb{A}$.

In [14], it is shown that a Banach algebra $\mathbb{A}$ is Connes amenable if and only if it has a σwc-virtual diagonal.

Definition 2.2. [9] Let $\mathbb{A}$ be a Banach algebra. $\mathbb{A}$-bimodule $\mathbb{E}$ is called normal if it is a dual space such that the module actions of $\mathbb{A}$ on $\mathbb{E}$ are separately w^*-continuous.

Recently the authors have introduced the new version, as called φ-Connes amenability of dual Banach algebra $\mathbb{A}$ that $\varphi \in \Delta(\mathbb{A})$, the set of all continuous homomorphisms from $\mathbb{A}$ onto $\mathbb{C}$, and also $\varphi \in \mathbb{A}_*$ [5], as follows:

A dual Banach algebra $\mathbb{A}$ is φ-Connes amenable that $\varphi \in \Delta(\mathbb{A})$ if, for every normal φ-bimodule $\mathbb{E}$, where the left action is of the form

$$a.\mathbf{e} = \varphi(a)\mathbf{e}, \qquad (a \in \mathbb{A}, \mathbf{e} \in \mathbb{E}) \tag{2.1}$$

every bounded w^*-continuous derivation $\mathcal{D} : \mathbb{A} \to \mathbb{E}$ is inner. Such derivations are called inner φ-derivations.

Suppose that $\mathbb{A}$ is a Banach algebra, $\mathcal{F}$ is a subspace of $\mathbb{A}^*$ and $\varphi \in \Delta(\mathbb{A}) \cap \mathcal{F}$. A linear functional $\mathbf{m}$ on $\mathcal{F}$ is said to be a mean if $\langle \mathbf{m}, \varphi\rangle = 1$. A mean $\mathbf{m}$ is φ-invariant if $\langle \mathbf{m}, a.f\rangle = \varphi(a)\langle \mathbf{m}, f\rangle$ for all $a \in \mathbb{A}$ and $f \in \mathcal{F}$. In this case, we note that φ-Connes amenability of $\mathbb{A}$ follows from $\mathcal{F} = \mathbb{A}_*$ (see [6, 5]). Also, the authors showed that a dual Banach algebra $\mathbb{A}$ is φ-Connes amenable if and only if the second dual, $\mathbb{A}^{**}$ has a φ-invariant mean on $\mathbb{A}_*$.

Suppose that $\mathbb{A}$ is a dual Banach algebra, and $\mathbb{U}$ is a Banach algebra such that $\mathbb{A}$ is a Banach $\mathbb{U}$-bimodule via the following actions,

$$v.(ab) = (v.a).b, \quad (vw).a = v.(w.a) \qquad (a, b \in \mathbb{A}, v, w \in \mathbb{U}).$$

Let $\mathbb{E}$ be a normal dual Banach $\mathbb{A}$-bimodule. Moreover, if $\mathbb{E}$ is an $\mathbb{U}$-bimodule via

$$v.(a.\mathbf{e}) = (v.a).\mathbf{e}, \quad (a.v).\mathbf{e} = a.(v.\mathbf{e}), \quad (v.\mathbf{e}).a = v.(\mathbf{e}.a),$$

for every $a \in \mathbb{A}$, $v \in \mathbb{U}$ and $\mathbf{e} \in \mathbb{E}$. Then we say that $\mathbb{E}$ is a normal Banach left $\mathbb{A}$-$\mathbb{U}$-bimodule. Similarly, for the right actions. Also, we say that $\mathbb{E}$ is symmetric, if $v.\mathbf{e} = \mathbf{e}.v$ for every $v \in \mathbb{U}$ and $\mathbf{e} \in \mathbb{E}$.

Let $\mathbb{U}$ and $\mathbb{A}$ be Banach algebras, and let $\mathbb{E}$ be a Banach $\mathbb{A}$-$\mathbb{U}$-bimodule. Then by using [13], $\mathbb{E}^*$ becomes a Banach $\mathbb{U}$-$\mathbb{A}$-bimodule via

$$\langle \mathbf{e}, a.f\rangle := \langle \mathbf{e}.a, f\rangle, \quad \langle \mathbf{e}, f.v\rangle := \langle v.\mathbf{e}, f\rangle, \qquad (v \in \mathbb{U}, a \in \mathbb{A}, f \in E^*, \mathbf{e} \in \mathbb{E}). \tag{2.2}$$

Definition 2.3. Let $\mathbb{A} = (\mathbb{A}_*)^*$ be a Banach algebra, $\mathbb{U}$ be a Banach algebra such that $\mathbb{A}$ is a Banach $\mathbb{U}$-bimodule, $\varphi \in \Delta(\mathbb{A}) \cap \mathbb{A}_*$ and $\mathbb{E}$ be a Banach $\mathbb{A}$-$\mathbb{U}$-bimodule. A bounded map $\mathcal{D}_{\mathbb{U}}$ from $\mathbb{A}$ to $\mathbb{E}$ is called a module φ-derivation if

$$\mathcal{D}_{\mathbb{U}}(v.a \pm b.w) = v.\mathcal{D}_{\mathbb{U}}(a) \pm \mathcal{D}_{\mathbb{U}}(b).w, \quad \mathcal{D}_{\mathbb{U}}(ab) = \mathcal{D}_{\mathbb{U}}(a).\varphi(b) + \varphi(a).\mathcal{D}_{\mathbb{U}}(b),$$

for all $a, b \in \mathbb{A}$ and $v, w \in \mathbb{U}$.

Definition 2.4. Suppose that $\mathbb{A}=(\mathbb{A}_*)^*$ and $\mathbb{U}$ are Banach algebras such that $\mathbb{A}$ is a Banach $\mathbb{U}$-bimodule and $\varphi\in\Delta(\mathbb{A})\cap\mathbb{A}_*$. Suppose that $\mathbb{E}$ is a symmetric Banach $\mathbb{A}$-$\mathbb{U}$-bimodule by given module action in (2.1). We say that $\mathcal{D}_{\mathbb{U}}(a)=\varphi(a).\mathbf{e}-\mathbf{e}.a$ is inner module φ-derivation for every $a\in\mathbb{A}$ and $\mathbf{e}\in\mathbb{E}$. In this case $\mathcal{D}_{\mathbb{U}}(a)$ denoted by $(\mathcal{D}_{\mathbb{U}})_{\mathbf{e}}$ or $(ad_{\mathbb{U}})_{\mathbf{e}}$.

Definition 2.5. Let $\mathbb{A}$ be a Banach algebra, $\mathbb{U}$ be a Banach algebra such that $\mathbb{A}$ is a Banach $\mathbb{U}$-bimodule and $\varphi\in\Delta(\mathbb{A})\cap\mathbb{A}_*$. We say that $\mathbb{A}$ is module φ-Connes amenable if for any symmetric normal Banach $\mathbb{A}$-$\mathbb{U}$-bimodule $\mathbb{E}$, each w^*-continuous module φ-derivation $\mathcal{D}_{\mathbb{U}}:\mathbb{A}\to\mathbb{E}$ is inner.

Remark 2.6. In [6], it is defined a certain version of module φ-Connes amenability where $\varphi:\mathbb{A}\to\mathbb{A}$ is a module homomorphism, as follows:

Let $\mathbb{A}$ be a Banach algebra, $\mathbb{U}$ be a Banach algebra such that $\mathbb{A}$ is a Banach $\mathbb{U}$-bimodule and $\varphi:\mathbb{A}\to\mathbb{A}$ is a map that satisfied

$$\varphi(v.a+b.w)=v.\varphi(a)+\varphi(b).w,\quad \varphi(ab)=\varphi(a)\varphi(b),$$

for every $a,b\in\mathbb{A}, v,w\in\mathbb{U}$.

In this case, $\mathbb{A}$ is called module φ-Connes amenable if for any symmetric normal Banach $\mathbb{A}$-$\mathbb{U}$-bimodule $\mathbb{E}$, each w^*-continuous module φ-derivation from $\mathbb{A}$ to $\mathbb{E}$ is inner.

3. Module $\varphi\otimes\psi$-Connes amenability and $\varphi\otimes\psi$-invariant mean

In this section, our aim is to study module $\varphi\otimes\psi$-Connes amenability of projective tensor product of Banach algebras. In the sequel, first by using Remark 2.6, we conclude the following result.

Proposition 3.1. *Let $\varphi:\mathbb{A}\to\mathbb{A}$ and $\psi:\mathbb{B}\to\mathbb{B}$ be two maps on Banach algebras. Suppose that $\mathbb{A}$ is module φ-Connes amenable and $\mathbb{B}$ is module ψ-Connes amenable. Then $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is module $\eta\circ(\varphi\otimes\psi)$-Connes amenable for any map $\eta:\mathbb{A}\widehat{\otimes}\mathbb{B}\to\mathbb{A}\widehat{\otimes}\mathbb{B}$.*

Proof. Let $\mathbb{U}$ be a Banach algebra such that $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is a Banach $\mathbb{U}$-bimodule. Let $\mathbb{E}$ be a symmetric normal Banach $\mathbb{A}\widehat{\otimes}\mathbb{B}$-$\mathbb{U}$-bimodule and $\mathcal{D}_{\mathbb{U}}:\mathbb{A}\widehat{\otimes}\mathbb{B}\to\mathbb{E}$ be a w^*-continuous module $\eta\circ(\varphi\otimes\psi)$-derivation. By the properties of Banach module in [3], it is clear that $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is module $\varphi\otimes\psi$-Connes amenable where $\varphi\otimes\psi(a\otimes b)=\varphi(a)\otimes\psi(b)$ for every $a\in\mathbb{A}, b\in\mathbb{B}$. We equip $\mathbb{E}$ for every $\mathbf{e}\in\mathbb{E}$ with the module operation via

$$(a\otimes b)\bullet\mathbf{e}=\eta(a\otimes b).\mathbf{e},\qquad \mathbf{e}\bullet(a\otimes b)=\mathbf{e}.\eta(a\otimes b)\tag{3.1}$$

for every $a\in\mathbb{A}$ and $b\in\mathbb{B}$. We obtain

$$\begin{aligned}\mathcal{D}_{\mathbb{U}}((a\otimes b)(c\otimes d))&=\mathcal{D}_{\mathbb{U}}(a\otimes b).\eta\circ(\varphi\otimes\psi)(c\otimes d)+\eta\circ(\varphi\otimes\psi)(a\otimes b).\mathcal{D}_{\mathbb{U}}(c\otimes d)\\&=\mathcal{D}_{\mathbb{U}}(a\otimes b)\bullet\varphi\otimes\psi(c\otimes d)+\varphi\otimes\psi(a\otimes b)\bullet\mathcal{D}_{\mathbb{U}}(c\otimes d)\end{aligned}$$

where $a,c\in\mathbb{A}$ and $b,d\in\mathbb{B}$. By hypothesis, there exists $\mathbf{y}\in\mathbb{E}$ such that

$$\begin{aligned}(\mathcal{D}_{\mathbb{U}})_{\mathbf{y}}(a\otimes b)&=\mathbf{y}\bullet(\varphi\otimes\psi)(a\otimes b)-(\varphi\otimes\psi)(a\otimes b)\bullet\mathbf{y}\\&=\mathbf{y}.\eta\circ(\varphi\otimes\psi)(a\otimes b)-\eta\circ(\varphi\otimes\psi)(a\otimes b).\mathbf{y}\end{aligned}$$

for $a,c\in\mathbb{A}$ and $b,d\in\mathbb{B}$. This shows that $\mathcal{D}_{\mathbb{U}}$ is an inner module $\eta\circ(\varphi\otimes\psi)$-derivation. □

Suppose that $\mathbb{A}=(\mathbb{A}_*)^*$ and $\mathbb{B}=(\mathbb{B}_*)^*$ are two Banach algebras, $\varphi\in\Delta(\mathbb{A})\cap\mathbb{A}_*$ and $\psi\in\Delta(\mathbb{B})\cap\mathbb{B}_*$. Using [14, Corollary 4.6], it follows that $\mathbb{A}_*\subset\sigma wc(\mathbb{A}^*)$ and $\mathbb{B}_*\subset\sigma wc(\mathbb{B}^*)$. Also, $\mathbb{A}\hat{\otimes}\mathbb{B}$ can be embedded in $(\sigma wc(\mathbb{A}\hat{\otimes}\mathbb{B})^*)^*$. Now, in the following, we prove several criteria for $(\mathbb{A}\hat{\otimes}\mathbb{B})^*$ to possess module $\varphi\otimes\psi$-Connes amenability.

Theorem 3.2. *With above notations, let $\mathbb{A}\widehat{\otimes}\mathbb{B}$ be a Banach algebra. Then $\mathbb{A}$ is module φ-Connes amenable and $\mathbb{B}$ is module ψ-Connes amenable if and only if $\sigma wc(\mathbb{A}\widehat{\otimes}\mathbb{B})$ is module $\varphi\otimes\psi$-Connes amenable.*

Proof. Let $\mathbb{A}$ be module φ-Connes amenable and $\mathbb{B}$ be module ψ-Connes amenable. Let $\mathbb{U}$ be a Banach algebra. Take $\mathbb{E}$ as a symmetric normal Banach $\mathbb{A}\widehat{\otimes}\mathbb{B}$-$\mathbb{U}$-bimodule and let $\mathcal{D}_{\mathbb{U}}:\mathbb{A}\widehat{\otimes}\mathbb{B}\to\mathbb{E}$ be a bounded w^*-continuous module $\varphi\otimes\psi$-derivation defined by

$$\mathcal{D}_{\mathbb{U}}\big((a\otimes b)(a'\otimes b')\big)=\mathcal{D}_{\mathbb{U}}(a\otimes b).(a'\otimes b')+(a\otimes b).\mathcal{D}_{\mathbb{U}}(a'\otimes b'),$$

for all $a,a'\in\mathbb{A}$ and $b,b'\in\mathbb{B}$. It is enough to show that $\mathcal{D}_{\mathbb{U}}$ is inner. Consider the module action

$$(a\otimes b).\mathbf{e}=\varphi\otimes\psi(a\otimes b)\mathbf{e} \tag{3.2}$$

for all $a\in\mathbb{A},b\in\mathbb{B}$ and $\mathbf{e}\in\mathbb{E}$. Since $\mathbb{E}$ is a Banach $\mathbb{A}\widehat{\otimes}\mathbb{B}$-$\mathbb{U}$-bimodule, then $\mathbb{E}_*$ can be equipped with the left and right module actions of $\mathbb{U}$ (see [15, Exercise 2.1.1]).

From [14, Corollary 4.6], the adjoint $\mathcal{D}^*_{\mathbb{U}}$ maps predual $\mathbb{E}_*$ into $\sigma wc(\mathbb{A}\widehat{\otimes}\mathbb{B})^*$. Assume that $d_{\mathbb{U}}:=\mathcal{D}^*_{\mathbb{U}}\mid_{\mathbb{E}_*}$ be the restriction of $\mathcal{D}^*_{\mathbb{U}}$ to $\mathbb{E}_*$. For all $a,a'\in\mathbb{A}$, $b,b'\in\mathbb{B}$ and $\mathbf{e}_0\in\mathbb{E}_*$, we get for the left action of $(\mathbb{A}\widehat{\otimes}\mathbb{B})$ on $\mathbb{E}_*$

$$\begin{aligned}\langle a'\otimes b',d_{\mathbb{U}}((a\otimes b).\mathbf{e}_0)\rangle&=\langle a'\otimes b',\mathcal{D}^*_{\mathbb{U}}|_{E_*}\big((a\otimes b).\mathbf{e}_0\big)\rangle\\&=\langle\mathcal{D}_{\mathbb{U}}(a'\otimes b'),(a\otimes b).\mathbf{e}_0\rangle\\&=\langle\mathcal{D}_{\mathbb{U}}(a'\otimes b').(a\otimes b),\mathbf{e}_0\rangle\\&=\langle\mathcal{D}_{\mathbb{U}}((a'\otimes b')(a\otimes b)),\mathbf{e}_0\rangle-\varphi\otimes\psi(a'\otimes b')\langle\mathcal{D}_{\mathbb{U}}(a\otimes b),\mathbf{e}_0\rangle\\&=\langle a'\otimes b',(a\otimes b).\mathcal{D}_{\mathbb{U}}(\mathbf{e}_0)\rangle-\varphi\otimes\psi(a'\otimes b')\langle\mathbf{e}_0,\mathcal{D}_{\mathbb{U}}(a\otimes b)\rangle.\end{aligned}$$

Therefore

$$d_{\mathbb{U}}((a\otimes b).\mathbf{e}_0)=(a\otimes b).d_{\mathbb{U}}(\mathbf{e}_0)-\langle\mathbf{e}_0,\mathcal{D}_{\mathbb{U}}(a\otimes b)\rangle\varphi\otimes\psi \tag{3.3}$$

for all $a\in\mathbb{A}$,$b\in\mathbb{B}$ and $\mathbf{e}_0\in\mathbb{E}_*$. Now take $\mathcal{D}'_{\mathbb{U}}=d^*_{\mathbb{U}}:(\sigma wc(\mathbb{A}\hat{\otimes}\mathbb{B})^*)^*\longrightarrow\mathbb{E}$ and consider those elements of $(\sigma wc(\mathbb{A}\hat{\otimes}\mathbb{B})^*)^*$ that lie in $\mathbb{A}\hat{\otimes}\mathbb{B}$ and set $\mathbf{y}=\mathcal{D}'_{\mathbb{U}}(\mathcal{M}\otimes\mathcal{N})\in\mathbb{E}$, where $\mathcal{M}$ and $\mathcal{N}$ are two means on $\mathbb{A}_*$ and $\mathbb{B}_*$, respectively. Using (3.3) it follows that

$$\begin{aligned}\langle\mathbf{y}.(a\otimes b),f\rangle&=\langle(a\otimes b).f,\mathcal{D}'_{\mathbb{U}}(\mathcal{M}\otimes\mathcal{N})\rangle=\langle d_{\mathbb{U}}((a\otimes b).f),\mathcal{M}\otimes\mathcal{N}\rangle\\&=\langle(a\otimes b).d_{\mathbb{U}}(f),\mathcal{M}\otimes\mathcal{N}\rangle-\langle f,\mathcal{D}_{\mathbb{U}}(a\otimes b)\rangle\langle\varphi\otimes\psi,\mathcal{M}\otimes\mathcal{N}\rangle\\&=\langle(a\otimes b).d_{\mathbb{U}}(f),\mathcal{M}\otimes\mathcal{N}\rangle-\langle f,\mathcal{D}_{\mathbb{U}}(a\otimes b)\rangle\langle\mathcal{M},\varphi\rangle\langle\mathcal{N},\psi\rangle\\&=\varphi\otimes\psi(a\otimes b)\langle d_{\mathbb{U}}(f),\mathcal{M}\otimes\mathcal{N}\rangle-\langle f,\mathcal{D}_{\mathbb{U}}(a\otimes b)\rangle\\&=\varphi\otimes\psi(a\otimes b)\langle f,\mathbf{y}\rangle-\langle f,\mathcal{D}_{\mathbb{U}}(a\otimes b)\rangle.\end{aligned}$$

Hence, by (3.2) we obtain

$$\begin{aligned}\mathcal{D}_{\mathbb{U}}(a\otimes b)&=\varphi\otimes\psi(a\otimes b).\mathbf{y}-\mathbf{y}.(a\otimes b)\\&=(a\otimes b).\mathbf{y}-\mathbf{y}.(a\otimes b)=(ad_{\mathbb{U}})_{\mathbf{y}}(a\otimes b),\end{aligned}$$

for all $a\in\mathbb{A}$ and $b\in\mathbb{B}$, as required. □

Example 3.3. Set $\mathbb{A}=\begin{pmatrix}0&0\\\mathbb{C}&\mathbb{C}\end{pmatrix}$. Considering usual matrix multiplication and l^1-norm, $\mathbb{A}$ is a dual Banach algebra. We consider $\varphi:\mathbb{A}\to\mathbb{C}$; $\varphi\begin{pmatrix}0&0\\z_1&z_2\end{pmatrix}=z_2$, for all $z_1,z_2\in\mathbb{C}$. Note that $\varphi\in\Delta(\mathbb{A})\cap\mathbb{A}_*$ is norm continuous and w^*-continuous. Let $u=\begin{pmatrix}0 & 0\\ 1 & -i\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ i & i\end{pmatrix}\in \mathbb{A}\widehat{\otimes}\mathbb{A}$. We can see that $u\in\sigma wc((\mathbb{A}\widehat{\otimes}\mathbb{A})^*)^*$ and therefore u has the property of $\varphi-\sigma wc-$virtual diagonal for Banach algebra $\mathbb{A}$. Consequently, there exists a $\varphi\otimes\varphi$-invariant mean on $(\mathbb{A}\widehat{\otimes}\mathbb{A})^*$ and so, on $(\mathbb{A}\widehat{\otimes}\mathbb{A})_*$ (for more details see [11]). Thus $(\mathbb{A}\widehat{\otimes}\mathbb{A})^*$ is module $\varphi\otimes\varphi$-Connes amenable. Also, $(\mathbb{A}\widehat{\otimes}\mathbb{A})_*$ is so.

Corollary 3.4. *Let $\mathbb{A}$ and $\mathbb{B}$ be Banach algebras, $\mathcal{F}$ be a subspace of $\mathbb{A}^*$ and $\mathcal{H}$ be a subspace of $\mathbb{B}^*$. Let $\mathcal{F}$ be module φ-Connes amenable and $\mathcal{H}$ be module ψ-Connes amenable, where $\varphi\in\Delta(\mathbb{A})\cap\mathcal{F}$ and $\psi\in\Delta(\mathbb{B})\cap\mathcal{H}$. Then $\mathcal{F}\widehat{\otimes}\mathcal{H}$ is module $\varphi\otimes\psi$-Connes amenable.*

Proof. Let $\mathbb{U}$ be a Banach algebra, where $\mathcal{F}\widehat{\otimes}\mathcal{H}$ is a Banach $\mathbb{U}$-bimodule. Let $\mathbb{E}_{\mathcal{F}}$ be a symmetric normal Banach $\mathcal{F}$-$\mathbb{U}$-bimodule and $\mathcal{D}^{\mathcal{F}}_{\mathbb{U}}:\mathcal{F}\to\mathbb{E}_{\mathcal{F}}$ be a w^*-continuous module φ-derivation. Also, let $\mathbb{E}_{\mathcal{H}}$ be a symmetric normal Banach $\mathcal{H}$-$\mathbb{U}$-bimodule and $\mathcal{D}^{\mathcal{H}}_{\mathbb{U}}:\mathcal{H}\to\mathbb{E}_{\mathcal{H}}$ be a w^*-continuous module ψ-derivation. We have

$$\mathcal{D}^{\mathcal{F}}_{\mathbb{U}}(f_1f_2)=\mathcal{D}^{\mathcal{F}}_{\mathbb{U}}(f_1).\varphi(f_2)+\varphi(f_1).\mathcal{D}^{\mathcal{F}}_{\mathbb{U}}(f_2)$$

and

$$\mathcal{D}^{\mathcal{H}}_{\mathbb{U}}(h_1h_2)=\mathcal{D}^{\mathcal{H}}_{\mathbb{U}}(h_1).\psi(h_2)+\psi(h_1).\mathcal{D}^{\mathcal{H}}_{\mathbb{U}}(h_2)$$

for every $f_1,f_2\in\mathcal{F}$ and $h_1,h_2\in\mathcal{H}$. It is clear that $\mathbb{E}_{\mathcal{F}}\widehat{\otimes}\mathbb{E}_{\mathcal{H}}$ is a symmetric normal Banach $\mathcal{F}\widehat{\otimes}\mathcal{H}$-$\mathbb{U}$-bimodule. By [15], there exists a w^*-continuous module $\varphi\otimes\psi$-derivation from $\mathcal{F}\widehat{\otimes}\mathcal{H}$ to $\mathbb{E}_{\mathcal{F}}\widehat{\otimes}\mathbb{E}_{\mathcal{H}}$, say $\mathcal{D}^{\mathcal{F}}_{\mathbb{U}}\otimes\mathcal{D}^{\mathcal{H}}_{\mathbb{U}}$. It is sufficient to show that each w^*-continuous module $\varphi\otimes\psi$-derivation such that $\mathcal{D}^{\mathcal{F}}_{\mathbb{U}}\otimes\mathcal{D}^{\mathcal{H}}_{\mathbb{U}}$ is inner. By hypothesis, there exists $x_{\mathcal{F}}\in\mathbb{E}_{\mathcal{F}}$ and $x_{\mathcal{H}}\in\mathbb{E}_{\mathcal{H}}$ such that

$$(\mathcal{D}^{\mathcal{F}}_{\mathbb{U}})_{x_{\mathcal{F}}}(f)=\varphi(f).x_{\mathcal{F}}-x_{\mathcal{F}}.\varphi(f),\qquad (f\in\mathcal{F})$$

and

$$(\mathcal{D}^{\mathcal{H}}_{\mathbb{U}})_{x_{\mathcal{H}}}(h)=\psi(h).x_{\mathcal{H}}-x_{\mathcal{H}}.\psi(h),\qquad (h\in\mathcal{H}).$$

Now we define

$$\mathcal{D}^{\mathcal{F}}_{\mathbb{U}}\otimes\mathcal{D}^{\mathcal{H}}_{\mathbb{U}}(f\otimes h):=(\mathcal{D}^{\mathcal{F}}_{\mathbb{U}})_{x_{\mathcal{F}}}(f)\otimes(\mathcal{D}^{\mathcal{H}}_{\mathbb{U}})_{x_{\mathcal{H}}}(h),\qquad (f\in\mathcal{F},h\in\mathcal{H}).$$

We can see that $\mathcal{D}^{\mathcal{F}}_{\mathbb{U}}\otimes\mathcal{D}^{\mathcal{H}}_{\mathbb{U}}$ is a module $\varphi\otimes\psi$-derivation. On the other side, there exists $x'_{\mathcal{F}}\in\mathbb{E}_{\mathcal{F}}$ and $x'_{\mathcal{H}}\in\mathbb{E}_{\mathcal{H}}$ such that

$$\begin{aligned}\mathcal{D}^{\mathcal{F}}_{\mathbb{U}}\otimes\mathcal{D}^{\mathcal{H}}_{\mathbb{U}}(f\otimes h)&=\big(\varphi(f).x'_{\mathcal{F}}-x'_{\mathcal{F}}.\varphi(f)\big)\otimes\big(\psi(h).x'_{\mathcal{H}}-x'_{\mathcal{H}}.\psi(h)\big)\\&=\varphi\otimes\psi(f\otimes h).(x'_{\mathcal{F}}\otimes x'_{\mathcal{H}})-(x'_{\mathcal{F}}\otimes x'_{\mathcal{H}}).\varphi\otimes\psi(f\otimes h).\end{aligned}$$

for some $x'_{\mathcal{F}}\otimes x'_{\mathcal{H}}\in\mathbb{E}_{\mathcal{F}}\widehat{\otimes}\mathbb{E}_{\mathcal{H}}$. Therefore, $\mathcal{D}^{\mathcal{F}}_{\mathbb{U}}\otimes\mathcal{D}^{\mathcal{H}}_{\mathbb{U}}$ is inner. □

Example 3.5. We set $Luc(\mathbb{N})$ to be the set of all bounded left uniformly continuous functions on $\mathbb{N}$. Now consider discrete semigroup $\mathbb{N}$ with trivial character 1. There is a 1-mean on $Luc(\mathbb{N})$. But $l^1(\mathbb{N})$ is not module 1-Connes amenable. Equivalently, there is not a 1-mean on $l^\infty(\mathbb{N})$. Therefore, for character $\varphi=1$ there exists a $\varphi\otimes\varphi$-invariant mean on subspace $Luc(\mathbb{N})\hat{\otimes}Luc(\mathbb{N})$ of $l^\infty(\mathbb{N})\hat{\otimes}l^\infty(\mathbb{N})$, but $l^1(\mathbb{N})\hat{\otimes}l^1(\mathbb{N})$ is not module $1\otimes 1$-Connes amenable.

The relation between amenability and Connes amenability for semigroup algebras is proved by Daws in [4]. Indeed, by [4, Theorem 5.14], it is shown that for a cancellative semigroup S and character $\varphi\in c_0(S)$, there exists a $\varphi\otimes\varphi$-invariant mean on $c_0(S)\otimes_w c_0(S)$ if there exists a $\varphi\otimes\varphi$-invariant mean on $l^\infty(S)\hat{\otimes}l^\infty(S)$ and vice versa. On the other hand, if S is a weakly cancellative semigroup such that $l^1(S)$ is not amenable and $l^1(S)$ is module Connes amenable, then there exists a $\varphi\otimes\varphi$-invariant mean on $c_0(S)\otimes_w c_0(S)$ but it is not a $\varphi\otimes\varphi$-invariant mean on $l^\infty(S)\hat{\otimes}l^\infty(S)$.

In the sequel, we denote the set of all w^*-continuous homomorphisms from a Banach algebra $\mathbb{A}$ onto $\mathbb{C}$ by $\Delta_{\omega^*}(\mathbb{A})$.

Theorem 3.6. *Let $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{A}\widehat{\otimes}\mathbb{B}$ be Banach algebras, and let $\varphi\in\Delta_{\omega^*}(\mathbb{A})\cap\mathbb{A}_*$ and $\psi\in\Delta_{\omega^*}(\mathbb{B})\cap\mathbb{B}_*$. Let $\mathbb{I}$, $\mathbb{J}$ be Banach algebras and closed two-sided ideals of $\mathbb{A}$, $\mathbb{B}$, respectively. If $\varphi\mid_{\mathbb{I}}\neq 0$, $\psi\mid_{\mathbb{J}}\neq 0$ and $\mathbb{I}\widehat{\otimes}\mathbb{J}$ is a Banach algebra that is module $\varphi\otimes\psi\mid_{\mathbb{I}\widehat{\otimes}\mathbb{J}}$-Connes amenable, then $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is module $\varphi\otimes\psi$-Connes amenable.*

Proof. Suppose that $\mathbb{U}$ is a Banach algebra and $\mathbb{E}$ is a symmetric normal Banach $\mathbb{A}\widehat{\otimes}\mathbb{B}$-$\mathbb{U}$-bimodule which,

$$(a\otimes b).\mathbf{e}=\varphi\otimes\psi(a\otimes b)\mathbf{e},\qquad (a\in\mathbb{A},b\in\mathbb{B},\mathbf{e}\in\mathbb{E}).$$

Suppose that $\mathcal{D}_{\mathbb{U}}:\mathbb{A}\widehat{\otimes}\mathbb{B}\to\mathbb{E}$ be a bounded w^*-continuous module $\varphi\otimes\psi$-derivation. It is clear that $\mathcal{D}_{\mathbb{U}}\mid_{\mathbb{I}\widehat{\otimes}\mathbb{J}}$ is a w^*-continuous module $\varphi\otimes\psi$-derivation. Since $\mathbb{I}\widehat{\otimes}\mathbb{J}$ is module $\varphi\otimes\psi\mid_{\mathbb{I}\widehat{\otimes}\mathbb{J}}$-Connes amenable, there exists $\mathbf{y}\in\mathbb{E}$ such that

$$(\mathcal{D}_{\mathbb{U}})_{\mathbf{y}}(i\otimes j)=\varphi\otimes\psi(i\otimes j).\mathbf{y}-\mathbf{y}.\varphi\otimes\psi(i\otimes j),$$

for all $i\in\mathbb{I},j\in\mathbb{J}$. Choose $i'\otimes j'\in\mathbb{I}\widehat{\otimes}\mathbb{J}$ with $\varphi(i')=1=\psi(j')$, and put $\mathbf{e}=\mathbf{y}.(i'\otimes j')$. Now, for $(a\otimes b)\in\mathbb{A}\widehat{\otimes}\mathbb{B}$ we obtain

$$\begin{aligned}\varphi\otimes\psi(a\otimes b).\mathbf{e}-\mathbf{e}.\varphi\otimes\psi(a\otimes b)&=\varphi\otimes\psi(a\otimes b)\mathbf{y}.(i'\otimes j')-\mathbf{y}.(i'\otimes j').\varphi\otimes\psi(a\otimes b)\\&\quad-\varphi\otimes\psi\big((a\otimes b)(i'\otimes j')\big)\mathbf{y}+\varphi\otimes\psi\big((a\otimes b)(i'\otimes j')\big)\mathbf{y}\\&=-\varphi\otimes\psi(a\otimes b).\big(\varphi\otimes\psi(i'\otimes j')\mathbf{y}-\mathbf{y}.(i'\otimes j')\big)\\&\quad+\varphi\otimes\psi\big((a\otimes b)(i'\otimes j')\big)\mathbf{y}-\mathbf{y}.(i'\otimes j').\varphi\otimes\psi(a\otimes b).\end{aligned}$$

By restricting $\mathcal{D}_{\mathbb{U}}$ on $\mathbb{I}\widehat{\otimes}\mathbb{J}$, it follows that

$$\begin{aligned}\varphi\otimes\psi(a\otimes b).\mathbf{e}-\mathbf{e}.\varphi\otimes\psi(a\otimes b)&=-\varphi\otimes\psi(a\otimes b)\mathcal{D}_{\mathbb{U}}\mid_{\mathbb{I}\widehat{\otimes}\mathbb{J}}(i'\otimes j')+\mathcal{D}_{\mathbb{U}}\mid_{\mathbb{I}\widehat{\otimes}\mathbb{J}}((i'\otimes j')(a\otimes b))\\&=-\varphi\otimes\psi(a\otimes b)\mathcal{D}_{\mathbb{U}}\mid_{\mathbb{I}\widehat{\otimes}\mathbb{J}}(i'\otimes j')+\varphi\otimes\psi(i'\otimes j').\mathcal{D}_{\mathbb{U}}(a\otimes b)\\&\quad+\mathcal{D}_{\mathbb{U}}\mid_{\mathbb{I}\widehat{\otimes}\mathbb{J}}(i'\otimes j').\varphi\otimes\psi(a\otimes b)\\&=-\varphi\otimes\psi(a\otimes b)\mathcal{D}_{\mathbb{U}}\mid_{\mathbb{I}\widehat{\otimes}\mathbb{J}}(i'\otimes j')+\mathcal{D}_{\mathbb{U}}(a\otimes b)\\&\quad+\mathcal{D}_{\mathbb{U}}\mid_{\mathbb{I}\widehat{\otimes}\mathbb{J}}(i'\otimes j').\varphi\otimes\psi(a\otimes b)\\&=\mathcal{D}_{\mathbb{U}}(a\otimes b)+\big(\varphi\otimes\psi(i'\otimes j')\mathbf{y}-\mathbf{y}.\varphi\otimes\psi(i'\otimes j')\big).(a\otimes b)\\&\quad-\varphi\otimes\psi(a\otimes b)\big(\varphi\otimes\psi(i'\otimes j').\mathbf{y}-\mathbf{y}.\varphi\otimes\psi(i'\otimes j')\big)\\&=\mathcal{D}_{\mathbb{U}}(a\otimes b)+\big(\mathbf{y}-\mathbf{y}.(i'\otimes j')\big).\varphi\otimes\psi(a\otimes b)\\&\quad-\varphi\otimes\psi(a\otimes b)\big(\mathbf{y}-\mathbf{y}.(i'\otimes j')\big).\end{aligned}$$

Therefore, $\mathcal{D}_{\mathbb{U}}(a\otimes b)=\varphi\otimes\psi(a\otimes b).\mathbf{y}-\mathbf{y}.(\varphi\otimes\psi)(a\otimes b)$ is inner. □

In the sequel, under the condition of existence of bounded approximate identities for the ideals, $\mathbb{I}$ and $\mathbb{J}$, we show that the converse of Theorem 3.6 holds.

Theorem 3.7. *Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{A}\widehat{\otimes}\mathbb{B}$, $\mathbb{I}$, $\mathbb{J}$, φ and ψ be as above. Let $\mathbb{I}$, $\mathbb{J}$ be with bounded approximate identities such that $\varphi\mid_{\mathbb{I}}$ and $\psi\mid_{\mathbb{J}}$ are non-zero and $\mathbb{I}\widehat{\otimes}\mathbb{J}$ be a Banach algebra. If $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is module $\varphi\otimes\psi$-Connes amenable, then $\mathbb{I}\widehat{\otimes}\mathbb{J}$ is module $\varphi\otimes\psi\mid_{\mathbb{I}\widehat{\otimes}\mathbb{J}}$-Connes amenable.*

Proof. Let $\mathbb{U}$ be a Banach algebra and $\mathbb{A}\widehat{\otimes}\mathbb{B}$ be module $\varphi\otimes\psi$-Connes amenable. Suppose that $\mathbb{E}_*$ is pseudo-unital and $\mathbb{E}$ is a symmetric normal Banach $\mathbb{I}\widehat{\otimes}\mathbb{J}$-$\mathbb{U}$-bimodule. Define the left action of $\mathbb{I}\widehat{\otimes}\mathbb{J}$ on $\mathbb{E}$ via

$$(a\otimes b).\mathbf{e}=\varphi\otimes\psi(a\otimes b)\mathbf{e},\qquad (a\in\mathbb{I},b\in\mathbb{J},\mathbf{e}\in\mathbb{E}).$$

We show that every bounded w^*-continuous module $\varphi\otimes\psi\mid_{\mathbb{I}\widehat{\otimes}\mathbb{J}}$-derivation $\mathcal{D}_{\mathbb{U}}:\mathbb{I}\widehat{\otimes}\mathbb{J}\to\mathbb{E}$ is inner. By similar argument as in [15, Proposition 2.1.6], it is easy to see that $\mathbb{E}$ is a symmetric normal Banach $\mathbb{A}\widehat{\otimes}\mathbb{B}$-$\mathbb{U}$-bimodule. Suppose that $\{a_\alpha\}_\alpha\subseteq\mathbb{A}$ and $\{b_\beta\}_\beta\subseteq\mathbb{B}$ are two nets that

$$\{a_\alpha\}_\alpha\xrightarrow{w^*}a,\qquad\qquad \{b_\beta\}_\beta\xrightarrow{w^*}b.$$

Let $\{a_\alpha\otimes b_\beta\}_{\alpha,\beta}$ be a net in $\mathbb{A}\widehat{\otimes}\mathbb{B}$ such that $\{a_\alpha\otimes b_\beta\}_{\alpha,\beta}\xrightarrow{w^*}a\otimes b$. Pick $\mathbf{e}_0\in\mathbb{E}_*$. There exists $i\otimes j\in\mathbb{I}\widehat{\otimes}\mathbb{J}$ and $\mathbf{y}_0\in\mathbb{E}_*$ such that $\mathbf{e}_0=(i\otimes j).\mathbf{y}_0$. We have

$$\begin{aligned}w^*-\lim_\alpha(w^*-\lim_\beta\langle\mathbf{e}.(a_\alpha\otimes b_\beta),\mathbf{e}_0\rangle)&=w^*-\lim_\alpha(w^*-\lim_\beta\langle\mathbf{e}.(a_\alpha\otimes b_\beta),(i\otimes j).\mathbf{y}_0\rangle)\\&=w^*-\lim_\alpha(w^*-\lim_\beta\langle\mathbf{e}.(a_\alpha\otimes b_\beta).(i\otimes j),\mathbf{y}_0\rangle)\\&=\langle\mathbf{e}.(a\otimes b),\mathbf{e}_0\rangle.\end{aligned}$$

Now we extend $\mathcal{D}_{\mathbb{U}}$. For this purpose, consider $\{e_\alpha\}_\alpha\subseteq\mathbb{A}$ and $\{e_\beta\}_\beta\subseteq\mathbb{B}$ as bounded approximate identities. Define $\widetilde{\mathcal{D}}_{\mathbb{U}}:\mathbb{A}\widehat{\otimes}\mathbb{B}\to\mathbb{E}$, via

$$a\otimes b\mapsto w^*-\lim_\alpha[w^*-\lim_\beta\mathcal{D}_{\mathbb{U}}\big((a\otimes b)(e_\alpha\otimes e_\beta)\big)-(a\otimes b).\mathcal{D}_{\mathbb{U}}(e_\alpha\otimes e_\beta)].$$

It is clear that $\widetilde{\mathcal{D}}_{\mathbb{U}}$ is a continuous module $\varphi\otimes\psi$-derivation. To see that $\widetilde{\mathcal{D}}_{\mathbb{U}}$ is continuous with respect to the w^*-topology, we have

$$\begin{aligned}w^*-\lim_\alpha(w^*-\lim_\beta\langle\widetilde{\mathcal{D}}_{\mathbb{U}}(a_\alpha\otimes b_\beta),\mathbf{e}_0\rangle)&=w^*-\lim_\alpha(w^*-\lim_\beta\langle\widetilde{\mathcal{D}}_{\mathbb{U}}(a_\alpha\otimes b_\beta),(i\otimes j).\mathbf{y}_0\rangle)\\&=w^*-\lim_\alpha[w^*-\lim_\beta\langle\widetilde{\mathcal{D}}_{\mathbb{U}}\big((a_\alpha\otimes b_\beta).(i\otimes j)\big)\\&-(a_\alpha\otimes b_\beta).\widetilde{\mathcal{D}}_{\mathbb{U}}(i\otimes j),\mathbf{y}_0\rangle]\\&=\langle\widetilde{\mathcal{D}}_{\mathbb{U}}\big((a\otimes b).(i\otimes j)\big)-(a\otimes b).\widetilde{\mathcal{D}}_{\mathbb{U}}(i\otimes j),\mathbf{y}_0\rangle\\&=\langle\widetilde{\mathcal{D}}_{\mathbb{U}}(a\otimes b).(i\otimes j),\mathbf{y}_0\rangle\\&=\langle\widetilde{\mathcal{D}}_{\mathbb{U}}(a\otimes b),\mathbf{e}_0\rangle,\end{aligned}$$

because $\mathcal{D}_{\mathbb{U}}$ is w^*-continuous and $\mathbb{E}$ is a normal Banach $\mathbb{A}\widehat{\otimes}\mathbb{B}$-$\mathbb{U}$-module. Therefore $\widetilde{\mathcal{D}}_{\mathbb{U}}$ is w^*-continuous. Since $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is module $\varphi\otimes\psi$-Connes amenable, then $\widetilde{\mathcal{D}}_{\mathbb{U}}$ is inner. Thus $\mathcal{D}_{\mathbb{U}}$ has the desired properties. □

4. Module (ψ,θ)-Connes amenability and (ψ,θ)-invariant mean

In this section, using the concept of (ψ,θ)-invariant mean, we study module (ψ,θ)-Connes amenability. Suppose that $\mathbb{A}$ is an unital dual Banach algebra with predual $\mathbb{A}_*$, the identity $e_{\mathbb{A}}$ and $\mathbb{B}$ is a dual Banach algebra with predual $\mathbb{B}_*$. Suppose that $\psi \in \Delta(\mathbb{A}) \cap \mathbb{A}_*$, $\theta \in \Delta(\mathbb{B}) \cap \mathbb{B}_*$ and $\Theta : \mathbb{B} \to \mathbb{A}$ is an algebraic homomorphism, which $\Theta(b) = \theta(b)e_{\mathbb{A}}$. The Θ-Lau product $\mathbb{A} \times_\Theta \mathbb{B}$ is defined by

$$(a,b).(a',b') = (a.a' + \Theta(b').a + \Theta(b).a', bb')$$

and the norm $\|(a,b)\|_{\mathbb{A}\times_\Theta\mathbb{B}} = \|a\|_{\mathbb{A}} + \|b\|_{\mathbb{B}}$ for all $a,a' \in \mathbb{A}$ and $b,b' \in \mathbb{B}$. This definition is a certain case of the product that is presented in [12]. In this section, we investigate the notions of module (ψ,θ)-Connes amenability and module $(0,\theta)$-Connes amenability. Since $\theta \in \Delta(\mathbb{B}) \cap \mathbb{B}_*$, then $\mathbb{A} \times_\Theta \mathbb{B}$ is a dual Banach algebra with predual $\mathbb{A}_* \times \mathbb{B}_*$. It is known that $(\mathbb{A} \times_\Theta \mathbb{B})^*$ is identified with $\mathbb{A}^* \times \mathbb{B}^*$ that $\langle (f,g),(a,b)\rangle = f(a) + g(b)$ for all $a \in \mathbb{A}, b \in \mathbb{B}$ and $f \in \mathbb{A}^*, g \in \mathbb{B}^*$. Let $\mathbb{A}^{**}$ and $\mathbb{B}^{**}$ be equipped by the first Arens product. Define $\Lambda := (\Gamma, 0)$ where, $\Lambda \in \mathbb{A}^{**} \times_{\Theta^{**}} \mathbb{B}^{**} = (\mathbb{A} \times_\Theta \mathbb{B})^{**}$ and Γ is ψ-invariant mean on $\mathbb{A}_*$. We define $\langle (\Gamma,0),(\psi,\theta)\rangle = \langle \Gamma,\psi\rangle + \langle 0,\theta\rangle = 1$. If we consider Banach algebra $\mathbb{A}$ with $\mathbb{A} \times \{0\}$, then $\mathbb{A}$ is a closed ideal in $\mathbb{A} \times_\Theta \mathbb{B}$ and $(\mathbb{A} \times_\Theta \mathbb{B})/\mathbb{A}$ is isometric isomorphism with $\mathbb{B}$, that means $\frac{\mathbb{A}\times_\Theta\mathbb{B}}{\mathbb{A}} \simeq \mathbb{B}$ (see [12]).

In the following we investigate module Connes amenability of $\mathbb{A} \times_\Theta \mathbb{B}$.

Lemma 4.1. *Let $\mathbb{A}$ be an unital dual Banach algebra with predual $\mathbb{A}_*$ and let $\mathbb{B}$ be a dual Banach algebra with predual $\mathbb{B}_*$. Then the following two statements are equivalent:*

(i) *$\mathbb{A}$ and $\mathbb{B}$ are module id-Connes amenable.*
(ii) *$\mathbb{A} \times_\Theta \mathbb{B}$ is module $id \otimes id$-Connes amenable.*

Proof. $(i) \Rightarrow (ii)$ Let $\mathbb{U}$ be a Banach algebra, $\mathbb{A}$ and $\mathbb{B}$ be module id-Connes amenable dual Banach algebras. Moreover, let $\mathbb{E}$ be a symmetric normal Banach $\mathbb{A} \times_\Theta \mathbb{B}$-$\mathbb{U}$-bimodule and $\mathcal{D}_{\mathbb{U}} : \mathbb{A} \times_\Theta \mathbb{B} \to \mathbb{E}$ be a bounded w^*-continuous module $id \otimes id$-derivation. We consider the restriction $\mathcal{D}_{\mathbb{U}}$ on $\mathbb{A}$ and put $\mathcal{D}_{\mathbb{U}}^{\mathbb{A}} = \mathcal{D}_{\mathbb{U}} \mid_{\mathbb{A}}: \mathbb{A} \times_\Theta \{0\} \to \mathbb{E}$. Since, $\mathbb{A}$ is module id-Connes amenable, so there exists $\mathbf{e} \in \mathbb{E}$ such that $\mathcal{D}_{\mathbb{U}}^{\mathbb{A}} = (ad_{\mathbb{U}})_{\mathbf{e}}$ and $\widetilde{\mathcal{D}}_{\mathbb{U}} = \mathcal{D}_{\mathbb{U}}^{\mathbb{A}} - (ad_{\mathbb{U}})_{\mathbf{e}}$ vanishes on $\mathbb{A}$. Then, $\widetilde{\mathcal{D}}_{\mathbb{U}}$ has a continuous extension, say $\mathcal{D}_{\mathbb{U}}^{\mathbb{B}} : \frac{\mathbb{A}\times_\Theta\mathbb{B}}{\mathbb{A}} \to \mathbb{E}$. Define

$$(\mathbb{A} \times_\Theta \{0\})_E = \{(a,0).\ \mathbf{y} - \mathbf{y}.(a,0) : a \in \mathbb{A},\ \mathbf{y} \in \mathbb{E}\}.$$

Note that $(\mathbb{A} \times_\Theta \{0\})_{\mathbb{E}}$ is a symmetric normal Banach $\{0\} \times_\Theta \mathbb{B}$-$\mathbb{U}$-module and image of $\mathcal{D}_{\mathbb{U}}^{\mathbb{B}}$ on $\mathbb{B}$ is orthogonal on $(\mathbb{A} \times_\Theta \{0\})_{\mathbb{E}}$. In fact, by concept of derivation and that $\mathbb{B}$ is module id-Connes amenable, it follows

$$\begin{aligned}\mathcal{D}_{\mathbb{U}}^{\mathbb{B}}((a,0) \times_\Theta (0,b)) &= \mathcal{D}_{\mathbb{U}}^{\mathbb{B}}(a,0).id(0,b) + id(a,0).\mathcal{D}_{\mathbb{U}}^{\mathbb{B}}(0,b)\\ &= (a,0).\mathcal{D}_{\mathbb{U}}^{\mathbb{B}}(0,b)\end{aligned} \tag{4.1}$$

$$\begin{aligned}\mathcal{D}_{\mathbb{U}}^{\mathbb{B}}((0,b) \times_\Theta (a,0)) &= \mathcal{D}_{\mathbb{U}}^{\mathbb{B}}(0,b).id(a,0) + id(0,b).\mathcal{D}_{\mathbb{U}}^{\mathbb{B}}(a,0)\\ &= \mathcal{D}_{\mathbb{U}}^{\mathbb{B}}(0,b).(a,0),\end{aligned} \tag{4.2}$$

for every $a \in \mathbb{A}$ and $b \in \mathbb{B}$. Then, by using (4.1) and (4.2) we have

$$\langle \mathcal{D}_{\mathbb{U}}^{\mathbb{B}}(0,b), (a,0).\ \mathbf{y} - \mathbf{y}.\ (a,0)\rangle = \langle \mathcal{D}_{\mathbb{U}}^{\mathbb{B}}(0,b).(a,0) - (a,0).\mathcal{D}_{\mathbb{U}}^{\mathbb{B}}(0,b), \mathbf{y}\rangle,$$

for every $\mathbf{y}\in\mathbb{E}$. It is easy to see that

$$(\mathbb{A}\times_{\Theta}\{0\})_{\mathbb{E}}{}^{\perp}=\{\lambda\in\mathbb{E}^{*}:\langle\lambda,\mathtt{f}\rangle=0,\forall\mathtt{f}\in(\mathbb{A}\times_{\Theta}\{0\})_{\mathbb{E}}\}$$

is symmetric normal dual Banach $\mathbb{B}$-$\mathbb{U}$-module with predual $\dfrac{\mathbb{E}_{*}}{(\mathbb{A}\times_{\Theta}\{0\})_{\mathbb{E}}}$. Since $\mathbb{B}$ is module id-Connes amenable, there exists $\delta\in(\mathbb{A}\times_{\Theta}\{0\})_{\mathbb{E}}{}^{\perp}$ which $\mathcal{D}^{\mathbb{B}}_{\mathbb{U}}(0,b)=(ad_{\mathbb{U}})_{\delta}$. Therefore, $\widetilde{\mathcal{D}}_{\mathbb{U}}$ is inner. All in all, we conclude that $\mathcal{D}_{\mathbb{U}}$ is inner.

$(ii)\Rightarrow(i)$ Let $\mathbb{A}\times_{\Theta}\mathbb{B}$ be module $id\otimes id$-Connes amenable. First, we show that $\mathbb{A}$ is module id-Connes amenable. For this purpose, let $\mathbb{E}$ be a symmetric normal dual Banach $\mathbb{A}$-$\mathbb{U}$-bimodule and let $\mathcal{D}_{\mathbb{U}}:\mathbb{A}\to\mathbb{E}$ be a bounded w^{*}-continuous module id-derivation. By [5, Lemma 3.3], and without loss of generality, suppose that $\mathbb{E}_{*}$ is pseudo-unital. Since $\mathbb{B}=(\mathbb{B}_{*})^{*}$ is a dual Banach algebra, $\mathbb{E}_{*}$ is an $\mathbb{A}\times_{\Theta}\mathbb{B}$-$\mathbb{U}$-bimodule. Pick $x_0\in\mathbb{E}_{*}$. Since $\mathbb{E}_{*}$ is pseudo-unital, there are $(c,0)\in\mathbb{A}\times_{\Theta}\{0\}$ and $\mathbf{y}_0\in\mathbb{E}_{*}$ such that $\mathbf{e}_0=(c,0).\mathbf{y}_0$. Define $(a,b).\mathbf{e}_0=(a,b).(c,0).\mathbf{y}_0$ for all $(a,b)\in\mathbb{A}\times_{\Theta}\mathbb{B}$. We claim that $\mathbb{E}$ is a symmetric normal Banach $\mathbb{A}\times_{\Theta}\mathbb{B}$-$\mathbb{U}$-bimodule. Indeed, let $\{(a_\alpha,b_\alpha)\}$ be a net in $\mathbb{A}\times_{\Theta}\mathbb{B}$ such that $(a_\alpha,b_\alpha)\xrightarrow{w^{*}}(a,b)$ on $\mathbb{A}\times_{\Theta}\mathbb{B}$ and let $\mathbf{e}\in\mathbb{E}$. It follows that

$$\begin{aligned}
w^{*}-\lim_{\alpha}\langle\mathbf{e}.(a_\alpha,b_\alpha),\mathbf{e}_0\rangle&=w^{*}-\lim_{\alpha}\langle\mathbf{e}.(a_\alpha,b_\alpha),(c,0).\mathbf{y}_0\rangle\\
&=w^{*}-\lim_{\alpha}\langle\mathbf{e}.(a_\alpha,b_\alpha)(c,0),\mathbf{y}_0\rangle\\
&=w^{*}-\lim_{\alpha}\langle\mathbf{e}.(a_\alpha c+\Theta(0).a_\alpha+\Theta(b_\alpha).c,b_\alpha.0),\mathbf{y}_0\rangle\\
&=w^{*}-\lim_{\alpha}\langle\mathbf{e}.(a_\alpha c+\theta(b_\alpha)e_{\mathbb{A}}.c,0),\mathbf{y}_0\rangle\\
&=\langle\mathbf{e}.(a,b),\mathbf{e}_0\rangle.
\end{aligned}$$

Now, we extend $\mathcal{D}_{\mathbb{U}}$ from $\mathbb{A}$ to $\mathbb{A}\times_{\Theta}\mathbb{B}$. Therefore, let

$$\widetilde{\mathcal{D}}_{\mathbb{U}}:\mathbb{A}\times_{\Theta}\mathbb{B}\to\mathbb{E},\quad(a,b)\mapsto\mathcal{D}_{\mathbb{U}}((a,b)(e_{\mathbb{A}},0))-(a,b).\,\mathcal{D}_{\mathbb{U}}(e_{\mathbb{A}},0).$$

$\widetilde{\mathcal{D}}_{\mathbb{U}}$ is a continuous module id-derivation. We denote the restriction of $\widetilde{\mathcal{D}}_{\mathbb{U}}$ on $\mathbb{A}$ by $\widetilde{\mathcal{D}}_{\mathbb{U}}|_{\mathbb{A}}$. We show that $\widetilde{\mathcal{D}}_{\mathbb{U}}$ is w^{*}-continuous. Similar to above argument, let $\{(a_\alpha,b_\alpha)\}$ be a net in $\mathbb{A}\times_{\Theta}\mathbb{B}$ such that $(a_\alpha,b_\alpha)\xrightarrow{w^{*}}(a,b)$ on $\mathbb{A}\times_{\Theta}\mathbb{B}$. For $\mathbf{e}_0\in\mathbb{E}_{*}$, let $(c,0)\in\mathbb{A}\times_{\Theta}\{0\}$ and $\mathbf{y}_0\in\mathbb{E}_{*}$ such that $\mathbf{e}_0=(c,0).\mathbf{y}_0$. Then

$$\begin{aligned}
w^{*}-\lim_{\alpha}\langle\widetilde{\mathcal{D}}_{\mathbb{U}}(a_\alpha,b_\alpha),\mathbf{e}_0\rangle&=w^{*}-\lim_{\alpha}\langle\widetilde{\mathcal{D}}_{\mathbb{U}}(a_\alpha,b_\alpha),(c,0).\mathbf{y}_0\rangle\\
&=w^{*}-\lim_{\alpha}\langle\widetilde{\mathcal{D}}_{\mathbb{U}}(a_\alpha,b_\alpha).(c,0),\mathbf{y}_0\rangle\\
&=w^{*}-\lim_{\alpha}\langle\widetilde{\mathcal{D}}_{\mathbb{U}}((a_\alpha,b_\alpha).(c,0))-(a_\alpha,b_\alpha).\widetilde{\mathcal{D}}_{\mathbb{U}}(c,0),\mathbf{y}_0\rangle\\
&=w^{*}-\lim_{\alpha}\langle\widetilde{\mathcal{D}}_{\mathbb{U}}|_{\mathbb{A}}(a_\alpha c+\Theta(0).a_\alpha+\Theta(b_\alpha).c,0)\\
&-(a_\alpha,b_\alpha).\widetilde{\mathcal{D}}_{\mathbb{U}}|_{\mathbb{A}}(c,0),\mathbf{y}_0\rangle\\
&=\langle\mathcal{D}_{\mathbb{U}}(ac+\Theta(0).a+\Theta(b).c,0)-(a,b).\mathcal{D}_{\mathbb{U}}(c,0),\mathbf{y}_0\rangle\\
&=\langle\mathcal{D}_{\mathbb{U}}((a,b).(c,0))-(a,b).\mathcal{D}_{\mathbb{U}}(c,0),\mathbf{y}_0\rangle\\
&=\langle\widetilde{\mathcal{D}}_{\mathbb{U}}((a,b).(c,0))-(a,b).\widetilde{\mathcal{D}}_{\mathbb{U}}(c,0),\mathbf{y}_0\rangle\\
&=\langle\widetilde{\mathcal{D}}_{\mathbb{U}}(a,b),(c,0).\mathbf{y}_0\rangle=\langle\widetilde{\mathcal{D}}_{\mathbb{U}}(a,b),\mathbf{e}_0\rangle,
\end{aligned}$$

because $\mathcal{D}_{\mathbb{U}}$ is w^*-continuous, $\mathbb{E}$ is an $\mathbb{A}\times_{\Theta}\{0\}$-$\mathbb{U}$-bimodule and $\mathbb{A}$ is normal dual Banach $\mathbb{A}$-$\mathbb{U}$-bimodule. From the previous argument and module $id\otimes id$-Connes amenability of $\mathbb{A}\times_{\Theta}\mathbb{B}$, we conclude that $\widetilde{\mathcal{D}}_{\mathbb{U}}$ and hence $\mathcal{D}_{\mathbb{U}}$ is inner.

By a similar argument, $\mathbb{B}$ is module id-Connes amenable. □

In [12], it is shown that

$$\Delta(\mathbb{A}\times_{\Theta}\mathbb{B})=\{(\psi,\theta):\psi\in\Delta(\mathbb{A})\}\cup\{(0,\varphi):\varphi\in\Delta(\mathbb{B})\},$$

where, $\theta\in\Delta(\mathbb{B})\cap\mathbb{B}_*$. In the following theorem we extend Lemma 4.1.

Theorem 4.2. *Let $\mathbb{A}$ be an unital dual Banach algebra that is Arens regular and $\mathbb{B}$ be a dual Banach algebra. Let $\psi\in\Delta(\mathbb{A})\cap\mathbb{A}_*$ and $\theta\in\Delta(\mathbb{B})\cap\mathbb{B}_*$. Then the following statements hold:*

(i) *If $\mathbb{A}\times_{\Theta}\mathbb{B}$ is module (ψ,θ)-Connes amenable, then $\mathbb{A}$ is module ψ-Connes amenable and vice versa.*

(ii) *If $\mathbb{A}\times_{\Theta}\mathbb{B}$ is module $(0,\theta)$-Connes amenable, then $\mathbb{B}$ is module θ-Connes amenable and vice versa.*

Proof. (i) Let $\mathbb{A}$ be module ψ-Connes amenable and $\mathbb{U}$ be a Banach algebra. Suppose that $\mathbb{E}$ is a symmetric normal Banach $\mathbb{A}$-$\mathbb{U}$-bimodule and $\mathcal{D}_{\mathbb{U}}$ from $\mathbb{A}$ to $\mathbb{E}$ is a module ψ-derivation. For extension $\mathcal{D}_{\mathbb{U}}$, we define

$$\widetilde{\mathcal{D}}_{\mathbb{U}}:\mathbb{A}\times_{\Theta}\mathbb{B}\to\mathbb{E},\qquad \widetilde{\mathcal{D}}_{\mathbb{U}}(a\otimes b)=\mathcal{D}_{\mathbb{U}}(a).b.$$

It is clear that $\mathbb{E}$ is a symmetric normal Banach $\mathbb{A}\times_{\Theta}\mathbb{B}$-$\mathbb{U}$-bimodule. We can extend $\psi\in\Delta(\mathbb{A})\cap\mathbb{A}_*$ to the character on the second dual of $(\mathbb{A}\times_{\Theta}\mathbb{B})^{**}$ and show that $\widetilde{\mathcal{D}}_{\mathbb{U}}$ is inner. Let $\mathbb{A}^{**}$ is equipped with the first Arens product and we set $\Lambda:=(\Gamma,0)$ where $\Lambda\in\mathbb{A}^{**}\times_{\Theta^{**}}\mathbb{B}^{**}=(\mathbb{A}\times_{\Theta}\mathbb{B})^{**}$ and Γ is ψ-invariant mean on $\mathbb{A}_*$. We have $\langle(\Gamma,0),(\psi,\theta)\rangle=1$. On the other hand, there exists a net $\{\Gamma_\alpha\}\subseteq\mathbb{A}$ such that $\Gamma_\alpha\xrightarrow{w^*}\Gamma$. We have

$$\begin{aligned}\langle(\Gamma,0).(a,b),(\gamma,\eta)\rangle&=w^*-\lim_\alpha\langle(\Gamma_\alpha,0).(a,b),(\gamma,\eta)\rangle\\&=w^*-\lim_\alpha\langle(\Gamma_\alpha.\ a+\Theta(b).\ \Gamma_\alpha+\Theta(0).\ a,0.\ b),(\gamma,\eta)\rangle\\&=w^*-\lim_\alpha\langle(\psi(a)\ \Gamma_\alpha+\theta(b)\ \Gamma_\alpha.\ e_{\mathbb{A}},0),(\gamma,\eta)\rangle\\&=w^*-\lim_\alpha(\psi(a)+\theta(b))\langle(\Gamma_\alpha,0),(\gamma,\eta)\rangle\\&=(\psi,\theta)(a,b)\langle(\Gamma,0),(\gamma,\eta)\rangle,\end{aligned}$$

for all $(\gamma,\eta)\in\mathbb{A}_*\times\mathbb{B}_*$. So, by using the first paragraph of Section 4, $(\Gamma,0)$ is a (ψ,θ)-invariant mean. Therefore $\widetilde{D}_{\mathbb{U}}$ is inner.

Conversely, suppose that $\mathbb{E}$ is a symmetric normal Banach $\mathbb{A}$-$\mathbb{U}$-bimodule such that $a.\mathbf{x}=\psi(a)\mathbf{x}$ for every $a\in\mathbb{A}$, $\mathbf{x}\in\mathbb{E}$ and let $\mathcal{D}_{\mathbb{U}}:\mathbb{A}\to\mathbb{E}$ be a bounded w^*-continuous module ψ-derivation. Suppose that $\mathbb{E}_*$ is pseudo-unital. As in the proof of Lemma 4.1, it is easy to see that $\mathbb{E}$ is a symmetric normal Banach $\mathbb{A}\times_{\Theta}\mathbb{B}$-$\mathbb{U}$-bimodule such that $(a,0).\mathbf{e}=(\psi,\theta)(a,0)\mathbf{e}$ for every $a\in\mathbb{A}$, $\mathbf{e}\in\mathbb{E}$. There exists $\mathbf{y}\in\mathbb{E}$ and $c\in\mathbb{A}$ such that $\mathbf{e}=(c,0).\mathbf{y}$. Then

$$\begin{aligned}(a,b).\mathbf{e}&=(a,b).(c,0).\mathbf{y}=(ac+\theta(b).c.e_{\mathbb{A}},0).\mathbf{y}\\&=(\psi(ac)+\theta(b)\psi(c))\ \mathbf{y}=((\psi(a)+\theta(b))\psi(c)\mathbf{y}\\&=(\psi(a)+\theta(b))(c,0).\mathbf{y}=(\psi,\theta)(a,b).\mathbf{e}.\end{aligned}$$

We extend $\mathcal{D}_{\mathbb{U}}$ by

$$\widetilde{D}_{\mathbb{U}}:\mathbb{A}\times_{\Theta}\mathbb{B}\to\mathbb{E},\qquad (a,b)\mapsto\mathcal{D}_{\mathbb{U}}((a,b)(e_{\mathbb{A}},0))-(a,b).\mathcal{D}_{\mathbb{U}}(e_{\mathbb{A}},0),$$

for every $(a,b)\in\mathbb{A}\times_{\Theta}\mathbb{B}$. As in the proof of Lemma 4.1, $\widetilde{D}_{\mathbb{U}}$ is inner.

(ii) It is also straightforward to verify this clause by the concept of invariant mean. Suppose that (Λ,Γ) is a $(0,\theta)$-invariant mean on $\mathbb{A}_*\times_{\Theta}\mathbb{B}_*$. Then

$$\langle(\Lambda,\Gamma).(a,b),(\gamma,\eta)\rangle=\theta(b)\langle(\Lambda,\Gamma),(\gamma,\eta)\rangle,\qquad \langle(\Lambda,\Gamma),(0,\theta)\rangle=1 \tag{4.3}$$

for all $\gamma\in\mathbb{A}_*,\eta\in\mathbb{B}_*$. From (4.3) we have

$$\langle(\Lambda.a+\Lambda.\theta(b)+\theta(\Lambda).a,\Gamma.b),(\gamma,\eta)\rangle=\langle(\theta(b)\Lambda,\theta(b)\Gamma),(\gamma,\eta)\rangle.$$

So, $\langle\Gamma.b,\eta\rangle=\theta(b)\langle\Gamma,\eta\rangle$ and $\langle\Gamma,\theta\rangle=1$. Thus, $\mathbb{B}$ is module θ-Connes amenable.

Conversely, let $\mathbb{B}$ be module θ-Connes amenable and Γ is a θ-invariant mean on $\mathbb{B}_*$. We claim that $(-e_{\mathbb{A}},\Gamma)$ is a $(0,\theta)$-invariant mean on $\mathbb{A}_*\times\mathbb{B}_*$. Let $\{\Gamma_{\alpha}\}$ be a net in $\mathbb{A}$ that $\Gamma_{\alpha}\xrightarrow{w^*}\Gamma$. So,

$$\begin{aligned}\langle(-e_{\mathbb{A}},\Gamma).(a,b),(\gamma,\eta)\rangle&=w^*-\lim_{\alpha}\langle(-e_{\mathbb{A}},\Gamma_{\alpha}).(a,b),(\gamma,\eta)\rangle\\&=w^*-\lim_{\alpha}\langle(-e_{\mathbb{A}}.a+\Theta(b)(-e_{\mathbb{A}})+\Theta(\Gamma_{\alpha}).a,\Gamma_{\alpha}.b),(\gamma,\eta)\rangle\\&=w^*-\lim_{\alpha}\langle(-e_{\mathbb{A}}.a+\theta(b)e_{\mathbb{A}}.(-e_{\mathbb{A}})+\theta(\Gamma_{\alpha}).e_{\mathbb{A}}.a,\Gamma_{\alpha}.b),(\gamma,\eta)\rangle\\&=\langle(-e_{\mathbb{A}}.a+\theta(b)e_{\mathbb{A}}.(-e_{\mathbb{A}})+\theta(\Gamma).e_{\mathbb{A}}.a,\Gamma.b),(\gamma,\eta)\rangle\\&=\langle(-e_{\mathbb{A}}\theta(b),\theta(b)\Gamma),(\gamma,\eta)\rangle\\&=(0,\theta)(a,b)\langle(-e_{\mathbb{A}},\Gamma),(\gamma,\eta)\rangle\end{aligned}$$

for all $(\gamma,\eta)\in\mathbb{A}_*\times\mathbb{B}_*$ and $(a,b)\in\mathbb{A}\times_{\Theta}\mathbb{B}$. This completes the proof. □

Conclusion

We gave necessary and sufficient conditions for module Connes amenability of $\mathbb{A}\widehat{\otimes}\mathbb{B}$ that $\mathbb{A}$ and $\mathbb{B}$ are two Banach algebras. As an application, we investigated some examples of invariant means on projective tensor product of Banach algebras. Moreover, we characterize the module (ψ,θ)-Connes amenability of Θ-Lau product $\mathbb{A}\times_{\Theta}\mathbb{B}$, which $\Theta:\mathbb{B}\to\mathbb{A}$ be an algebraic homomorphism and ψ and θ are homomorphisms in $\mathbb{A}_*$ and $\mathbb{B}_*$, respectively.

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