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Research Paper

CONTINUOUS RIESZ BASES IN HILBERT C^*-MODULES

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Abstract. The paper is devoted to continuous frames and continuous Riesz basis in Hilbert C^*-modules. We define a continuous Riesz basis in Hilbert C^*-modules, and investigate the relationship between a continuous Riesz basis and an L^2-independent Bessel mapping. Also, we show that a continuous frame is a continuous Riesz basis if and only if it is a Riesz-type frame. Finally, we give the relation between two continuous Riesz bases in Hilbert C^*-modules.

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1. Introduction and Background

Frame theory is nowadays a fundamental research area in mathematics, computer science and engineering with many interesting applications in a variety of different fields. Frames were first introduced by Duffin and Schaeffer [7] in the context of nonharmonic fourier series. Then Daubecheies, Grassman and Mayer [6] reintroduced and developed them. The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by G. Kaiser [13] and independently by Ali, Antoine and Gazeau [2]. These frames are known as continuous frames. For a discussion of continuous frames, we refer to Refs.[16, 15]. Arefijamaal and et al. [4] introduced continuous Riesz bases and give some equivalent conditions for a continuous frame to be a continuous Riesz basis.

One reason to study frames in Hilbert C^*-modules is that there are some differences between Hilbert spaces and Hilbert C^*-modules. For example, in general, every bounded operator on a Hilbert space has an unique adjoint, while this fact not hold for bounded operators on a Hilbert C^*-module. Thus it is more difficult to make a discussion of the theory of Hilbert C^*-modules than that of Hilbert spaces in general. We refer the readers to [14], for more details on Hilbert C^*-modules. Frank and Larson [9] presented a general approach to the frame theory in Hilbert C^*-modules. The theory of frames has been extended from Hilbert spaces to Hilbert C^*-modules, see [1, 10, 12, 9, 17, 18].

The paper is organized as follows. First, we recall the basic definitions and some notations about Hilbert C^*-modules, and we also give some properties of them. In section 2, we recall the notion of continuous frames in Hilbert C^*-modules and their operators. In section 3, by defining a μ-complete Bessel mapping we give the definition of a continuous Riesz basis in Hilbert C^*-modules. We investigate the relation between a continuous Riesz basis and an

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L^2-independent Bessel mapping. Also, we show that a continuous Riesz basis is a continuous exact frame and a continuous frame is a continuous Riesz basis if and only if it is a Riesz-type frame. Finally, we give the relation between two continuous Riesz bases in Hilbert C^*-modules.

2. Preliminaries

First, we recall some definitions and basic properties of Hilbert C^*-modules. We give only a brief introduction to the theory of Hilbert C^*-modules to make our explanations self-contained. For comprehensive accounts, we refer to [14, 19]. Throughout this paper, $\mathcal{A}$ shows a unital C^*-algebra.

Definition 2.1. A *pre-Hilbert module* over unital C^*-algebra $\mathcal{A}$ is a complex vector space U which is also a left $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : U \times U \to \mathcal{A}$ which is $\mathbb{C}$-linear and $\mathcal{A}$-linear in its first variable and satisfies the following conditions:

(i) $\langle f, f \rangle \geq 0$,
(ii) $\langle f, f \rangle = 0 \quad \text{iff} \quad f = 0$,
(iii) $\langle f, g \rangle^* = \langle g, f \rangle$,
(iv) $\langle af, g \rangle = a\langle f, g \rangle$,

for all $f, g \in U$ and $a \in \mathcal{A}$.

A pre-Hilbert $\mathcal{A}$-module U is called *Hilbert $\mathcal{A}$-module* if U is complete with respect to the topology determined by the norm $\|f\| = \|\langle f, f \rangle\|^{\frac{1}{2}}$.

By [12, Example 2.46], if $\mathcal{A}$ is an C^*-algebra, then it is a Hilbert $\mathcal{A}$-module with respect to the inner product

$$\langle a, b \rangle = ab^*, \quad (a, b \in \mathcal{A}).$$

Example 2.2. [19, Page 237] Let $l^2(\mathcal{A})$ be the set of all sequences $\{a_n\}_{n\in\mathbb{N}}$ of elements of an C^*-algebra $\mathcal{A}$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ is convergent in $\mathcal{A}$. Then $l^2(\mathcal{A})$ is a Hilbert $\mathcal{A}$-module with respect to the pointwise operations and inner product defined by

$$\langle \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \rangle = \sum_{n=1}^{\infty} a_n b_n^*.$$

In the following lemma the *Cauchy-Schwartz inequality* reconstructed in Hilbert C^*-modules.

Lemma 2.3. [19, Lemma 15.1.3] ***(Cauchy-Schwartz inequality)*** *Let U be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. Then*

$$\|\langle f, g \rangle\|^2 \leq \|\langle f, f \rangle\| \, \|\langle g, g \rangle\|,$$

for all $f, g \in U$.

Definition 2.4. [14, Page 8] Let U and V be two Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$. A map $T : U \to V$ is said to be *adjointable* if there exists a map $T^* : V \to U$ satisfying

$$\langle Tf, g \rangle = \langle f, T^*g \rangle,$$

for all $f \in U, g \in V$. Such a map T^* is called the *adjoint* of T. By $End^*_{\mathcal{A}}(U)$ we denote the set of all adjointable maps on U.

It is surprising that an adjointable operator is automatically linear and bounded.

Lemma 2.5. [20, Lemma 1.1] *Let U and V be two Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$ and $T \in End^*_{\mathcal{A}}(U,V)$ has closed range. Then T^* has closed range and*

$$U = Ker(T) \oplus R(T^*)\ , \quad V = Ker(T^*) \oplus R(T)$$

.

Lemma 2.6. [3, Lemma 0.1] *Let U and V be two Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$ and $T \in End^*_{\mathcal{A}}(U,V)$. Then*

(i) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and*

$$\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2.$$

(ii) *If T is surjective, then the adjointable map TT^* is invertible and*

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.$$

Now, we introduce continuous frames in Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$, and then we give some results for these frames.

Let $\mathcal{Y}$ be a Banach space, $(\mathcal{X}, \mu)$ a measure space, and $f : \mathcal{X} \to \mathcal{Y}$ a measurable function. The integral of the Banach-valued function f has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions (see [8, 21]). Since every C^*-algebra and Hilbert C^*-module is a Banach space, hence we can use this integral in these spaces. In the following, we assume that $\mathcal{A}$ is a unital C^*-algebra and U is a Hilbert C^*-module over $\mathcal{A}$ and (Ω, μ) is a measure space.

Definition 2.7. [11] Let (Ω, μ) be a measure space and $\mathcal{A}$ is a unital C^*-algebra. We define,

$$L^2(\Omega, \mathcal{A}) = \{\varphi : \Omega \to \mathcal{A}\ ; \quad \int_\Omega \|\phi(\omega)(\varphi(\omega))^*\|\, d\mu(\omega) < \infty\}.$$

For any $\varphi, \psi \in L^2(\Omega, \mathcal{A})$, the inner product is defined by

$$\langle \varphi, \psi \rangle = \int_\Omega \langle \varphi(\omega), \psi(\omega)\rangle d\mu(\omega) = \int_\Omega \varphi(\omega)\psi(\omega)^* d\mu(\omega),$$

and the norm is defined by $\|\varphi\| = \|\langle \varphi, \varphi\rangle\|^{\frac{1}{2}}$. It was shown in [14] $L^2(\Omega, \mathcal{A})$ is a Hilbert $\mathcal{A}$-module.

Continuous frames for Hilbert $\mathcal{A}$-modules are defined as follows.

Definition 2.8. [10] A mapping $F : \Omega \to U$ is called a continuous frame for U if

(i) F is weakly-measurable, i.e, for any $f \in U$, the mapping $\omega \longmapsto \langle f, F(\omega)\rangle$ is measurable on Ω.

(ii) There exist constants $A, B > 0$ such that

$$A\langle f, f\rangle \leq \int_\Omega \langle f, F(\omega)\rangle\langle F(\omega), f\rangle d\mu(\omega) \leq B\langle f, f\rangle, \quad (f \in U). \tag{2.1}$$

The constants A, B are called *lower* and *upper* frame bounds, respectively. The mapping F is called *Bessel* if the right inequality in (2.1) holds and is called *tight* if $A = B$.

Definition 2.9. [11] A continuous frame $F : \Omega \to U$ is called *exact* if for every measurable subset $\Omega_1 \subseteq \Omega$ with $0 < \mu(\Omega_1) < \infty$, the mapping $F|_{\Omega\setminus\Omega_1}$ is not a continuous frame for U.

Example 2.10. Let $U=\left\{\begin{pmatrix} a & 0 & 0\\ 0 & 0 & b\end{pmatrix} : a,b\in\mathbb{C}\right\}$, and $\mathcal{A}=\left\{\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix} : x,y\in\mathbb{C}\right\}$ which is an C^*-algebra. We define the inner product

$$\begin{aligned}\langle .,.\rangle : U\times U &\rightarrow \quad \mathcal{A}\\ (M,N) &\longmapsto M(\overline{N})^t.\end{aligned}$$

This inner product makes U an C^*-module on $\mathcal{A}$. We consider a measure space $(\Omega=[0,1],\mu)$ whose μ is the Lebesgue measure. Also $F:\Omega\rightarrow U$ defined by $F(\omega)=\begin{pmatrix}\sqrt{3}\omega & 0 & 0\\ 0 & 0 & \sqrt{3}\omega\end{pmatrix}$, for any $\omega\in\Omega$.

For each $f=\begin{pmatrix} a & 0 & 0\\ 0 & 0 & b\end{pmatrix}\in U$, we have

$$\begin{aligned}\int_{[0,1]}\langle f,F(\omega)\rangle\langle F(\omega),f\rangle d\mu(\omega) &= \int_{[0,1]} 3\omega^2\begin{pmatrix}|a|^2 & 0\\ 0 & |b|^2\end{pmatrix} d\mu(\omega)\\ &= \begin{pmatrix}|a|^2 & 0\\ 0 & |b|^2\end{pmatrix} = \langle f,f\rangle.\end{aligned}$$

Therefore F is a continuous tight frame with bounds $A=B=1$.

The following operators for Bessel mappings and continuous frames in Hilbert C^*-modules are defined in [11].

Let $F:\Omega\rightarrow U$ be a Bessel mapping . Then

(i) The *pre-frame operator* or *synthesis operator operator* $T_F : L^2(\Omega,\mathcal{A}) \rightarrow U$ weakly defined by

$$\langle T_F\varphi, f\rangle = \int_\Omega \varphi(\omega)\langle F(\omega),f\rangle d\mu(\omega),\quad (f\in U). \tag{2.2}$$

(ii) The adjoit of T, called the *analysis operator* $T_F^* : U \rightarrow L^2(\Omega,\mathcal{A})$ is defined by

$$(T_F^* f)(\omega) = \langle f, F(\omega)\rangle,\quad (\omega\in\Omega). \tag{2.3}$$

The pre-frame operator is a well defined, surjective, adjointable $\mathcal{A}$-linear map and is bounded with $\|T_F\|\leq\sqrt{B}$ and the analysis operator $T_F^* : U \rightarrow L^2(\Omega, A)$ is injective and has closed range [11].

If $F:\Omega\rightarrow U$ is a continuous frame for Hilbert C^*-module U. Then the frame operator $S_F : U \rightarrow U$ is weakly defined by

$$\langle S_F f, f\rangle = \int_\Omega \langle f,F(\omega)\rangle\langle F(\omega),f\rangle d\mu(\omega),\quad (f\in U). \tag{2.4}$$

In [11] we prove that the frame operator S_F is positive, adjointable, selfadjoit and invertible and the lower and the upper bounds of F are, respectively $\|S_F^{-1}\|^{-1}$ and $\|T_F\|^2$. Now we introduce the concept of the duals of continuous frames in Hilbert C^*-modules and give some important properties of continuous frames and their duals.

Definition 2.11. Let $F:\Omega\rightarrow U$ be a continuous Bessel mapping. A continuous Bessel mapping $G:\Omega\rightarrow U$ is called a *dual* for F if for every $f\in U$

$$f = \int_\Omega \langle f, G(\omega)\rangle F(\omega) d\mu(\omega), \tag{2.5}$$

or

$$\langle f,g\rangle=\int_\Omega\langle f,G(\omega)\rangle\langle F(\omega),g\rangle d\mu(\omega)\qquad (f,g\in U). \tag{2.6}$$

In this case (F,G) is called a *dual pair*. If T_F and T_G denote the pre-frame operators of F and G, respectively, then (2.6) is equivalent to $T_FT_G^*=I_U$. The condition

$$\langle f,g\rangle=\int_\Omega\langle f,G(\omega)\rangle\langle F(\omega),g\rangle d\mu(\omega)\qquad (f,g\in U),$$

is equivalent

$$\langle f,g\rangle=\int_\Omega\langle f,F(\omega)\rangle\langle G(\omega),g\rangle d\mu(\omega)\qquad (f,g\in U),$$

because $T_FT_G^*=I_U$ if and only if $T_GT_F^*=I_U$.
Also, by reconstruction formula we have

$$f=S_F^{-1}S_Ff=S^{-1}\int_\Omega\langle f,F(\omega)\rangle F(\omega)d\mu(\omega)=\int_\Omega\langle f,F(\omega)\rangle S_F^{-1}F(\omega)d\mu(\omega),$$

and

$$f=S_FS_F^{-1}f=\int_\Omega\langle S_F^{-1}f,F(\omega)\rangle F(\omega)d\mu(\omega)=\int_\Omega\langle f,S_F^{-1}F(\omega)\rangle F(\omega)d\mu(\omega).$$

Then $S_F^{-1}F$ is a dual for F, which is called *canonical dual*.

3. Main Results

In this section, we introduce the concept of continuous Riesz bases in Hilbert C^*-modules and give some important properties of them. First, we give the notion of a Riesz-type frame that is introduced in [11].

Definition 3.1. Let $F:\Omega\to U$ be a continuous frame for Hilbert C^*-module U. If F has only one dual, we call F a *Riesz-type frame*.

Theorem 3.2. [11, Theorem 3.4] *Let $F:\Omega\to U$ be a continuous frame for Hilbert C^*-module U over a unital C^*-algebra $\mathcal{A}$. Then F is a Riesz-type frame if and only if the analysis operator $T_F^*:U\to L^2(\Omega,\mathcal{A})$ is surjective.*

Definition 3.3. Let U be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. A Bessel mapping $F:\Omega\to U$ is called a *μ-complete* if

$$\{f\in U;\quad \langle f,F(\omega)\rangle=0\quad a.e.\ [\mu]\}=\{0\}.$$

$\langle f,F(\omega)\rangle=0\quad a.e.\ [\mu]$ means that $\mu\{f;\ \langle f,F(\omega)\rangle\neq 0\}=0$.
Now, we define a continuous Riesz basis for Hilbert C^*-modules. Recall that $|a|^2=a^*a$, for any element a in C^*-algebra $\mathcal{A}$.

Definition 3.4. Let U be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. A weakly-measurable mapping $F:\Omega\to U$ is called a *continuous Riesz basis* for Hilbert C^*-module U, if the following conditions are satisfied.

(i) F is μ-complete.
(ii) There are two constants $A,B>0$ such that

$$A\left\|\int_{\Omega_1}|\varphi(\omega)^*|^2d\mu(\omega)\right\|^{\frac12}\le\left\|\int_{\Omega_1}\varphi(\omega)F(\omega)d\mu(\omega)\right\|\le B\left\|\int_{\Omega_1}|\varphi(\omega)^*|^2d\mu(\omega)\right\|^{\frac12}, \tag{3.1}$$

for every $\varphi \in L^2(\Omega, \mathcal{A})$ and measurable subset $\Omega_1 \subseteq \Omega$ with $\mu(\Omega_1) < +\infty$.

Remark 3.5. Let $F : \Omega \to U$ be a continuous Riesz basis for Hilbert C^*-module U. Define

$$T : L^2(\Omega, \mathcal{A}) \longrightarrow U$$
$$\varphi \longmapsto \int_\Omega \varphi(\omega)F(\omega)d\mu(\omega)$$

Then T is well-defined, adjointable map with $T^*f = \{\langle f, F(\omega)\rangle\}_{\omega\in\Omega}$ and bounded such that

$$A\|\varphi\|^2 \leq \|T\varphi\|^2 \leq B\|\varphi\|^2.$$

Hence F is a continuous Bessel mapping. Also by μ-completeness of F we have

$$Ker(T^*) = \{f \in U \ ; \ \langle f, F(\omega)\rangle = 0 \quad \forall\omega \in \Omega\} = \{0\},$$

so by lemma 2.5, $R(T) = Ker(T^*)^\perp = U$. Then T is surjective and by [11, Theorem 2.15], F is a continuous frame for U.

Definition 3.6. Let U be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. A Bessel mapping $F : \Omega \to U$ is said to be L^2-*independent* if for $\varphi \in L^2(\Omega, \mathcal{A})$, $\int_\Omega \varphi(\omega)F(\omega)d\mu(\omega) = 0$ implies that $\varphi(\omega) = 0$, for each $\omega \in \Omega$.

We give the following result.

Theorem 3.7. *Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U over a unital C^*-algebra $\mathcal{A}$ with bounds $A, B > 0$. Then the following are equivalent*

(i) *F is a continuous Riesz basis.*
(ii) *F is μ-complete and L^2-independent.*

Proof. $(i) \implies (ii)$ Let F be a continuous Riesz basis and $\int_\Omega \varphi(\omega)F(\omega)d\mu(\omega) = 0$ for some $\varphi \in L^2(\Omega, \mathcal{A})$. Since

$$A\left\|\int_\Omega |\varphi(\omega)^*|^2 d\mu(\omega)\right\| \leq \left\|\int_\Omega \varphi(\omega)F(\omega)d\mu(\omega)\right\|^2 = 0,$$

so

$$\langle\{\varphi(\omega)\}_{\omega\in\Omega}, \{\varphi(\omega)\}_{\omega\in\Omega}\rangle = \int_\Omega |\varphi(\omega)^*|^2 d\mu(\omega) = 0.$$

Hence $\{\varphi(\omega)\}_{\omega\in\Omega} = 0$ and $\varphi = 0$ i.e. F is L^2-independent.

$(ii) \implies (i)$ Let F be a L^2-independent continuous frame for Hilbert C^*-module U with bounds $A, B > 0$. For $\varphi \in L^2(\Omega, \mathcal{A})$ and measurable subset $\Omega_1 \subseteq \Omega$ with $\mu(\Omega_1) < +\infty$, put $f = \int_{\Omega_1} \varphi(\omega)F(\omega)d\mu(\omega)$. Then we have,

$$f = \int_{\Omega_1} \varphi(\omega)F(\omega)d\mu(\omega) = \int_\Omega \varphi(\omega)\chi_{\Omega_1}(\omega)F(\omega)d\mu(\omega).$$

Also $f = \int_\Omega \langle f, S^{-1}F(\omega)\rangle F(\omega)d\mu(\omega)$ where S is the continuous frame operator of F. Since F is L^2-independent, so

$$\varphi(\omega)\chi_{\Omega_1}(\omega) = \langle f, S^{-1}F(\omega)\rangle, \qquad (\omega \in \Omega).$$

and by [11, Corollary 2.11],

$$B^{-1}\langle f, f\rangle \leq \langle S^{-1}f, f\rangle \leq A^{-1}\langle f, f\rangle,$$

and so

$$A\|\langle S^{-1}f, f\rangle\| \leq \|\langle f, f\rangle\| \leq B\|\langle S^{-1}f, f\rangle\|.$$

Now we show that $\langle S^{-1}f, f\rangle = \int_{\Omega_1} |\varphi(\omega)^*|^2 d\mu(\omega)$.

$$\begin{aligned}\langle S^{-1}f, f\rangle &= \langle f, S^{-1}f\rangle \\ &= \int_{\Omega_1} \langle f, \varphi(\omega)S^{-1}F(\omega)\rangle d\mu(\omega) \\ &= \int_{\Omega_1} \langle f, S^{-1}F(\omega)\rangle \varphi(\omega)^* d\mu(\omega) \\ &= \int_{\Omega_1} \varphi(\omega)\varphi(\omega)^* d\mu(\omega) = \int_{\Omega_1} |\varphi(\omega)^*|^2 d\mu(\omega).\end{aligned}$$

Therefore,

$$A\left\|\int_{\Omega_1} |\varphi(\omega)^*|^2 d\mu(\omega)\right\| \leq \left\|\int_{\Omega_1} \varphi(\omega)F(\omega)d\mu(\omega)\right\|^2 \leq B\left\|\int_{\Omega_1} |\varphi(\omega)^*|^2 d\mu(\omega)\right\|,$$

i.e. F is a continuous Riesz basis for U with bounds A, B. □

Theorem 3.8. *Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U over a unital C^*-algebra $\mathcal{A}$. If F is a continuous Riesz basis for U, then it is a continuous exact frame.*

Proof. Let $\Omega_1 \subseteq \Omega$ be a measurable subset of Ω with $0 < \mu(\Omega_1) < \infty$. For $\varphi = \chi_{\Omega_1} \in L^2(\Omega, \mathcal{A})$ we have,

$$\begin{aligned}\|\int_{\Omega_1} F(\omega)d\mu(\omega)\|^2 &= \|\int_{\Omega_1} \chi_{\Omega_1}(\omega)F(\omega)d\mu(\omega)\|^2 \\ &\geq A\|\int_{\Omega_1} |\chi_{\Omega_1}(\omega)|^2 d\mu(\omega)\| \\ &= A\|\mu(\Omega_1)\| > 0.\end{aligned}$$

Hence $\int_{\Omega_1} F(\omega)d\mu(\omega) \neq 0$.

Now suppose that $F : \Omega \setminus \Omega_1 \to U$ is a continuous frame for U. Then by completeness of $F\mid_{\Omega\setminus\Omega_1}$ there exists $\varphi_0 \in L^2(\Omega \setminus \Omega_1, \mathcal{A})$ such that

$$\int_{\Omega_1} F(\omega)d\mu(\omega) = \int_{\Omega\setminus\Omega_1} \varphi_0(\omega)F(\omega)d\mu(\omega).$$

Define $\varphi : \Omega \to A$ where

$$\varphi(\omega) = \begin{cases}\varphi_0(\omega) & \omega \in \Omega \setminus \Omega_1 \\ 1 & \omega \in \Omega_1.\end{cases}$$

Then $\varphi \in L^2(\Omega, \mathcal{A})$ and

$$\int_\Omega \chi_{\Omega_1}(\omega)F(\omega)d\mu(\omega) = \int_\Omega \varphi(\omega)F(\omega)d\mu(\omega),$$

so $\int_\Omega(\varphi - \chi_{\Omega_1})(\omega)F(\omega)d\mu(\omega) = 0$. Hence L^2-independent shows that $\varphi = \chi_{\Omega_1}$ and so $\varphi_0 = 0$.
Therefore

$$\int_{\Omega_1} F(\omega)d\mu(\omega) = \int_{\Omega\setminus\Omega_1} \varphi_0(\omega)F(\omega)d\mu(\omega) = 0,$$

which is a contradiction. □

Proposition 3.9. *Let $F:\Omega\to U$ be a continuous Bessel mapping for Hilbert C^*-module U over a unital C^*-algebra $\mathcal{A}$ with pre-frame operator T. Suppose that F is μ-complete and the mapping*

$$V:\ L^2(\Omega,\mathcal{A})\longrightarrow L^2(\Omega,\mathcal{A})$$
$$\varphi\longmapsto\int_\Omega\varphi(\omega)\langle F(\omega),F(.)\rangle d\mu(\omega)$$

defines a bounded, adjointable and invertible operator. Then F is a continuous Riesz basis for U.

Proof. Since F is Bessel, so the pre-frame operator T is well-defined and bounded and adjointable with $T^*f=\{\langle f,F(\omega)\rangle\}_{\omega\in\Omega}$ for $f\in U$.
Also $T^*T=V$, because

$$\begin{aligned}(T^*T)(\varphi)&=T^*(\int_\Omega\varphi(\omega)F(\omega)d\mu(\omega))\\&=\{\langle\int_\Omega\varphi(\omega)F(\omega)d\mu(\omega),F(\gamma)\rangle\}_{\gamma\in\Omega}\\&=\{\int_\Omega\varphi(\omega)\langle F(\omega),F(\gamma)\rangle d\mu(\omega)\}_{\gamma\in\Omega}.\end{aligned}$$

Since T is bounded, so there exist $B>0$ such that $\|T\varphi\|^2\leq B\|\varphi\|^2$ i.e.

$$\|\int_\Omega\varphi(\omega)F(\omega)d\mu(\omega)\|^2\leq B\|\int_\Omega|\varphi(\omega)^*|^2d\mu(\omega)\|.$$

Since T^*T is positive, so

$$\begin{aligned}\left\|\int_\Omega\varphi(\omega)F(\omega)d\mu(\omega)\right\|^2&=\|T\varphi\|^2\\&=\|\langle T^*T\varphi,\varphi\rangle\|\\&=\left\|\langle(T^*T)^{\frac{1}{2}}\varphi,(T^*T)^{\frac{1}{2}}\varphi\rangle\right\|\\&=\left\|(T^*T)^{\frac{1}{2}}\varphi\right\|^2\\&\geq\left\|(T^*T)^{\frac{-1}{2}}\right\|^{-2}\|\varphi\|^2.\end{aligned}$$

Therefore F is continuous Riesz basis with lower and upper bounds $\left\|(T^*T)^{\frac{-1}{2}}\right\|^{-2}$ and B, respectively. □

Theorem 3.10. *Let $F:\Omega\to U$ be a continuous frame for Hilbert C^*-module U over a unital C^*-algebra $\mathcal{A}$ with pre-frame operator operator T. Then F is a continuous Riesz basis for U if and only if F is a Riesz-type frame.*

Proof. ($\Longrightarrow$) Let $F_1\neq F_2$ be two duals of F. Then for each $f\in U$,

$$\begin{aligned}\int_\Omega\langle f,F_1(\omega)-F_2(\omega)\rangle F(\omega)d\mu(\omega)&=\int_\Omega\langle f,F_1(\omega)\rangle F(\omega)d\mu(\omega)-\int_\Omega\langle f,F_2(\omega)\rangle F(\omega)d\mu(\omega)\\&=f-f=0.\end{aligned}$$

Since F is continuous Riesz basis, so is L^2-independent and

$$\langle f, F_1(\omega) - F_2(\omega)\rangle = 0 \quad \Longrightarrow \quad \langle f, F_1(\omega)\rangle = \langle f, F_2(\omega)\rangle \quad (\omega \in \Omega).$$

Therefore $F_1 = F_2$.

($\Longleftarrow$) Let F be Riesz-type frame and $\varphi \in L^2(\Omega, \mathcal{A})$ such that $\int_\Omega \varphi(\omega)F(\omega)d\mu(\omega) = 0$. Since F is Riesz-type, so $R(T^*) = L^2(\Omega, \mathcal{A})$. Also $L^2(\Omega, \mathcal{A}) = Ker(T) \oplus R(T^*)$. Then

$$\varphi \in Ker(T) = R(T^*)^\perp = \{0\},$$

then $\varphi = 0$ and so F is L^2-independent. Therefore F is a continuous Riesz basis. □

Corollary 3.11. *Let $F : \Omega \to U$ be a continuous frame for Hilbert C^*-module U over a unital C^*-algebra $\mathcal{A}$. If F is a Riesz-type frame, then it is a continuons exact frame.*

Duo to the Theorem 3.10, the converse of the Proposition 3.9 holds as follows.

Corollary 3.12. *Let $F : \Omega \to U$ be a continuous Riesz basis for Hilbert C^*-module U over a unital C^*-algebra $\mathcal{A}$ with bounds $A, B > 0$ and pre-frame operator T. Then F is μ-complete and the mapping*

$$\begin{aligned} V :\ L^2(\Omega, \mathcal{A}) &\longrightarrow L^2(\Omega, \mathcal{A}) \\ \varphi &\longmapsto \int_\Omega \varphi(\omega)\langle F(\omega), F(.)\rangle d\mu(\omega) \end{aligned}$$

defines a bounded, adjointable and invertible operator.

Proof. Let F be a continuous Riesz basis for U with bounds $A, B > 0$. Then the pre-frame operator T satisfies $\|T\| \leq \sqrt{B}$. Also,

$$\begin{aligned} (T^*T)(\varphi) &= T^*(\int_\Omega \varphi(\omega)F(\omega)d\mu(\omega)) \\ &= \{\langle \int_\Omega \varphi(\omega)F(\omega)d\mu(\omega), F(\gamma)\rangle\}_{\gamma\in\Omega} \\ &= \{\int_\Omega \varphi(\omega)\langle F(\omega), F(\gamma)\rangle d\mu(\omega)\}_{\gamma\in\Omega}. \end{aligned}$$

Then $V = T^*T$. Moreover, F is Riesz-type and T^* is surjective. Then by lemma 2.6, V is adjointable and invertible operator and

$$\left\|(T^*T)^{-1}\right\|^{-1} \leq V \leq \|T^*\|^2 \leq B.$$

□

According to the Theorem 3.10, in the next corollary we show the relation between two continuous Riesz bases for a Hilbert C^*-module U.

Corollary 3.13. *Let $F, G : \Omega \to U$ be two continuous Riesz bases for Hilbert C^*-module U over a unital C^*-algebra $\mathcal{A}$ and T_F, T_G, S_G be the pre-frame operator of F, the pre-frame operator of G and the frame operator of G, respectively. Then there exists an invertible operator $K \in End^*_{\mathcal{A}}(U)$ such that $G = S_G K^* F$.*

Proof. Let $f \in U$ such that $(T_G T_F^*)f = 0$. Then $T_G((T_F^* f)(\omega)) = 0$ for all $\omega \in \Omega$ and $\int_\Omega \langle f, F(\omega)\rangle G(\omega)d\mu(\omega) = 0$.

Since G is L^2-independence, so $\langle f, F(\omega)\rangle = 0$ for all $\omega \in \Omega$ and the completeness of F implies that $f = 0$ and hence $T_G T_F^*$ is injective. Moreover, T_F^* and T_G are surjective, therefore $T_G T_F^*$ is invertible.

Put $K := (T_G T_F^*)^{-1}$. Then for any $f, g \in U$,

$$\begin{aligned}\langle f, g\rangle &= \langle K^{-1}Kf, g\rangle \\ &= \langle T_F^* Kf, T_G^* g\rangle \\ &= \int_\Omega \langle Kf, F(\omega)\rangle\langle G(\omega), g\rangle d\mu(\omega) \\ &= \int_\Omega \langle f, K^*F(\omega)\rangle\langle G(\omega), g\rangle d\mu(\omega).\end{aligned}$$

Thus K^*F is a dual of G. But G is a Riesz-type frame, then $S_G^{-1}G = K^*F$ and hence $G = S_G K^* F$. □

Conclusion

In this paper, we have introduced a μ-complete Bessel mapping and a continuous Riesz basis in Hilbert C^*-modules, by recalling some definitions and properties of continuous frames in Hilbert C^*-modules. We have investigated the relation between a continuous Riesz basis and an L^2-independent Bessel mapping. Also, we have shown that a continuous Riesz basis is a continuous exact frame and a continuous frame is a continuous Riesz basis if and only if it is a Riesz-type frame. Finally, we have given the relation between two continuous Riesz bases in Hilbert C^*-modules.

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