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Research Paper

CLASSIFICATION OF 2-DIMENSIONAL BRYANT-TYPE METRICS

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ABSTRACT. The class of Bryant-type metrics is a natural extension of the class of 4-th root Finsler metrics which are used in Biology as ecological metrics. In this paper, we classify Bryant-type metrics admitting an (α, β)-metric on a two-dimensional manifold and show that it contains two classes of non-Riemannian (α, β)-metrics, specially Randers-type metrics. This might provide fine insights into a possible theory of deformations of Finsler norms.

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Keywords: Bryant metric, generalized 4-th root metric, 4-th root metric, (α, β)-metric, Randers-type metric.

1. Introduction

There are two important classes of Finsler metrics, namely, the class of m-th root metrics and the class of (α, β)-metrics. Let (M, F) be a Finsler manifold of dimension n, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. Let $F = \sqrt[m]{A}$, where A is given by $A := a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ with $a_{i_1 \dots i_m}$ symmetric in all its indices [21]. Then F is called an m-th root metric. The theory of m-th root metrics has been developed by Shimada [21], and applied to Biology as an ecological metric [3]. Recently studies show that the theory of m-th root Finsler metrics plays a very important role in Gravitation, Cancer and Seismic Ray Theory [2][4][7][8][14][15][24][25][28]. An (α, β)-metric is a Finsler metric of the form $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on M ([23][26][27]). These metrics have important applications in Physics, Mechanics and Seismology, etc, see for instance [3]. The Randers metric $F = \alpha + \beta$ is a significant (α, β)-metric which was introduced by Randers in the context of general relativity [12][19].

Let

$$F = \sqrt{\sqrt{A} + B} + C, \tag{1.1}$$

where $A = a_{ijkl}(x) y^i y^j y^k y^l$ with a_{ijkl} symmetric in all its indices, $B = b_{ij}(x) y^i y^j$ and $C = c_i(x) y^i$. The Finsler metrics (1.1) are called *Bryant-type metrics*. In (1.1), if $C = 0$ then F is called a generalized 4-th root metric [22]. Thus, the Bryant-type metrics can be considered as the Randers change of generalized 4-th root metrics. In [15], Li-Shen studied the locally projectively flat generalized 4-th root metrics. In [30], Xu-Li has constructed a family of projectively flat Bryant-type metrics with flag curvature $\mathbf{K} = 1$. For other progress on the

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class of generalized m-th root metrics, see [28] and [29]. If $B = C = 0$, then F reduces to a 4-th root metric.

As an special example of Bryant-type metric, let us put

$$A := \frac{\sqrt{\mathcal{A}} + \mathcal{B}}{2\mathcal{D}}, \quad B := \Big(\frac{\mathcal{C}}{\mathcal{D}}\Big)^2, \quad C := \frac{\mathcal{C}}{\mathcal{D}}$$

where

$$\mathcal{A} := (|y|^2\cos(2\alpha) + |x|^2|y|^2 - \langle x,y\rangle^2)^2 + (|y|^2\sin(2\alpha))^2,$$
$$\mathcal{B} := |y|^2\cos(2\alpha) + |x|^2|y|^2 - \langle x,y\rangle^2, \quad \mathcal{C} := \langle x,y\rangle\sin(2\alpha), \quad \mathcal{D} := |x|^4 + 2|x|^2\cos(2\alpha) + 1,$$

and $0 < \alpha < \pi/2$. Then we get the Bryant metric in [10][11] which the first time obtained by Bryant in the following form

$$F(x,y) = \mathcal{I}m\left[\frac{-\langle x,y\rangle + i\sqrt{(e^{2i\alpha} + |x|^2)|y|^2 - \langle x,y\rangle^2}}{e^{2i\alpha} + |x|^2}\right], \tag{1.2}$$

where $\mathcal{I}m[\cdot]$ denotes the imaginary part of a complex number. This metric has been introduced by Bryant when he classified the global structures of projectively flat Finsler metrics of flag curvature $\mathbf{K} = 1$ on $\mathbb{S}^2$ [9][10][11]. In [20], by using the algebraic equations, Shen gave the expression (1.1) of Bryant metric including higher dimensions. By this expression, Shen obtained several interesting projectively flat Finsler metrics of constant flag curvature which can be used as models in certain problems.

In [18], Matsumoto-Numata considered the arbitrary cubic metric $F = \sqrt[3]{a_{ijk}y^iy^jy^k}$ on an n-dimensional manifold M, which admits an (α,β)-metric. They proved that for the case of $n > 2$, if F is an (α,β)-metric where α is non-degenerate, then F^3 can be written in the form $F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}$ with constants c_1 and c_2. For $n = 2$, they showed that it can be written in the form $F = \sqrt[3]{\alpha^2\beta}$ by choosing suitable quadratic form $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and one-form $\beta = b_i(x)y^i$, where α^2 may be degenerate. In [1], Abazari-Khoshdani proved that a 4-th root metric admitting an (α,β)-metric on n-dimensional manifold M can be written as the (α,β)-metric $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$, where c_1, c_2 and c_3 are real constants. For other progress on m-th root metrics, see [13], [15], [16] and [17]. It is interesting to characterize the Bryant-type metric (1.1) admitting an (α,β)-metric on a Finsler surface M. Then, we prove the following.

Theorem 1.1. *Let $F = \sqrt{\sqrt{A} + B} + C$ be the Bryant-type metric on a 2-dimensional manifold M. Then, F can be expressed in the form $F = \alpha\phi(s)$, $s = \beta/\alpha$, which $\phi(s)$ is given by one of the following forms*

$$\phi(s) := c_1 + c_2 s + c_3\sqrt{1 + c_4 s}, \qquad \textit{(Randers-type metrics)} \tag{1.3}$$

$$\phi(s) := d_1\sqrt{-1 \pm \sqrt{d_2 + d_3 s^2}} \mp d_4\sqrt{d_5 \pm d_6\sqrt{d_2 + d_3 s^2} + \frac{d_7 s}{-1 \pm \sqrt{d_2 + d_3 s^2}}} + s, \tag{1.4}$$

where c_i and d_i are non-zero real constants.

Theorem 1.1 provide fine insights into a possible theory of deformations of Finsler norms. These deformations obtain a mutual bridge between the class of (α,β)-metrics and the class of m-th root metrics [5][6].

2. Proof of Theorem 1.1

To prove the Theorem 1.1, we need to remark some facts about the class of m-th root (α, β)-metrics. In [18], Matsumoto-Numata studied the cubic (α, β)-metrics and proved the following.

Lemma 2.1. ([18]) Let $F = \sqrt[3]{a_{ijk}y^iy^jy^k}$ be the cubic Finsler metric on a Finsler surface M. Then it can be written in the form $F = \sqrt[3]{\alpha^2\beta}$ by choosing suitable quadratic form $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and one-form $\beta = b_i(x)y^i$, where α^2 may be degenerate.

In [1], Abazari-Khoshdani consider the class of 4-th root (α, β)-metrics and prove the following.

Lemma 2.2. ([1]) Let $F = \sqrt[4]{a_{ijkl}y^iy^jy^ky^l}$ be a quartic Finsler metric on an n-dimensional Finsler manifold M. Then it can be written in the form $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$ with real constants c_1, c_2, c_3 by choosing suitable quadratic form α^2 and one-form β of y^i, where α^2 may be degenerate.

Now, we are ready to prove the Theorem 1.1.

Proof of Theorem 1.1: Let $F = \sqrt{\sqrt{A} + B} + C$ be a Bryant-type metric on a 2-dimensional Finsler manifold M, where $A = a_{ijkl}(x)y^iy^jy^ky^l, B = b_{ij}(x)y^iy^j, C = c_i(x)y^i$. So, we have

$$\begin{aligned} &F^4 - 4F^3c_iy^i + 2F^2\Big\{3c_ic_j - b_{ij}\Big\}y^iy^j + F\Big\{4b_{ij}c_k - 4c_ic_jc_k\Big\}y^iy^jy^k \\ &+\Big\{c_ic_jc_kc_l + b_{ij}b_{kl} - 2b_{ij}c_kc_l - a_{ijkl}\Big\}y^iy^jy^ky^l = 0. \end{aligned} \tag{2.1}$$

By permutation in (2.1), we get

$$\begin{aligned} 16F^4 &- 32(c_1y^1 + c_2y^2)F^3 + \Big\{8(y^1)^2(3c_1^2 - b_{11}) + 16y^1y^2(3c_1c_2 - b_{12}) \\ &+8(y^2)^2(3c_2^2 - b_{22})\Big\}F^2 + 8\Big\{y^1\ (y^2)^2(c_1b_{22} + 2c_2b_{12} - 3c_1c_2^2) \\ &+(y^1)^3(c_1b_{11} - c_1^3) + (y^2)^3(c_2b_{22} - c_2^3) + y^2(y^1)^2(c_2b_{11} + 2c_1b_{21} - 3c_2c_1^2)\Big\}F \\ &+(y^1)^4(d_0 - a_{1111}) + (y^1)^3\ y^2(d_1 - 4a_{1112}) + (y^2)^2(y^1)^2(d_2 - 6a_{1122}) \\ &+y^1(y^2)^3(d_3 - 4a_{1222}) + (y^2)^4(d_4 - a_{2222}) = 0, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} d_0 &:= (c_1^2 - b_{11})^2, \\ d_4 &:= (c_2^2 - b_{22})^2, \\ d_1 &:= 4(c_2c_1^3 + b_{11}b_{12} - b_{12}c_1^2 - b_{11}c_1c_2), \\ d_3 &:= 4(c_1c_2^3 + b_{22}b_{12} - b_{12}c_2^2 - b_{22}c_1c_2), \\ d_2 &:= 6c_1^2c_2^2 - 2(b_{11}c_2^2 + 4b_{12}c_1c_2 + b_{22}c_1^2) + 2(b_{11}b_{22} + 2b_{12}^2). \end{aligned}$$

Let us put

$$f := \frac{F}{y^1} \quad \text{and} \quad t := \frac{y^2}{y^1}.$$

Then (2.2) can be written as follows

$$f^4 + \mathfrak{A}f^3 + \mathfrak{B}f^2 + \mathfrak{C}f + \mathfrak{D} = 0, \tag{2.3}$$

where

$$\begin{aligned}
\mathfrak{A} :=& -2(c_1 + c_2 t),\\
\mathfrak{B} :=& \frac{1}{2}\Big\{(3c_1^2 - b_{11}) + 2t(3c_1c_2 - b_{12}) + t^2(3c_2^2 - b_{22})\Big\},\\
\mathfrak{C} :=& \frac{1}{2}\Big\{c_1(b_{11} - c_1^2) + t(c_2b_{11} + 2c_1b_{21} - 3c_1^2c_2) + t^2(c_1b_{22} + 2c_2b_{12} - 3c_1c_2^2) + t^3(c_2b_{22} - c_2^3)\Big\},\\
\mathfrak{D} :=& \frac{1}{16}\Big\{(d_0 - a_{1111}) + t(d_1 - 4a_{1112}) + t^2(d_2 - 6a_{1122}) + t^3(d_3 - 4a_{1222}) + t^4(d_4 - a_{2222})\Big\}.
\end{aligned}$$

Let us define

$$g := f + \frac{1}{4}\mathfrak{A}.$$

Then (2.3) is written as follows

$$g^4 + g^2\Phi + g\Psi + \Gamma = 0, \tag{2.4}$$

where

$$\begin{aligned}
\Phi &:= \mathfrak{B} - \frac{3}{8}\mathfrak{A}^2,\\
\Psi &:= \mathfrak{C} - \frac{1}{2}\mathfrak{A}\mathfrak{B} + \frac{1}{8}\mathfrak{A}^3,\\
\Gamma &:= \mathfrak{D} - \frac{1}{4}\mathfrak{A}\mathfrak{C} + \frac{1}{16}\mathfrak{A}^2\mathfrak{B} - \frac{3}{256}\mathfrak{A}^4.
\end{aligned}$$

The equation (2.4) is a 4-degree equation without including the third degree sentence. So, we can decompose it as follows

$$g^4 + g^2\Phi + g\Psi + \Gamma = (\bar{a}g^2 + \bar{b}g + \bar{c})(\tilde{a}g^2 + \tilde{b}g + \tilde{c}), \tag{2.5}$$

where $\bar{a}$, $\bar{b}$, $\bar{c}$, $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ are unknown real numbers. We are going to find these coefficients. By (2.5), we get

$$\bar{a}\tilde{a} = 1, \tag{2.6}$$

$$\bar{a}\tilde{b} + \bar{b}\tilde{a} = 0, \tag{2.7}$$

$$\bar{a}\tilde{c} + \bar{b}\tilde{b} + \bar{c}\tilde{a} = \Phi, \tag{2.8}$$

$$\bar{b}\tilde{c} + \bar{c}\tilde{b} = \Psi, \tag{2.9}$$

$$\bar{c}\tilde{c} = \Gamma. \tag{2.10}$$

According to (2.6)-(2.10), we obtain

$$\bar{a} = \tilde{a} = 1, \tag{2.11}$$

$$\tilde{b} = -\bar{b}, \tag{2.12}$$

$$\tilde{c} + \bar{c} = \Phi + \bar{b}^2, \tag{2.13}$$

$$\tilde{c} - \bar{c} = \frac{\Psi}{\bar{b}}. \tag{2.14}$$

By (2.13) and (2.14), one can get

$$\tilde{c} = \frac{1}{2}\Big(\Phi + \bar{b}^2 + \frac{\Psi}{\bar{b}}\Big), \qquad \bar{c} = \frac{1}{2}\Big(\Phi + \bar{b}^2 - \frac{\Psi}{\bar{b}}\Big). \tag{2.15}$$

All of coefficients are computed along $\bar{b}$. If we find $\bar{b}$, then the equation (2.5) can be solved. In order to find $\bar{b}$, we use the relations (2.6)-(2.15). First, (2.10) and (2.15) imply that

$$z^3 + \phi z^2 + \psi z + \eta = 0, \tag{2.16}$$

where

$$z := 4\bar{b}^2, \tag{2.17}$$

$$\phi := 8\mathfrak{B} - 3\mathfrak{A}^2, \tag{2.18}$$

$$\psi := 16\mathfrak{B}^2 + 3\mathfrak{A}^4 - 16\mathfrak{A}^2\mathfrak{B} - 64\mathfrak{D} + 16\mathfrak{A}\mathfrak{C}, \tag{2.19}$$

$$\eta := 8\mathfrak{A}^4\mathfrak{B} - 16\mathfrak{A}^3\mathfrak{C} + 64\mathfrak{A}\mathfrak{B}\mathfrak{C} - \mathfrak{A}^6 - 16\mathfrak{A}^2\mathfrak{B}^2 - 64\mathfrak{C}^2. \tag{2.20}$$

It is remarkable that, in (2.16), ϕ, ψ, η are polynomials of degree 2, 4 and 6, respectively. So, we can decompose (2.16) as follows

$$(z + \mathfrak{a})(z^2 + \mathfrak{b}z + \mathfrak{c}) = 0, \tag{2.21}$$

where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are polynomials in terms of degree 2, 2 and 4, respectively. Now, we discuss about the solutions of (2.21). For the equation (2.21), we have two major cases:

Case 1 : $z + \mathfrak{a} = 0$. In this case, we get

$$\bar{b} = \pm\frac{1}{2}\sqrt{d_1 + d_2 t + d_3 t^2}, \tag{2.22}$$

where d_1, d_2, d_3 are real numbers. Putting $t = \frac{y^2}{y^1}$ in (2.22), we obtain

$$\bar{b} = \pm\frac{1}{2y^1}\alpha, \tag{2.23}$$

where

$$\alpha := \sqrt{d_1(y^1)^2 + d_2 y^1 y^2 + d_3 (y^2)^2}$$

is a Riemannian metric. By definition, Ψ is a cubic form. Then, by using the Lemma 2.1 and putting (2.23) in (2.15), we get

$$\tilde{c} = \frac{1}{(y^1)^2}\Big\{\bar{c}_1\alpha^2 + \bar{c}_2\alpha\beta\Big\}, \qquad \bar{c} = \frac{1}{(y^1)^2}\Big\{\bar{c}_1\alpha^2 + \bar{c}_3\alpha\beta\Big\}, \tag{2.24}$$

where $\beta = b_i(x)y^i$ is a 1-form on M and $\bar{c}_i$ are real constants.

By (2.5), one of the following holds

$$\bar{a}g^2 + \bar{b}g + \bar{c} = 0, \tag{2.25}$$

$$\tilde{a}g^2 + \tilde{b}g + \tilde{c} = 0. \tag{2.26}$$

In the case of (2.25), the solutions of (2.5) are given by

$$g_{1,2} = \frac{1}{y^1}\Big\{e_1\alpha \pm e_2\sqrt{\lambda_1\alpha^2 + \lambda_2\alpha\beta}\Big\},$$

where e_i and λ_i are real numbers. Similarly, in the case of (2.26), the solutions of (2.5) are given by

$$g_{3,4} = \frac{1}{y^1}\Big\{e_3\alpha \pm e_4\sqrt{\lambda_1\alpha^2 + \lambda_3\alpha\beta}\Big\},$$

where λ_3 is a real number. On the other hand, according to definition, we have $f := g - \frac{A}{4}$. Thus, for the case of (2.27), we get

$$f_1 = \frac{1}{y^1}\Big\{e_1\alpha + e_2\sqrt{\lambda_1\alpha^2 + \lambda_2\alpha\beta} - \beta\Big\},$$
$$f_2 = \frac{1}{y^1}\Big\{e_1\alpha - e_2\sqrt{\lambda_1\alpha^2 + \lambda_2\alpha\beta} - \beta\Big\}.$$

Similarly for (2.27), we obtain

$$f_3 = \frac{1}{y^1}\Big\{e_3\alpha + e_4\sqrt{\lambda_1\alpha^2 + \lambda_3\alpha\beta} - \beta\Big\},$$
$$f_4 = \frac{1}{y^1}\Big\{e_3\alpha - e_4\sqrt{\lambda_1\alpha^2 + \lambda_3\alpha\beta} - \beta\Big\}.$$

By (2.3), we have $F = y^1 f$. Then

$$F = (e_1\alpha - \beta) \pm e_2\sqrt{\lambda_1\alpha^2 + \lambda_2\alpha\beta}, \tag{2.27}$$
$$F = (e_3\alpha - \beta) \pm e_4\sqrt{\lambda_1\alpha^2 + \lambda_3\alpha\beta}. \tag{2.28}$$

It is easy to see that, (2.27) and (2.28) can be written as (1.3) which are called Randers-type Finsler metrics.

Case 2 : $z^2 + \mathfrak{b}z + \mathfrak{c} = 0$. By considering $t := \frac{y^2}{y^1}$, the solution of $z^2 + \mathfrak{b}z + \mathfrak{c} = 0$ are given by following

$$z_{1,2} = \frac{1}{2(y^1)^2}\Big\{ - \big(k_1(y^2)^2 + m_1y^1y^2 + n_1(y^1)^2\big) \pm \sqrt{\omega}\Big\},$$

where

$$\omega := \kappa_1(y^1)^4 + \kappa_2(y^1)^3y^2 + \kappa_3(y^1)^2(y^2)^2 + \kappa_4y^1(y^2)^3 + \kappa_5(y^2)^4$$

and k_1, m_1, n_1 and κ_i are real constants. Let us define

$$\overline{\alpha} := \sqrt{k_1(y^2)^2 + m_1y^1y^2 + n_1(y^1)^2}$$

which is a Riemannain metric. Since ω is a quartic form, then by using Lemma 2.2, we can get two following solutions

$$z_{1,2} = \frac{1}{2(y^1)^2}\Big\{ - \overline{\alpha}^2 \pm \overline{\alpha}\sqrt{h_1\overline{\alpha}^2 + h_2\beta^2}\Big\}.$$

By (2.17), we have $z = 4\,\bar{b}^2$. Then we obtain

$$\bar{b}^2 = \frac{1}{8(y^1)^2}\Big\{ - \overline{\alpha}^2 \pm \overline{\alpha}\sqrt{h_1\overline{\alpha}^2 + h_2\beta^2}\Big\}. \tag{2.29}$$

Putting (2.29) in (2.15) implies that

$$\bar{c} = \frac{1}{(y^1)^2}\left\{\mu_1\overline{\alpha}^2 \pm \mu_2\overline{\alpha}\sqrt{h_1\overline{\alpha}^2 + h_2\beta^2} + \frac{\mu_3\overline{\alpha}^2\beta}{\sqrt{-\overline{\alpha}^2 \pm \sqrt{h_1\overline{\alpha}^2 + h_2\beta^2}}}\right\}, \tag{2.30}$$

$$\tilde{c} = \frac{1}{(y^1)^2}\left\{\mu_1\overline{\alpha}^2 \pm \mu_2\alpha\sqrt{h_1\overline{\alpha}^2 + h_2\beta^2} + \frac{\mu_4\overline{\alpha}^2\beta}{\sqrt{-\overline{\alpha}^2 \pm \sqrt{h_1\overline{\alpha}^2 + h_2\beta^2}}}\right\}, \tag{2.31}$$

where μ_i are real constants. According to (2.5), (2.30) and (2.31), the solutions of (2.25) and (2.26) are as follows

$$g_{1,2} = \frac{1}{y^1}\left\{p_1\sqrt{-\overline{\alpha}^2 \pm \widehat{\alpha}} \pm p_2\sqrt{\bar{e}_1\overline{\alpha}^2 \pm \bar{e}_2\widehat{\alpha} + \bar{e}_3\frac{\overline{\alpha}^2\beta}{-\overline{\alpha}^2 \pm \widehat{\alpha}}}\right\},$$

$$g_{3,4} = \frac{1}{y^1}\left\{p_1\sqrt{-\overline{\alpha}^2 \pm \widehat{\alpha}} \pm p_2\sqrt{\bar{e}_1\overline{\alpha}^2 \pm \bar{e}_2\widehat{\alpha} + \bar{e}_4\frac{\overline{\alpha}^2\beta}{-\overline{\alpha}^2 \pm \widehat{\alpha}}}\right\},$$

where p_i and $\bar{e}_i$ are real constants and

$$\widehat{\alpha} := \overline{\alpha}\sqrt{h_1\overline{\alpha}^2 + h_2\beta^2}.$$

Since $f = g - \frac{A}{4}$, then we get

$$f_{1,2} = \frac{1}{y^1}\left\{\beta + p_1\sqrt{-\overline{\alpha}^2 \pm \widehat{\alpha}} \mp p_2\sqrt{\bar{e}_1\overline{\alpha}^2 \pm \bar{e}_2\widehat{\alpha} + \bar{e}_3\frac{\overline{\alpha}^2\beta}{-\overline{\alpha}^2 \pm \widehat{\alpha}}}\right\}, \tag{2.32}$$

$$f_{3,4} = \frac{1}{y^1}\left\{\beta + p_1\sqrt{-\overline{\alpha}^2 \pm \widehat{\alpha}} \mp p_2\sqrt{\bar{e}_1\overline{\alpha}^2 \pm \bar{e}_2\widehat{\alpha} + \bar{e}_4\frac{\overline{\alpha}^2\beta}{-\overline{\alpha}^2 \pm \widehat{\alpha}}}\right\}. \tag{2.33}$$

By considering $F = y^1 f$, (2.32) and (2.33), we get (1.4). This completes the proof. □

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