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Research Paper

ON PRESSURE OF ASYMPTOTICALLY SUB-ADDITIVE POTENTIALS WITH MISTAKES VIA WEIGHTED METRICS

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Abstract. In this paper, we use some bi-sequences of positive numbers to define weighted dynamical metrics. Then we show that, replacing the Bowen dynamical metric by the weighted metric, the definition of pressure for asymptotically sub-additive potentials, including measure-theoretic and topological, is not affected. This generalizes some known results for pressure, defined using mean metrics and continuous potentials.

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1. Introduction

The concepts of measure-theoretic and topological pressure of a topological dynamical system are introduced by Ruelle [15] and Walters [17] and are applied in formulation of the thermodynamic formalism in ergodic theory, dynamical systems and dimension theory [2, 10].

These concepts are extended for sub-additive potentials [1, 3, 5] and asymptotically sub-additive potentials with mistake functions [18].

The definition of measure-theoretic and topological pressure for continuous, sub-additive and asymptotically sub-additive potentials are mainly given, using the Bowen metric

$$d_n(x,y) := \max\{d(T^i(x),T^i(y)) : 0 \le i \le n-1\} \qquad (x,y \in X,\ n \in \mathbb{N}).$$

and its corresponding dynamical ball

$$B_n(x,\epsilon) = \{y \in X : \quad d_n(x,y) < \epsilon\}.$$

More precisely, the concepts of (n,ϵ)-separated and spanning sets, applied in the definition of measure-theoretic and topological pressure, are based on the Bowen metric.

In [9], the definition of measure-theoretic and topological pressure for continuous potentials are formulated using the mean metric

$$\widehat{d}_n(x,y) := \frac{1}{n}\sum_{i=0}^{n-1} d(T^i(x),T^i(y)) \qquad (x,y \in X,\ n \in \mathbb{N}).$$

The idea of replacing the Bowen metric by the mean metric is also applied for the entropy of dynamical systems [8]. In [13, 14], the concept of mean metric is generalized to a more general

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family of weighted metrics $d_{n,\Gamma}$, induced by a probability bi-sequence $\Gamma = \{\gamma_{m,n}\}_{m,n\geq 0} \subset \mathbb{R}$, in the sense that $\sum_{i=0}^{n} \gamma_{i,n} = 1$ for any $n \in \mathbb{N}$, and are applied for the pressure and conditional entropy of a dynamical system.

In this paper, we attempt to formulate the measure-theoretic and topological pressure for asymptotically sub-additive potentials, using the weighted metric $d_{n,\Gamma}$ induced by a probability bi-sequence. We show that, the quantities defined via the metric $d_{n,\Gamma}$ coincide to the ones defined via the Bowen metric d_n.

In Section 2, we present some preliminary concepts and results which are used in this paper. In Section 3, we formulate the measure-theoretic pressure for asymptotically sub-additive potentials, using the weighted metrics. In Section 4, we formulate the topological pressure for asymptotically sub-additive potentials, using the weighted metrics. Section 5 is a discussion and concluding remarks.

In the rest of the paper, a topological dynamical system (abbreviated by TDS) is a continuous map $T : X \to X$ on a compact metric space X. The space X is naturally equipped by the Borel σ-algebra. The set of T-invariant and T-ergodic probability measures are denoted by $M(X,T)$ and $E(X,T)$ respectively.

2. Preliminary concepts and facts

In this section, we review some preliminary facts which are required for the rest of the paper. Let d_n be the Bowen metric and $B_n(x,\epsilon)$ the corresponding ball centered at x and with radius ϵ. Given $Z \subseteq X$, $n \in \mathbb{N}$ and $\epsilon > 0$, A subset $F \subseteq Z$ is an (n,ϵ)-spanning set for Z if for any $x \in Z$, there exists $y \in F$ such that $d_n(x,y) \leq \epsilon$. A set $E \subseteq Z$ is an (n,ϵ)-separated set for Z if for any $x, y \in E$ with $x \neq y$ implies $d_n(x,y) > \epsilon$. Given $\delta > 0$ and $\mu \in M(X,T)$, a set S is an (n,ϵ,δ)-spanning set if $\mu(\bigcup_{x\in S} B_n(x,\epsilon)) > 1-\delta$.

A sequence of continuous real valued functions $\mathcal{F} = \{f_n\}_{n\geq 1}$ is an asymptotically sub-additive potential (abbreviated by ASP) on X, if for each $k \geq 1$ there exists a sub-additive potential $\Phi_k = \{\phi_n^k\}_{n\geq 1}$, in the sense that, $\phi_{n+m}^k(x) \leq \phi_n^k(x) + \phi_m^k(T^n(x))$, $\forall x \in X$, $\forall m,n \in \mathbb{N}$, such that

$$\limsup_{n\to+\infty} \frac{1}{n} ||f_n - \phi_n^k|| < \frac{1}{k},$$

where $||f_n - \phi_n^k|| := \sup_{x\in X} |f_n(x) - \phi_n^k(x)|$. In [6], the asymptotically sub-additive topological pressure of T with respect to $\mathcal{F}$ is defined as follows:

$$P(T,\mathcal{F}) := \lim_{\epsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log P(T,\mathcal{F},n,\epsilon)$$

where,

$$P(T,\mathcal{F},n,\epsilon) := \sup\{\sum_{y\in E} e^{f_n(y)} : E \text{ is an } (n,\epsilon)-\text{separated subset for } X\}. \tag{2.1}$$

If $\mathcal{F} = \{f_n\}_{n\geq 1}$ is an ASP, given $\mu \in M(X,T)$, set

$$\mathcal{F}_*(\mu) := \lim_{n\to+\infty} \int_X f_n d\mu.$$

When $\mu \in E(X,T)$, the limit above exists μ-almost everywhere without integrating with respect to μ [6]. Indeed one may easily see that $\mathcal{F}_*(\mu) = \lim_{k\to+\infty} \lim_{n\to+\infty} \frac{1}{n}\int_X \phi_n^k d\mu$. We have the following variational principle [6].

Theorem 2.1. *Let $T : X \to X$ be a TDS, and $\mathcal{F} = \{f_n\}_{n\geq 1}$ an ASP. Then,*

$$P(T,\mathcal{F}) = \begin{cases} -\infty & \text{if } \mathcal{F}_*(\mu) = -\infty \text{ for all } \mu \in M(X,T) \\ \sup\{h_\mu(T) + \mathcal{F}_*(\mu) : \mu \in M(X,T),\ \mathcal{F}_*(\mu) \neq -\infty\} & \text{otherwise} \end{cases}$$

Remark 2.2. Note that, In light of the ergodic decomposition for T-invariant measures and the Jacob's theorem, the supremum on the right hand side of the previous equality may be taken over all $\mu \in E(X,T)$.

Before we proceed, we review the concept of mistake function which will be used in the definition of Γ-mistake balls as a generalization of the mistake balls [11, 12, 16].

Definition 2.3. Given $\epsilon_0 > 0$, a function $g : \mathbb{N} \times (0,\epsilon_0] \to \mathbb{N}$ is called a mistake function if for every $\epsilon \in (0,\epsilon_0]$ and $n \in \mathbb{N}$, $g(n,\epsilon) \leq g(n+1,\epsilon)$ and $\lim_{n\to+\infty} \frac{g(n,\epsilon)}{n} = 0$. Note that, one may extend the definition of a mistake function on all positive real line by setting $g(n,\epsilon) = g(n,\epsilon_0)$ for $\epsilon > \epsilon_0$.

The measure-theoretic and topological pressures for asymptotic sub-additive potentials under a mistake function are defined in [4]. These quantities are also connected together via a variational principle. (See Corollary 1 in [4]). It is proved that, the pressures under a mistake function coincides to the corresponding quantities in the absence of any mistake function. (See Theorems 2.3 and 2.4 in [4]).

In Sections 3 and 4, we extend the results in [4], for the pressures defined using a family of weighted metrics including the mean metric.

3. Weighted metrics and measure-theoretic pressure

In this section, we define weighted metrics corresponding to a probability bi-sequence. We first review the concept of probability bi-sequence.

Definition 3.1. A sequence $\Gamma = \{\gamma_{m,n}\}_{m,n\geq 0}$ is said to be a probability bi-sequence if $\sum_{i=0}^{n} \gamma_{i,n} = 1$ for all $n \geq 1$.

Example 3.2. $\Gamma_0 = \{\gamma_{m,n}\}_{m,n\geq 0}$ with $\gamma_{m,n} = \frac{1}{n+1}$ is a probability bi-sequence. Also, given every sequence of positive numbers $\{a_n\}_{n\geq 0}$, if $S_n := \sum_{j=0}^{n} a_j$, then $\Gamma_1 = \{\gamma_{m,n}\}_{m,n\geq 0}$ with $\gamma_{m,n} := \frac{a_m}{S_n}$ is a probability bi-sequence.

Definition 3.3. Let $T : X \to X$ be a compact dynamical system and $\Gamma = \{\gamma_{m,n}\}_{m,n\geqslant 0}$ be a probability bi-sequence. Given $n \in \mathbb{N}$ and $x, y \in X$, set:

$$d_{n,\Gamma}(x,y) := \sum_{i=0}^{n-1} \gamma_{i,n-1} d(T^i(x), T^i(y)).$$

Obviously, $d_{n,\Gamma}$ is a metric on X. For $\epsilon > 0$, $x \in X$ and $n \in \mathbb{N}$, the (Γ,n)-ball centered at x with radius ϵ and length n is defined by

$$B_{n,\Gamma}(x,\epsilon) := \{y \in X : d_{n,\Gamma}(y,x) < \epsilon\}.$$

Note that, if $\Gamma_0 = \{\gamma_{m,n}\}_{m,n\geq 0}$ with $\gamma_{m,n} = \frac{1}{n+1}$ then $d_{n,\Gamma_0} = \widehat{d}_n$ is the mean metric.

Given $n \geq 1$ and $\epsilon > 0$, a subset $F \subseteq X$ is said to be a (Γ, n, ϵ)-spanning set if for any $x \in X$ there exists $y \in F$ such that $d_{n,\Gamma}(x, y) \leq \epsilon$. Also, a subset $E \subseteq X$ is said to be a (Γ, n, ϵ)-separated set if $x, y \in E$ and $x \neq y$ implies $d_{n,\Gamma}(x, y) > \epsilon$.

Let $\mu \in E(X, T)$. For $0 < \delta < 1$, $n \geq 1$ and $\epsilon > 0$, a set $F \subseteq X$ is called a $(\Gamma, \mu, n, \epsilon, \delta)$-spanning set if $\mu(\bigcup_{x\in F} B_{n,\Gamma}(x, \epsilon)) \geq 1 - \delta$.

Let $\Lambda_n := \{0, 1, 2, \cdots, n-1\}$. Given any subset $\Lambda \subseteq \Lambda_n$, set $d_\Lambda(x, y) := \max\{d(T^i(x), T^i(y)) : i \in \Lambda\}$ and $B_\Lambda(x, \epsilon) := \{y \in X : d_\Lambda(x, y) < \epsilon\}$. Now we are ready to define the Γ-mistake balls.

Definition 3.4. Let $T : X \to X$ be a TDS. For $n \geq 1$, $\epsilon > 0$ and any probability bi-sequence $\Gamma = \{\gamma_{m,n}\}_{m,n\geq 0}$, the Γ-mistake ball with center $x \in X$ and radius ϵ and of length n is defined as follows:

$$\begin{aligned} B_{n,\Gamma}(g; x, \epsilon) &:= \{y \in X : \sum_{i=0}^{n-1} \gamma_{i,n-1}\, \chi_{B(T^i(x),\epsilon)}(T^i(y)) > 1 - \frac{g(n, \epsilon)}{n}\} \\ &= \bigcup_{\Lambda \in I_\Gamma(g;n,\epsilon)} B_\Lambda(x, \epsilon), \end{aligned}$$

where

$$I_\Gamma(g; n, \epsilon) := \{\Lambda \subseteq \Lambda_n : \sum_{i\in\Lambda} \gamma_{i,n-1} > 1 - \frac{g(n, \epsilon)}{n}\}.$$

The following lemma is an analogous version of Lemma 3.1 in [4].

Lemma 3.5. *Let X be a TDS, g a mistake function, $\mathcal{F} = \{f_n\}_{n\geq 1}$ an ASP and $\Gamma = \{\gamma_{m,n}\}_{m,n\geq 0}$ be a probability bi-sequence. Given $k \geq 1$, there exists sub-additive potential $\Phi_k = \{\phi^k_n\}_{n\geq 1}$ such that for any $l \geq 1$ and small $\eta > 0$, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, the following inequalities hold for large n:*

$$\sup_{y\in B_{n,\Gamma}(g;x,\epsilon)} f_n(y) \leq \sum_{i=0}^{n-1} \left(\frac{1}{l}\phi^k_l(T^i(x)) + \eta\right) + C\left(1 + \frac{(\gamma^*_{n-1})^{-1}}{n} g(n, \epsilon)\right) + \frac{n}{k},$$

where $\gamma^*_{n-1} := \min_{0\leq i\leq n-1} \gamma_{i,n-1}$ and $C := \max\{2||\frac{1}{l}\phi^k_l|| + \eta, 4\max_{1\leq i\leq 2l} |\phi^k_j(x)|\}$.

Proof. Given $k \geq 1$, since $\mathcal{F} = \{f_n\}_{n\geq 1}$ is an ASP, there exists sub-additive potentials $\Phi_k = \{\phi^k_n\}_{n\geq}$ such that $\limsup_{n\to+\infty} \frac{1}{n}||f_n - \phi^k_n|| < \frac{1}{k}$. This implies that,

$$f_n(x) \leq \phi^k_n(x) + \frac{n}{k} \qquad \forall x \in X,$$

for large $n \in \mathbb{N}$. Now, fix any positive integer $l \geq 1$. Since $\frac{1}{l}\phi^k_l$ is continuous, for every $\eta > 0$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, we have

$$d(x, y) < \epsilon \Longrightarrow \left|\frac{1}{l}\phi^k_l(x) - \frac{1}{l}\phi^k_l(y)\right| < \eta. \tag{3.1}$$

For every $y \in B_{n,\Gamma}(g, x, \epsilon)$, there exists $\Lambda \in I_\Gamma(g, n, \epsilon)$ such that $y \in B_\Lambda(x, \epsilon)$. Note that, if $\Lambda \in I_\Gamma(g, n, \epsilon)$ then $|\Lambda^c| \le \frac{(\gamma^*_{n-1})^{-1}}{n} g(n, \epsilon)$. Therefore,

$$\begin{aligned} \sum_{i=0}^{n-1} \frac{1}{l}\phi_l^k(T^i(y)) &= \sum_{i\in\Lambda} \frac{1}{l}\phi_l^k(T^i(y)) + \sum_{i\notin\Lambda} \frac{1}{l}\phi_l^k(T^i(y)) \\ &\le \sum_{i\in\Lambda} \left(\frac{1}{l}\phi_l^k(T^i(x) + \eta\right) + \sum_{i\in\Lambda^c} ||\frac{1}{l}\phi_l^k|| \\ &\le \sum_{i=0}^{n-1} \left(\frac{1}{l}\phi_l^k(T^i(x)) + \eta\right) + |\Lambda^c| \left(2||\frac{1}{l}\phi_l^k|| + \eta\right) \\ &\le \sum_{i=0}^{n-1} \left(\frac{1}{l}\phi_l^k(T^i(x)) + \eta\right) + \frac{(\gamma^*_{n-1})^{-1}}{n} \left(2||\frac{1}{l}\phi_l^k|| + \eta\right) g(n, \epsilon). \qquad (3.2) \end{aligned}$$

For $n \ge 1$, large enough, one may write $n = sl + r$ where $0 \le r < l$ and $s \ge 0$. Then, for $0 \le j < l$, setting $\phi_0^k = 0$, we have

$$\phi_n^k(x) \le \phi_j^k(x) + \sum_{i=0}^{s-2} T^{il}(T^j(x)) + \phi_{l+r-j}^k(T^{(s-1)l}(T^j(x))).$$

Summing over $j = 0, 1, 2, \cdots, l-1$, one has

$$l\phi_n^k(x) \le 2lC_0 + \sum_{i=0}^{(s-1)l-1} \phi_l^k(T^i(x)),$$

where $C_0 := \max_{1\le j\le 2l} |\phi_j^k(x)|$. Hence,

$$\phi_n^k(x) \le 2C_0 + \sum_{i=0}^{(s-1)l-1} \frac{1}{l}\phi_l^k(T^i(x)) \le 4C_0 + \sum_{i=0}^{n-1} \frac{1}{l}\phi_l^k(T^i(x)). \qquad (3.3)$$

Set $C := \max\{4C_0, 2||\frac{1}{l}\phi_l^k|| + \eta\}$. Then, combining (3.1), (3.2) and (3.3), we have

$$\begin{aligned} \sup_{y\in B_{n,\Gamma}(g;x,\epsilon)} f_n(y) &\le \sup_{y\in B_{n,\Gamma}(g;x,\epsilon)} \left(\phi_n^k(y) + \frac{n}{k}\right) \\ &\le \sup_{y\in B_{n,\Gamma}(g;x,\epsilon)} \left(4C_0 + \sum_{i=0}^{n-1} \frac{1}{l}\phi_l^k(T^i(y)) + \frac{n}{k}\right) \\ &\le 4C_0 + \frac{n}{k} + \sum_{i=0}^{n-1} \left(\frac{1}{l}\phi_l^k(T^i(x)) + \eta\right) + \frac{(\gamma^*_{n-1})^{-1}}{n} \left(2||\frac{1}{l}\phi_l^k|| + \eta\right) g(n, \epsilon) \\ &\le \sum_{i=0}^{n-1} \left(\frac{1}{l}\phi_l^k(T^i(x)) + \eta\right) + C\left(1 + \frac{(\gamma^*_{n-1})^{-1}}{n} g(n, \epsilon)\right) + \frac{n}{k}. \end{aligned}$$

This completes the proof. □

In the following, we first present some definitions.

Definition 3.6. Given $n \geq 1$ and $\epsilon > 0$, a set $F \subseteq X$ is said to be a (Γ, g, n, ϵ)-spanning set for X if for any $x \in X$ there exists $y \in F$ and $\Lambda \in I_\Gamma(g, n, \epsilon)$ such that $d_\Lambda(x, y) \leq \epsilon$. Also, a set $E \subseteq X$ is said to be a (Γ, g, n, ϵ)-separated set for X if for any $x, y \in E$, $x \neq y$ implies that $d_\Lambda(x, y) > \epsilon$ for any $\Lambda \in I_\Gamma(g, n, \epsilon)$.

Definition 3.7. Given $0 < \delta < 1$, $n \geq 1$, $\epsilon > 0$ and $\mu \in E(X, T)$, a set $S \subseteq X$ is said to be a $(\Gamma, g, n, \epsilon, \delta, \mu)$-spanning set for X, if $\mu(\bigcup_{x \in S} B_{n,\Gamma}(g, x, \epsilon)) > 1 - \delta$.

Definition 3.8. Let $T : X \to X$ be a TDS, $\mathcal{F} = \{f_n\}_{n \geq 1}$ an ASP and $\Gamma = \{\gamma_{m,n}\}_{m,n \geq 0}$ a probability bi-sequence. Define

$$P_{\Gamma,\mu}(g, T, \mathcal{F}) := \lim_{\delta \to 0} \lim_{\epsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log P_{\Gamma,\mu}(g, T, \mathcal{F}, n, \epsilon, \delta)$$

where,

$$P_{\Gamma,\mu}(g, T, \mathcal{F}, n, \epsilon, \delta) := \inf\{\sum_{x \in S} \exp(\sup_{y \in B_{n,\Gamma}(g,x,\epsilon)} f_n(y)) : S \text{ is a } (\Gamma, g, n, \epsilon, \delta, \mu){-}\text{spanning set for } X\}.$$

Before we proceed, we review the following lemma.

Lemma 3.9. *([10], Appendix II, Lemma 1) For each $\eta > 0$, there exists $0 < \gamma \leq \eta$, a finite partition $\xi = \{C_1, C_2, \cdots, C_m\}$ and a finite open cover $\mathcal{U} = \{U_1, U_2, \cdots, U_k\}$ of X, where $k \geq m$, such that the following properties hold:*

(1) $\operatorname{diam}(U_i) \leq \eta$ *and* $\operatorname{diam}(C_j) \leq \eta$, $1 \leq i \leq k$, $1 \leq j \leq m$.
(2) $\overline{U_i} \subset C_i$, $1 \leq i \leq m$ *where* $\overline{U_i}$ *denotes the closure of the set* U_i.
(3) $\mu(C_i \setminus U_i) \leq \gamma$, $1 \leq i \leq m$ *and* $\mu(\bigcup_{i=m+1}^{k} U_i) \leq \gamma$.
(4) $2\gamma \log m \leq \eta$.

Now, we are ready to state and prove our main result.

Theorem 3.10. *Let $T : X \to X$ be a TDS, $0 < \delta < 1$, g a mistake function, $\mu \in E(X, T)$, $\mathcal{F} = \{f_n\}_{n \geq 1}$ an ASP such that $\mathcal{F}_*(\mu) \neq -\infty$ and $\Gamma = \{\gamma_{m,n}\}_{m,n \geq 0}$ a probability bi-sequence with $K := \limsup_{n \to +\infty} \frac{(\gamma_n^*)^{-1}}{n} < +\infty$. Then,*

$$P_{\Gamma,\mu}(g, T, \mathcal{F}) = \lim_{\epsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log P_{\Gamma,\mu}(g, T, \mathcal{F}, n, \epsilon, \delta) = h_\mu(T) + \mathcal{F}_*(\mu).$$

Proof. Let $k \geq 1$, $l \geq 1$ and $\eta > 0$ be given.

Step 1. Since $B_n(x, \epsilon) \subseteq B_{n,\Gamma}(g, x, \epsilon)$, each $(n, \epsilon, \delta, \mu)$-spanning set is a $(\Gamma, g, n, \epsilon, \delta, \mu)$-spanning set and so,

$$\begin{aligned} P_{\Gamma,\mu}(g, T, \mathcal{F}, n, \epsilon, \delta) &\leq \inf\{\sum_{x \in S} (\sup_{y \in B_{n,\Gamma}(g,x,\epsilon)} f_n(y)) : S \text{ is an } (n, \epsilon, \delta) - \text{spanning set for } X\} \\ &\leq \exp\{n(\eta + \frac{1}{k}) + C\left(1 + \frac{(\gamma_{n-1}^*)^{-1}}{n} g(n, \epsilon)\right)\} \\ &\times \inf\{\sum_{x \in S} \exp\{\sum_{i=0}^{n-1} \frac{1}{l} \phi_l^k(T^i(x))\} : S \text{ is an } (n, \epsilon, \delta) - \text{spanning set for } X\}. \end{aligned}$$

Therefore,

$$\frac{1}{n}\log P_{\Gamma,\mu}(g,T,\mathcal{F},n,\epsilon,\delta) \leq \eta+\frac{1}{k}+\frac{C}{n}\left(1+\frac{(\gamma^*_{n-1})^{-1}}{n}g(n,\epsilon)\right)$$
$$+\ \frac{1}{n}\log\inf\{\sum_{x\in S}\exp\{\sum_{i=0}^{n-1}\frac{1}{l}\phi_l^k(T^i(x))\}: S \text{ is an } (n,\epsilon,\delta)-\text{spanning set for } X\},$$

where C is as in Lemma 3.5. Since $K:=\limsup_{n\to+\infty}\frac{(\gamma^*_{n-1})^{-1}}{n}<+\infty$, letting $n\to+\infty$, $\epsilon\to 0$ and applying Theorem 2.1 in [7], we conclude that,

$$P_{\Gamma,\mu}(g,T,\mathcal{F})\leq \eta+\frac{1}{k}+h_\mu(T)+\int_X\frac{1}{l}\phi_l^k(x)d\mu(x).$$

Finally, letting $l\to+\infty$, $k\to+\infty$ and using the fact that $\eta>0$ is arbitrary, we will have

$$P_{\Gamma,\mu}(g,T,\mathcal{F})\leq h_\mu(T)+\mathcal{F}_*(\mu).$$

Step 2. We show that

$$\lim_{\epsilon\to0}\liminf_{n\to+\infty}\frac{1}{n}\log P_{\Gamma,\mu}(g,T,\mathcal{F},\epsilon,\delta)\geq h_\mu(T)+\mathcal{F}_*(\mu).$$

To do this, we need to modify the method used in [4]. Let $0<\eta<1-\delta$ be given. By Lemma 3.9, there exists $0<\gamma\leq\eta$, a finite Borel partition $\xi=\{C_1,C_2,\cdots,C_m\}$ and a finite open cover $\mathcal{U}=\{U_1,U_2,\cdots,U_k\}$ with $k\geq n$ such that,

1. $\mathrm{diam}(\xi)\leq\eta$ and $\mathrm{diam}(\mathcal{U})\leq\eta$.
2. $\overline{U}_i\subset C_i$ for $1\leq i\leq m$.
3. $\mu(C_i\setminus U_i)\leq\gamma$, for $1\leq i\leq m$ and $\mu(\bigcup_{i=m+1}^k U_i)\leq\gamma$.
4. $2\gamma\log m\leq\eta$.

Fix a set $Z\subseteq X$ with $\mu(Z)\geq 1-\delta$ and set $t_n(x):=|\{0\leq l<n: T^l(x)\in\bigcup_{i=m+1}^k U_i\}|$. Since $\mu\in E(X,T)$, as in the proof of Theorem 2.3 in [4], applying the Birkhöff's ergodic theorem, the Shannon-McMillan-Brieman theorem, the sub-additive and ergodic Egorov's ergodic theorem, there exists $A\subseteq Z$ and $N\geq1$ with $\mu(A)\geq\mu(Z)-\gamma$ such that for every $x\in A$ and $n\geq N$,

1^*. $t_n(x)\leq 2\gamma n$.
2^*. $\mu(\xi_n(x))\leq\exp\{-(h_\mu(T,\xi)-\gamma)n\}$.
3^*. $\mathcal{F}_*(\mu)-\gamma\leq\frac{1}{n}f_n(x)\leq\mathcal{F}_*(\mu)+\gamma$,

where $\xi_n:=\bigvee_{i=0}^{n-1}T^{-i}\xi$. In light of part 2^*, for any $n\geq N$, we have $|\xi^*_n|\geq\mu(A)\exp\{(h_\mu(T,\xi)-\gamma)n\}$ where $\xi^*_n:=\{C\in\xi_n: C\cap A\neq\emptyset\}$. Assume that S is a (Γ,g,n,ϵ)-spanning set for Z. Obviously, S is a $(\Gamma,g,n,\epsilon,\delta\mu)$ spanning set for X. If we set $S':=\{x\in S:\overline{B}_{n,\Gamma}(g,x,\epsilon)\cap A\neq\emptyset\}$, then $A\subseteq\bigcup_{x\in S'}\overline{B}_{n,\Gamma}(g,x,\epsilon)$.

Now, fix $x\in S'$ and set

$$\mathcal{Q}_{n,\epsilon}:=\{\Lambda_x\in I_\Gamma(g,n,\epsilon):\overline{B}_{n,\Gamma}(g,x,\epsilon)\cap A\neq\emptyset\}.$$

Since $\gamma^*_{n-1}:=\min_{0\leq i\leq n-1}\gamma_{i,n-1}$, one may easily see that $|\Lambda_x^c|\leq\frac{(\gamma^*_{n-1})^{-1}}{n}g(n,\epsilon)$. Set $\xi_{\Lambda_x}:=\bigvee_{j\in\Lambda_x}T^{-j}\xi$ and denote by $N(x,\Lambda_x)$, the number of atoms of ξ_{Λ_x} which intersect $A\cap\overline{B}_{\Lambda_x}(x,\epsilon)$.

Let also $N(x,\xi_n)$ denotes the number of atoms of ξ_n which intersect $A\cap\overline{B}_{\Lambda_x}(x,\epsilon)$. In light of part 1^* above, we have $N(x,\Lambda_x)\le m^{2\gamma n}$ and so,

$$N(x,\xi_n)\le N(x,\Lambda_x)m^{\frac{(\gamma^*_{n-1})^{-1}}{n}g(n,\epsilon)}\le m^{2\gamma n+\frac{(\gamma^*_{n-1})^{-1}}{n}g(n,\epsilon)}.$$

Hence,

$$|\xi^*_n|\le\sum_{x\in S'}N(x,\xi_n)\le|S'|\exp\{(2\gamma n+\frac{(\gamma^*_{n-1})^{-1}}{n}g(n,\epsilon))\log m\}.$$

Therefore,

$$\begin{aligned}\sum_{x\in S}\exp\left(\sup_{y\in B_{n,\Gamma}(g,x,\epsilon)}f_n(y)\right) &\ge \sum_{x\in S}\exp\left(\sup_{y\in B_{n,\Gamma}(g,x,\epsilon)}f_n(y)\right)\\ &\ge |S'|\exp\{n(\mathcal{F}_*(\mu)-\gamma)\}\\ &\ge \mu(A)\exp\{(h_\mu(T,\xi)+\mathcal{F}_*(\mu)-2\gamma)n-\left(2\gamma n+\frac{(\gamma^*_{n-1})^{-1}}{n}g(n,\epsilon)\right)\log m\}.\end{aligned}$$

This easily results in

$$\frac{1}{n}\log P_{\Gamma,\mu}(g,T,\mathcal{F},n,\epsilon,\delta)\ge\frac{1}{n}\log\mu(A)+h_\mu(T,\xi)+\mathcal{F}_*(\mu)-2\gamma-\left(2\gamma+\frac{(\gamma^*_{n-1})^{-1}}{n}\frac{g(n,\epsilon)}{n}\right)\log m.$$

Letting $n\to+\infty$ and $\epsilon\to0$, since $\eta>0$ is arbitrary and $K:=\limsup_{n\to+\infty}\frac{(\gamma^*_n)^{-1}}{n}<+\infty$, we will have

$$\lim_{\epsilon\to0}\liminf_{n\to+\infty}\frac{1}{n}\log P_{\Gamma,\mu}(g,T,\mathcal{F},\epsilon,\delta)\ge h_\mu(T)+\mathcal{F}_*(\mu),$$

which completes the proof. □

4. Weighted metrics and topological pressure

In this section, we define topological version of the concepts defined in the previous section. As in the previous section, let $T:X\to X$ be a TDS, $\Gamma=\{\gamma_{m,n}\}_{m,n\ge0}$ be a probability bi-sequence and $\mathcal{F}=\{f_n\}_{n\ge1}$ be an ASP.

Definition 4.1. Given $\epsilon>0$ and $n\in\mathbb{N}$, set

$$P_\Gamma(T,\mathcal{F},n,\epsilon):=\sup\{\sum_{y\in E}e^{f_n(y)}:E\text{ is a }(\Gamma,n,\epsilon)-\text{separated set for }X\}.$$

Then, we define

$$P_\Gamma(T,\mathcal{F}):=\lim_{\epsilon\to0}\limsup_{n\to+\infty}\frac{1}{n}\log P_\Gamma(T,\mathcal{F},n,\epsilon).$$

We also set

$$P^*_\Gamma(T,\mathcal{F},n,\epsilon):=\inf\{\sum_{x\in F}\exp\{\sup_{y\in B_{n,\Gamma}(x,\epsilon)}f_n(y)\}:F\text{ is a }(\Gamma,n,\epsilon)-\text{spanning set for }X\},$$

and

$$P^*_\Gamma(T,\mathcal{F}):=\lim_{\epsilon\to0}\limsup_{n\to+\infty}\frac{1}{n}\log P^*_\Gamma(T,\mathcal{F},n,\epsilon).$$

Definition 4.2. Let g be a mistake function on X. Given $\epsilon > 0$ and $n \in \mathbb{N}$, set

$$P_\Gamma(g,T,\mathcal{F},n,\epsilon) := \sup\{\sum_{x\in E} e^{f_n(x)} : E \text{ is a } (\Gamma,g,n,\epsilon)-\text{separated set for } X\},$$

and then define

$$P_\Gamma(g,T,\mathcal{F}) := \lim_{\epsilon\to 0}\limsup_{n\to+\infty}\frac{1}{n}\log P_\Gamma(g,T,\mathcal{F},n,\epsilon).$$

Set also,

$$P^*_\Gamma(g,T,\mathcal{F},n,\epsilon) := \inf\{\sum_{x\in F}\exp\{\sup_{y\in B_{n,\Gamma}(g,x,\epsilon)} f_n(y)\} : F \text{ is a } (\Gamma,g,n,\epsilon)-\text{spanning set for } X\},$$

and define

$$P^*_\Gamma(g,T,\mathcal{F}) := \lim_{\epsilon\to 0}\limsup_{n\to+\infty}\frac{1}{n}\log P^*_\Gamma(g,T,\mathcal{F},n,\epsilon).$$

The proof of the following theorem is similar to the proof of Proposition 1 in [4], replacing (n,ϵ)-separated sets by (Γ,n,ϵ)-separated sets and (n,ϵ)-spanning sets by (Γ,n,ϵ)-spanning sets.

Theorem 4.3. *Let $T : X \to X$ be a TDS, $\Gamma = \{\gamma_{m,n}\}_{m,n\geq 0}$ be a probability bi-sequence and $\mathcal{F} = \{f_n\}_{n\geq 1}$ be an ASP. Then $P^*_\Gamma(T,\mathcal{F}) = P_\Gamma(T,\mathcal{F})$.*

We also have the following theorem.

Theorem 4.4. *Let $T : X \to X$ be a TDS, $\Gamma = \{\gamma_{m,n}\}_{m,n\geq 0}$ be a probability bi-sequence and $\mathcal{F} = \{f_n\}_{n\geq 1}$ be an ASP. Then $P_\Gamma(T,\mathcal{F}) = P(T,\mathcal{F})$.*

Proof. Let E be a (Γ,n,ϵ)-separated set for X. Then, E is also an (n,ϵ)-separated set for X. So,

$$P_\Gamma(T,\mathcal{F},n,\epsilon) \leq \sup\{\sum_{x\in E} e^{f_n(x)} : E \text{ is an } (n,\epsilon)-\text{sepatrated set for } X\} = P(T,\mathcal{F},n,\epsilon).$$

This easily results in $P_\Gamma(T,\mathcal{F},n,\epsilon) \leq P(T,\mathcal{F},n,\epsilon)$.

Conversely, let $\mu \in E(X,T)$ and $0 < \delta < 1$. Let F be a (Γ,n,ϵ)-spanning set for X. Then F is a $(\Gamma,n,\epsilon,\delta,\mu)$-spanning set for X. Therefore, $P^*_\Gamma(T,\mathcal{F},n,\epsilon) \geq P_{\Gamma,\mu}(T,\mathcal{F},n,\epsilon,\delta)$. Applying Theorem 4.3, we have

$$P^*_\Gamma(T,\mathcal{F}) = P_\Gamma(T,\mathcal{F}) \geq h_\mu(T) + \mathcal{F}_*(\mu).$$

Taking supremume over all $\mu \in E(X,T)$, and applying Theorem 2.1, we conclude that $P_\Gamma(T,\mathcal{F}) \geq P(T,\mathcal{F})$, which completes the proof. □

Theorem 4.5. *Let $T : X \to X$ be a TDS, g a mistake function on X, $\Gamma = \{\gamma_{m,n}\}_{m,n\geq 0}$ a probability bi-sequence and $\mathcal{F} = \{f_n\}_{n\geq 1}$ an ASP. Then $P_\Gamma(g,T,\mathcal{F}) = P_\Gamma(T,\mathcal{F})$.*

Proof. Since each (Γ,g,n,ϵ)-separated set is a (Γ,n,ϵ)-separated set then,

$$P_\Gamma(g,T,\mathcal{F},n,\epsilon) \leq \sup\{\sum_{y\in E} e^{f_n(y)} : E \text{ is a } (\Gamma,n,\epsilon)-\text{separated set for } X\} = P_\Gamma(T,\mathcal{F},n,\epsilon),$$

and consequently, $P_\Gamma(g,T,\mathcal{F}) \leq P_\Gamma(T,\mathcal{F})$.

Now, let $\mu \in E(X,T)$ with $\mathcal{F}_*(\mu) \neq -\infty$. Since each (Γ, g, n, ϵ)-spanning set is a $(\Gamma, g, n, \epsilon, \delta, \mu)$-spanning set, then $P^*_\Gamma(g, T, \mathcal{F}, n, \epsilon) \geq P_{\Gamma,\mu}(g, T, \mathcal{F}, n, \epsilon, \delta)$. Applying Theorems 3.10 and 4.4, we conclude that

$$P^*_\Gamma(g, T, \mathcal{F}) \geq h_\mu(T) + \mathcal{F}_*(\mu). \tag{4.1}$$

Given $n \in \mathbb{N}$ and $\epsilon > 0$, one may choose a set $D = \{x_1, x_2, \cdots, x_k\} \subseteq X$ such that,

(i) $x_m \in X \setminus \bigcup_{i=1}^{m-1} B_{n,\Gamma}(g, x, \epsilon)$ for $2 \leq m \leq k$.
(ii) $f_n(x_1) = \sup_{x \in X} f_n(x)$ and $f_n(x_m) = \sup_{x \in X \setminus \bigcup_{i=1}^{m-1} B_{n,\Gamma}(g,x,\epsilon)} f_n(x)$ for $2 \leq m \leq k$.

Clearly D is a maximal (Γ, g, n, ϵ)-separated set for X and so a (Γ, g, n, ϵ)-spanning set for X. Consequently,

$$\begin{aligned} P^*_\Gamma(g, T, \mathcal{F}, n\epsilon) &\leq \sum_{x \in D} \exp\left(\sup_{y \in B_{n,\Gamma}} f_n(y)\right) \\ &= \sum_{x \in D} e^{f_n(x)} \leq \sup\{\sum_{x \in E} e^{f_n(x)} : E \text{ is a } (\Gamma, g, n, \epsilon) - \text{separated set for } X\} \\ &= P_\Gamma(g, T, \mathcal{F}, n, \epsilon). \end{aligned}$$

This easily results in

$$P^*_\Gamma(g, T, \mathcal{F}) \leq P_\Gamma(g, T, \mathcal{F}). \tag{4.2}$$

Combining (4.1) and (4.2), we obtain $P_\Gamma(g, T, \mathcal{F}) \geq h_\mu(T) + \mathcal{F}_*(\mu)$. Finally, applying Theorems 4.4 and 2.1, we will have $P_\Gamma(g, T, \mathcal{F}) \geq P_\Gamma(T, \mathcal{F})$ which completes the proof. □

Remark 4.6. One should note that, the special case $\Gamma_0 = \{\gamma_{m,n}\}_{m.n \geq 0}$ with $\gamma_{m,n} = \frac{1}{n+1}$ results in mean metric $\widehat{d}_n(x,y) := \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(x), T^i(y))$. So, applying Theorem 3.10 with Γ_0 and $g(n, \epsilon) = n\epsilon$ also extends the results in [9] for ASP potentials.

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