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Research Paper

EXISTENCE OF THREE CLASSICAL SOLUTIONS FOR IMPULSIVE FRACTIONAL BOUNDARY VALUE PROBLEMS WITH p-LAPLACIAN

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ABSTRACT. In this paper, we study the existence of at least three distinct solutions for a class of impulsive fractional boundary value problems with p-Laplacian with Dirichlet boundary conditions. Our approach is based on recent variational methods for smooth functionals defined on reflexive Banach spaces. One example is presented to demonstrate the application of our main results.

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1. Introduction and Background

In this paper, we consider the following nonlinear impulsive fractional boundary value problem with Dirichlet boundary condition

$(P^{f,g}_{\lambda,\mu})$

$$\begin{cases} D^{\alpha}_{-T}\Phi_p({}^cD^{\alpha}_{0^+}u(t)) + |u(t)|^{p-2}u(t) = \lambda f(t,u(t)) + \mu g(t,u(t)), \quad t \neq t_j, \quad t \in (0,T), \\ \Delta(D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0^+}u))(t_j) = I_j(u(t_j)) \\ u(0) = u(T) = 0 \end{cases}$$

where $\alpha \in (\frac{1}{p}, 1]$, $p > 1$, $\Phi_p(s) = |s|^{p-2}s$ $(s \neq 0)$, D^{α}_{-T} represents the right Riemann–Liouville fractional derivative of order α and ${}^cD^{\alpha}_{0^+}$ represents the left Caputo fractional derivative of order α,

$$\Delta(D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0^+}u))(t_j) = D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0^+}u)(t_j^+) - D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0^+}u)(t_j^-),$$

$$D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0^+}u)(t_j^+) = \lim_{t\to t_j^+} D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0^+}u)(t),$$

$$D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0^+}u)(t_j^-) = \lim_{t\to t_j^-} D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0^+}u)(t),$$

$\lambda > 0$, $\mu \geq 0$, $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, m$ are Lipschitz continuous functions with the Lipschitz constants $L_j > 0$, i.e

$$|I_j(x_2) - I_j(x_1)| \leq L_j|x_2 - x_1|$$

for every $x_1, x_2 \in \mathbb{R}$ and $I_j(0) = 0$.

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Fractional differential equations appear naturally in a number of fields such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, Bode analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, etc. [24, 27, 28]. Recently, fractional differential equations have been of great interest due to the intensive development of the theory of fractional calculus itself and its applications. For some recent works on fractional differential equations, see [2, 6, 8, 14] and the references therein.

Nonlinear boundary value problems involving p-Laplacian operator Δ_p occur in a variety of physical phenomena, such as: non-Newtonian fluids, reaction-diffusion problems, petroleum extraction, flow through porous media, etc. Thus, the study of such problems and their far reaching generalizations have attracted several mathematicians in recent years, we refer the reader to [10, 12, 15, 26, 29] and the references therein.

The theory of impulsive differential equations is emerging as an important area of investigation since it is a lot richer than the corresponding theory of non-impulsive differential equations. Many evolutionary processes in nature are characterized by the fact that at certain moments in time an abrupt change of state is experienced. That is the reason for the rapid development of the theory of impulsive differential equations, for instance, see the two books [7].

For an introduction of the basic theory of impulsive differential equation, we refer the reader to [25]. Among previous research, little is concerned with differential equations with fractional order with impulses [23]. Ahmad and Sivasundaram [4, 5] gave some existence results for two point boundary value problems involving nonlinear impulsive hybrid differential equations of fractional order $1 < \alpha \leq 2$. Ahmad and Nieto in [3] established sufficient conditions for the existence of solutions of the anti periodic boundary value problem for impulsive differential equations with the Caputo derivative of order $q \in (1, 2]$.

The study of impulsive fractional boundary value problem has already been extended to the case involving the p-Laplacian. For details, see [1, 17, 18, 30, 32] and the references therein. For example Wang et al. in [30] based on a variant fountain theorem, the existence of infinitely many nontrivial high or small energy solutions for the problem ($P^{f,g}_{\lambda,\mu}$). Zhao and Tang in [32] by employing critical point theory and variational methods studied the existence and multiplicity of solutions for the problem ($P^{f,g}_{\lambda,\mu}$). In [17] using variational methods several sufficient conditions for the existence of at least one classical solution to impulsive fractional differential equations with a p-Laplacian and Dirichlet boundary conditions were presented. In we study the existence of multiple non-trivial classical solution to the problem In [1], applying Ricceri's variational principle, we ensured the existence of infinitely many solutions for the problem ($P^{f,g}_{\lambda,\mu}$).

Corresponding to the functions f and g, we introduce the functions $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $G : [0, T] \times \mathbb{R} \to \mathbb{R}$, respectively, as follow

$$F(t, \xi) = \int_0^{\xi} f(t, x)dx \quad \text{for all } (t, \xi) \in [0, T] \times \mathbb{R}$$

and

$$G(t, \xi) = \int_0^{\xi} g(t, x)dx \quad \text{for all } (t, \xi) \in [0, T] \times \mathbb{R}.$$

We present two consequences of Theorems 3.1 and 3.2, respectively, in the case $p = 2$. For our convenience, set

$$G^{\theta} := \int_0^T \max_{|\xi|\le\theta} G(t,\xi)dt \quad \text{for all } \theta > 0$$

and

$$G_{\sigma} := T \inf_{[0,T]\times[0,\sigma]} G(t,x) \quad \text{for all } \sigma > 0.$$

If g is sign-changing, then clearly $G^{\theta} \ge 0$ and $G_{\sigma} \le 0$. Put

$$\bar{k} = \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}.$$

Theorem 1.1. *Assume that there exist positive constants θ and δ with $\theta < \bar{k}\sigma$ such that*

$(\mathcal{A}_1)$ $$\frac{\int_0^T \sup_{|\xi|\le\theta_1} F(t,\xi)dt}{\theta_1^2} < \frac{1-LT\bar{k}^2}{\bar{k}^2\left(1+LT\bar{k}^2\right)}\frac{\int_0^T F(t,\sigma)dt}{\sigma^2};$$

$(\mathcal{A}_2)$ $$\limsup_{x\to+\infty}\frac{\max_{t\in[0,T]}F(t,x)}{|x|^2} \le 0.$$

Then, for every

$$\lambda \in \left(\frac{\frac{1+LT\bar{k}^2}{2}\sigma^2}{\int_0^T F(t,\sigma)dt}, \frac{\frac{1-LT\bar{k}^2}{2\bar{k}^2}\theta^2}{\int_0^T \sup_{|\xi|\le\theta} F(t,\xi)dt}\right)$$

where

$$\frac{(1+LT\bar{k}^2)\sigma^2}{\int_0^T F(t,\sigma)dt} < \frac{\frac{1-LT\bar{k}^2}{\bar{k}^2}\theta^2}{\int_0^T \sup_{|\xi|\le\theta} F(t,\xi)dt}$$

and for every continuous function $g : [0,T]\times\mathbb{R}\to\mathbb{R}$, satisfying the condition

$$\limsup_{x\to+\infty}\frac{\max_{t\in[0,T]}G(t,x)}{|x|^2} < +\infty,$$

and for every

$$\mu \in \left[0, \min\left\{\min\left\{\frac{(1-LT\bar{k}^2)\theta^2 - 2\lambda\bar{k}^2\int_0^T \sup_{|\xi|\le\theta} F(t,\xi)dt}{2\bar{k}^2G^{\theta}}, \frac{\frac{1+LT\bar{k}^2}{2}\sigma^2 - \lambda\int_0^T F(t,\sigma)dt}{G_{\sigma}}\right\}, \frac{1}{\max\left\{0, \frac{2\bar{k}^2T}{1-LT\bar{k}^2}\limsup_{x\to+\infty}\frac{\sup_{t\in[0,T]}G(t,x)}{|x|^2}\right\}}\right\}\right),$$

the problem

$$\left\{\begin{array}{l} D^{\alpha}_{-T}({}^{c}D^{\alpha}_{0+}u(t)) + |u(t)|^2 = \lambda f(t,u(t)) + \mu g(t,u(t)), \quad t \neq t_j, \quad t \in (0,T), \\ \Delta(D^{\alpha-1}_{-T}({}^{c}D^{\alpha}_{0+}u))(t_j) = I_j(u(t_j)) \\ u(0) = u(T) = 0 \end{array}\right. \tag{1.1}$$

possesses at least three solutions in $E^{\alpha,2}_0$.

Theorem 1.2. *Assume that there exist positive constants* θ_1, θ_2 *and* σ *with* $\theta_1 < \bar{k}\sigma$ *and* $\max\left\{\sigma, \sqrt{\frac{1+LT\bar{k}^2}{1-LT\bar{k}^2}}\bar{k}\sigma\right\} < \theta_2$ *such that*

($\mathcal{B}_1$) $f(t,x) \geq 0$ *for each* $(t,x) \in [0,T] \times [-\theta_2, \theta_2]$;

($\mathcal{B}_2$)

$$\max\left\{\frac{\int_0^T F(t,\theta_1)dt}{\theta_1^2}, \frac{2\int_0^T F(t,\theta_2)dt}{\theta_2^2}\right\} < \frac{1-LT\bar{k}^2}{\bar{k}^2\left(1+LT\bar{k}^2\right)}\frac{\int_0^T F(t,\sigma)dt}{\sigma^2}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{3(1+LT\bar{k}^2)}{4}\sigma^2}{\int_0^T F(t,\sigma)dt}, \min\left\{\frac{\frac{1-LT\bar{k}^2}{2\bar{k}^2}\theta_1^2}{\int_0^T \sup_{|\xi|\leq\theta_1} F(t,\xi)dt}, \frac{\frac{1-LT\bar{k}^2}{2\bar{k}^2}\theta_2^2}{2\int_0^T \sup_{|\xi|\leq\theta_2} F(t,\xi)dt}\right\}\right)$$

where

$$\frac{\frac{3(1+LT\bar{k}^2)}{2}\sigma^2}{\int_0^T F(t,\sigma)dt} < \min\left\{\frac{\frac{1-LT\bar{k}^2}{\bar{k}^2}\theta_1^2}{\int_0^T \sup_{|\xi|\leq\theta_1} F(t,\xi)dt}, \frac{\frac{1-LT\bar{k}^2}{\bar{k}^2}\theta_2^2}{2\int_0^T \sup_{|\xi|\leq\theta_2} F(t,\xi)dt}\right\}$$

and for every non-negative continuous function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ *for each*

$$\mu \in \left(0, \frac{1}{2\bar{k}^2}\min\left\{\frac{(1-LT\bar{k}^2)\theta_1^2 - 2\lambda\bar{k}^2\int_0^T \sup_{|\xi|\leq\theta_1} F(t,\xi)dt}{G^{\theta_1}}, \frac{(1-LT\bar{k}^2)\theta_2^2 - 4\lambda\bar{k}^2\int_0^T \sup_{|\xi|\leq\theta_2} F(t,\xi)dt}{2G^{\theta_2}}\right\}\right],$$

the problem (1.1) *possesses at least three non-negative solutions* u_1, u_2 *and* u_3 *such that*

$$u_i(t) < \theta_2, \ \forall t \in [0,T] \ (i = 1,2,3).$$

Motivated by the above facts, in the present paper, using three kinds of three critical points theorems obtained in [9, 11] which we recall in the next section (Theorems 2.1 and 2.2), we establish the existence of at least three solutions for the problem ($P^{f,g}_{\lambda,\mu}$) in which two parameters are involved. Precise estimates of these two parameters λ and μ will be given, see Theorems 3.1- 3.2. We present Example 3.5 in which the hypotheses of Theorem 3.1 is fulfilled. Theorems 3.6 and 3.7 are two consequence of Theorems 3.1 and 3.2, respectively.

This work is organized as follows: in section 2 we present some preliminary results and in the section 3 we state and prove the main result.

2. Preliminaries

Let X be a real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two functions of class C^1 on X, and let λ be a positive real parameter. In order to study the problem ($P^{f,g}_{\lambda,\mu}$), our main tools are critical points theorems for functional of type $I_\lambda = \Phi - \lambda\Psi$ which insure the existence at least three critical points for every λ belonging to well-defined open intervals.

Theorem 2.1. [11, Theorem 2.6] *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$.*
Assume that there exist $r > 0$ and $\overline{v} \in X$, with $r < \Phi(\overline{v})$ such that

(a_1) $$\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})};$$

(a_2) *for each* $\lambda \in \Lambda_r := \left]\dfrac{\Phi(\overline{v})}{\Psi(\overline{v})}, \dfrac{r}{\sup_{\Phi(u)\le r} \Psi(u)}\right[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.2. [9, Corollary 3.1] *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
2. *for each $\lambda > 0$ and for every $u_1, u_2 \in X$ which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \ge 0$ and $\Psi(u_2) \ge 0$, one has*
$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \ge 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{v} \in X$, with $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$, such that

(b_1) $$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})};$$

(b_2) $$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}.$$

Then, for each
$$\lambda \in \left]\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})},\ \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)},\ \frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}\right\}\right[,$$
the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

Theorems 2.1 and 2.2 have been successfully used to ensure the existence of at least three solutions for perturbed boundary value problems in the papers [10, 13, 16, 19, 20, 21]. In this section, we will introduce several basic definitions, notations, lemmas, and propositions used all over this paper.

Let $AC[a,b]$ be the space of absolutely continuous functions on $[a,b]$.

Definition 2.3. [22] Let f be a function defined on $[a,b]$ and $0<\alpha\leq 1$. The left and right Riemann-Liouville fractional integrals of order α for the function f are defined by

$$D_{a^+}^{-\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds,\quad t\in[a,b],$$
$$D_{b^-}^{-\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)ds,\quad t\in[a,b]$$

provided the right-hand sides are pointwise defined on $[a,b]$ where $\Gamma(\alpha)$ is the standard gamma function given by

$$\Gamma(\alpha)=\int_0^{+\infty} z^{\alpha-1}e^{-z}dz.$$

Definition 2.4. [22] Let f be a function defined on $[a,b]$ and $0<\alpha\leq 1$. The left and right Riemann-Liouville fractional integrals of order α for the function f are defined by

$$D_{a^+}^{\alpha}f(t)=\frac{d}{dt}D_{a^+}^{\alpha-1}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-s)^{-\alpha}f(s)ds,\quad t\in[a,b],$$
$$D_{b^-}^{\alpha}f(t)=-\frac{d}{dt}D_{b^-}^{\alpha-1}f(t)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b (s-t)^{-\alpha}f(s)ds,\quad t\in[a,b].$$

Definition 2.5. [22] Let f be a function defined on $[a,b]$ and $0<\alpha\leq 1$. The left and right Riemann-Liouville fractional integrals of order α for the function f are defined by

$${}^cD_{a^+}^{\alpha}f(t)=D_{a^+}^{\alpha-1}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}f'(s)ds,\quad t\in[a,b],$$
$${}^cD_{b^-}^{\alpha}f(t)=-D_{b^-}^{\alpha-1}f(t)=-\frac{1}{\Gamma(1-\alpha)}\int_t^b (s-t)^{-\alpha}f'(s)ds,\quad t\in[a,b].$$

In particular, when $\alpha=1$, we have ${}^cD_{a^+}^{1}f(t)=f'(t)$ and ${}^cD_{b^-}^{1}f(t)=-f'(t)$.

Proposition 2.6. [33]

(1) *If* $u\in L^p([0,T],\mathbb{R})$, $v\in L^q([0,T],\mathbb{R})$ *and* $p\geq 1$, $q\geq 1$, $\dfrac{1}{p}+\dfrac{1}{q}\leq 1+\theta$ *or* $p\neq 1$, $q\neq 1$, $\dfrac{1}{p}+\dfrac{1}{q}=1+\theta$, *then we have*

$$\int_a^b [D_t^{-\theta}u(t)]v(t)dt=\int_a^b [v(t)D_b^{-\theta}]u(t)dt,\quad \theta>0.$$

(2) *If* $0<\alpha\leq 1$, $u\in AC[a,b]$, *and* $v\in L^p[a,b]$ $(1\leq p<\infty)$, *then*

$$\int_a^b u(t)({}^cD_{a^+}^{\alpha}f(t))dt=D_b^{\alpha-1}u(t)v(t)\mid_{t=a}^{t=b}+\int_a^b D_b^{\alpha}u(t)v(t)dt.$$

Let $C_0^\infty([0,T],\mathbb{R}^N)$ be the set of all functions $u\in C^\infty([0,T],\mathbb{R}^N)$ with $u(a)=u(b)=0$ and the norm

$$\|u\|_\infty=\max_{t\in[a,b]}|u(t)|.$$

Denote the norm of the space $L^p([0,T],\mathbb{R}^N)$ for $1 \le p < \infty$ by

$$\|u\|_{L^p} = \Big(\int_a^b |u(s)|^p ds\Big)^{\frac{1}{p}}.$$

The following lemma yields the boundedness of the Riemann-Liouville fractional integral operators from the space $L^p([a,b],\mathbb{R}^N)$ to the space $L^p([a,b],\mathbb{R}^N)$ where $1 \le p < \infty$.

Definition 2.7. Let $0 < \alpha \le 1$, $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure $C_0^\infty([0,T],\mathbb{R})$, that is

$$E_0^{\alpha,p} = \overline{C_0^\infty([0,T],\mathbb{R})}$$

with respect to the weighted norm

$$\|u\|_{E_0^{\alpha,p}} = \Big(\int_0^T |{}^cD_{0^+}^\alpha u(t)|^p dt + \int_0^T |u(t)|^p dt\Big)^{\frac{1}{p}} \tag{2.1}$$

for every $u \in E_0^{\alpha,p}$.

Remark 2.8. It is obvious that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u \in L^2([0,T],\mathbb{R})$ having an α-order Riemann-Loiuville fractional derivative ${}^cD_t^\alpha u \in L^2([0,T],\mathbb{R})$ and $u(0) = u(T) = 0$. From [22, Propostion 3.1], we know for $0 < \alpha \le 1$, the space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

Lemma 2.9. [33] *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For any $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}^cD_{0^+}^\alpha u(t)\|_{L^p}. \tag{2.2}$$

In addition, for $\frac{1}{p} < \alpha \le 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\|u\|_\infty \le k \|{}^cD_{0^+}^\alpha u(t)\|_{L^p} \tag{2.3}$$

where $k = \dfrac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}}$.

Remark 2.10. According to Lemma 2.9, it is easy to see that the norm of $E_0^{\alpha,p}$ defined in (2.1) is equivalent to the following norm:

$$\|u\|_{\alpha,p} = \Big(\int_0^T |{}^cD_{0^+}^\alpha u(t)|^p dt\Big)^{\frac{1}{p}}. \tag{2.4}$$

Lemma 2.11. *Let $\frac{1}{p} < \alpha \le 1$. If the sequence $\{u_k\}$ converges weakly to u in $E_0^{\alpha,p}$, i.e., $u_k \rightharpoonup u$, then $u_k \longrightarrow u$ in $C[0,T]$, i.e., $\|u - u_k\|_\infty \longrightarrow 0$ as $k \longrightarrow \infty$.*

Lemma 2.12. *A function*

$$u \in \left\{ u \in AC[0,T] : \left(\int_{t_j}^{t_{j+1}} |({}^cD_{0^+}^\alpha u(t)|^p + |u(t)|^p) dt \right) < \infty,\ j = 1,2,\ldots,m \right\}$$

is called a classical solution of BVP ($P_{\lambda,\mu}^{f,g}$) if

(1) *u satisfies ($P_{\lambda,\mu}^{f,g}$).*
(2) *The limits $D_{T^-}^{\alpha-1}\Phi_p({}^cD_{0^+}^\alpha u)(t_j^+)$, $D_{T^-}^{\alpha-1}\Phi_p({}^cD_{0^+}^\alpha u)(t_j^-)$ exist.*

Definition 2.13. We mean by a (weak) solution of the BVP ($P^{f,g}_{\lambda,\mu}$), any function $u \in E_0^{\alpha,p}$ such that

$$\int_0^T |^cD^\alpha_{0+}u(t)|^{p-2}(^cD^\alpha_{0+}u(t))(^cD^\alpha_{0+}v(t))dt + \int_0^T |u(t)|^{p-2}u(t)v(t)dt$$
$$+\sum_{j=1}^m I_j(u(t_j))v(t_j) - \lambda\int_0^T f(t,u(t))v(t)dt - \mu\int_0^T g(t,u(t))v(t)dt = 0$$

for every $v \in E_0^{\alpha,p}$.

Lemma 2.14. [32] *If $u \in E_0^{\alpha,p}$ is a weak solution of BVP ($P^{f,g}_{\lambda,\mu}$), then u is a classical solution of BVP ($P^{f,g}_{\lambda,\mu}$)*

We assume throughout and without further mention, that $1 > LTk^p$ where $L = \sum_{j=1}^m L_j$. We need the following proposition for existence our main results.

Proposition 2.15. *Let $S : \mathrm{E}^\alpha \longrightarrow (\mathrm{E}^\alpha)^*$ be the operator defined by*

$$S(u)(v) = \int_0^T |^cD^\alpha_{0+}u(t)|^{p-2}(^cD^\alpha_{0+}u(t))(^cD^\alpha_{0+}v(t))dt$$
$$+\int_0^T |u(t)|^{p-2}u(t)v(t)dt + \sum_{j=1}^m I_j(u(t_j))v(t_j)$$

for every $u, v \in \mathrm{E}^\alpha$. Then, S admits a continuous inverse on $(\mathrm{E}^\alpha)^*$.

Proof. It is obvious that

$$S(u)(u) = \int_0^T |^cD^\alpha_{0+}u(t)|^p dt + \int_0^T |u(t)|^p dt + \sum_{j=1}^m I_j(u(t_j))u(t_j)$$
$$\geq (1 - LTk^p)\|u\|^p_{\alpha,p}.$$

This follows that S is coercive. Owing to our assumptions on the data, one has

$$\langle S(u) - S(v), u - v\rangle = \int_0^T |^cD^\alpha_{0+}(u(t) - v(t))|^p dt + \int_0^T |(u(t) - v(t))|^p dt$$
$$+\sum_{j=1}^m I_j((u(t_j) - v(t_j))(u(t_j) - v(t_j))$$
$$\geq (1 - LTk^p)\|u - v\|^p_{\alpha,p} > 0$$

for every $u, v \in \mathrm{E}^\alpha$, which means that S is strictly monotone. Moreover, since E^α is reflexive, for $u_n \longrightarrow u$ strongly in E^α as $n \to +\infty$, one has $S(u_n) \to S(u)$ weakly in $(E^\alpha)^*$ as $n \to \infty$. Hence, S is demicontinuous, so by [31, Theorem 26.A(d)], the inverse operator S^{-1} of S exists and it is continuous. Indeed, let e_n be a sequence of $(\mathrm{E}^\alpha)^*$ such that $e_n \to e$ strongly in $(\mathrm{E}^\alpha)^*$ as $n \to \infty$. Let u_n and u in E^α such that $S^{-1}(e_n) = u_n$ and $S^{-1}(e) = u$. Taking into account that S is coercive, one has that the sequence u_n is bounded in the reflexive space E^α. For a suitable subsequence, we have $u_n \to \hat{u}$ weakly in E^α as $n \to \infty$, which concludes

$$\langle S(u_n) - S(u), u_n - \hat{u}\rangle = \langle e_n - e, u_n - \hat{u}\rangle = 0.$$

Note that if $u_n \to \hat{u}$ weakly in E^α as $n \to +\infty$ and $S(u_n) \to S(\hat{u})$ strongly in $(\mathrm{E}^\alpha)^*$ as $n \to +\infty$, one has $u_n \to \hat{u}$ strongly in E^α as $n \to +\infty$, and since S is continuous, we have

$u_n \to \hat{u}$ weakly in E^α as $n \to +\infty$ and $S(u_n) \to S(\hat{u}) = S(u)$ strongly in $(\mathrm{E}^\alpha)^*$ as $n \to +\infty$. Hence, taking into account that S is an injection, we have $u = \hat{u}$. □

3. Main result

Fixing positive constants θ and σ, put

$$\frac{(1+LTk^p)\sigma^p}{\displaystyle\int_0^T F(t,\sigma)dt} < \frac{\dfrac{1-LTk^p}{k^p}\theta^p}{\displaystyle\int_0^T \sup_{|\xi|\le\theta} F(t,\xi)dt}$$

and taking

$$\lambda \in \Lambda := \left(\frac{\dfrac{1+LTk^p}{p}\sigma^p}{\displaystyle\int_0^T F(t,\sigma)dt}, \frac{\dfrac{1-LTk^p}{pk^p}\theta^p}{\displaystyle\int_0^T \sup_{|\xi|\le\theta} F(t,\xi)dt} \right)$$

set $\delta_{\lambda,g}$ given by

$$\delta_{\lambda,g} := \min\left\{ \frac{(1-LTk^p)\theta^p - \lambda p k^p \displaystyle\int_0^T \sup_{|\xi|\le\theta} F(t,\xi)dt}{pk^p G^\theta}, \frac{\dfrac{1+LTk^p}{p}\sigma^p - \lambda \displaystyle\int_0^T F(t,\sigma)dt}{G_\sigma} \right\}$$

and

$$\overline{\delta}_{\lambda,g} := \min\left\{ \delta_{\lambda,g}, \frac{1}{\max\left\{0, \dfrac{pk^pT}{1-LTk^p} \displaystyle\limsup_{x\to+\infty} \frac{\sup_{t\in[0,T]} G(t,x)}{|x|^p}\right\}} \right\} \tag{3.1}$$

where we read $\epsilon/0 = +\infty$, so that, for instance, $\overline{\delta}_{\lambda,g} = +\infty$ when

$$\limsup_{x\to+\infty} \frac{\sup_{|t|\in[0,T]} G(t,x)}{x^p} \le 0,$$

and $G_\sigma = G^\theta = 0$. We formulate our main result as follows.

Theorem 3.1. *Assume that there exist positive constants θ and δ with $\theta < k\sigma$ such that*

(A$_1$) $$\frac{\displaystyle\int_0^T \sup_{|\xi|\le\theta_1} F(t,\xi)dt}{\theta_1^p} < \frac{1-LTk^p}{k^p\,(1+LTk^p)} \frac{\displaystyle\int_0^T F(t,\sigma)dt}{\sigma^p};$$

(A$_2$) $$\limsup_{x\to+\infty} \frac{\max_{t\in[0,T]} F(t,x)}{|x|^p} \le 0.$$

Then, for every $\lambda \in \Lambda$ and for every continuous function $g : [0,T]\times\mathbb{R} \to \mathbb{R}$, satisfying the condition

$$\limsup_{|x|\to+\infty} \frac{\max_{t\in[0,T]} G(t,x)}{x^p} < +\infty,$$

there exists $\overline{\delta}_{\lambda,g} > 0$ given by (3.1) such that, for each $\mu \in [0, \overline{\delta}_{\lambda,g})$, the problem $(P^{f,g}_{\lambda,\mu})$ possesses at least three solutions in $E_0^{\alpha,p}$.

Proof. Our aim is to apply Theorem 2.1 to the problem $(P^{f,g}_{\lambda,\mu})$. Take $X = E_0^{\alpha,p}$. Let the functionals Φ, Ψ for every $u \in X$, defined by

$$\Phi(u) = \frac{1}{p}\|u\|^p_{\alpha,p} - \sum_{j=1}^{m}\int_0^{u(t_j)} I_j(s)ds \tag{3.2}$$

and

$$\Psi(u) = \int_0^T F(t, u(t))dt + \frac{\mu}{\lambda}\int_0^T G(t, u(t))dt.$$

Let us prove that the functionals Φ and Ψ satisfy the required conditions in Theorem 2.1. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^T f(t, u(t))v(t)dt + \frac{\mu}{\lambda}\int_0^T g(t, u(t))v(t)dt$$

for every $v \in X$, as well as is sequentially weakly upper semicontinuous. Now from the facts $-L_j|\xi| \leq I_j(\xi) \leq L_j|\xi|$ for every $\xi \in \mathbb{R}$, $j = 1, \ldots, m$, and taking (2.3) into account, for every $u \in X$, we have

$$\begin{aligned}\frac{1 - LTk^p}{p}\|u\|^p_{\alpha,p} &\leq \frac{1}{p}\|u\|^p_{\alpha,p} - \frac{LTk^p}{p}\|u\|^p_{\alpha,p} \leq \Phi(u)\\ &\leq \frac{1}{p}\|u\|^p_{\alpha,p} + \frac{LTk^p}{p}\|u\|^p_{\alpha,p} \leq \frac{1 + LTk^p}{p}\|u\|^p_{\alpha,p},\end{aligned} \tag{3.3}$$

by using the condition $1 > LTk^p$, and the first inequality in (3.3), it follows $\lim_{\|u\|\to+\infty}\Phi(u) = +\infty$, namely Φ is coercive. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\begin{aligned}\Phi'(u)(v) &= \int_0^T |{}^cD^\alpha_{0^+}u(t)|^{p-2}({}^cD^\alpha_{0^+}u(t))({}^cD^\alpha_{0^+}v(t))dt\\ &\quad + \int_0^T |u(t)|^{p-2}u(t)v(t)dt + \sum_{j=1}^{m} I_j(u(t_j))v(t_j)\end{aligned}$$

for every $v \in X$. Furthermore, Proposition 2.15 gives that Φ' admits a continuous inverse on X^*. Moreover, Φ is sequentially weakly lower semicontinuous. Therefore, we observe that the regularity assumptions on Φ and Ψ, as requested in Theorem 2.1, are verified. Define w by setting

$$w(t) = \begin{cases} 0, & \text{if } t = 0,\\ \sigma, & \text{if } t \in (0, T),\\ 0, & \text{if } t = 0.\end{cases}$$

Clearly, $w \in X$, from (3.2) and (3.3), we have

$$\frac{1 - LTk^p}{p}\sigma^p \leq \Phi(w) \leq \frac{1 + LTk^p}{p}\sigma^p.$$

Choose

$$r = \frac{1 - LTk^p}{pk^p}\theta^p.$$

From the condition $\theta < k\sigma$, we achieve $r < \Phi(w)$. From the definition of Φ, the estimate $\Phi(u) < r$ implies that

$$\Phi^{-1}(-\infty, r] = \{u \in X; \Phi(u) \leq r\} \subseteq \{u \in X; |u| \leq \theta\}. \tag{3.4}$$

and it concludes that

$$\begin{aligned}\sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u) &= \sup_{u\in\Phi^{-1}(-\infty,r]} \int_0^T \left[F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\right] dt \\ &\leq \int_0^T \sup_{|\xi|\leq\theta} F(t,\xi)dt + \frac{\mu}{\lambda}G^\theta.\end{aligned}$$

On the other hand, we have

$$\Psi(w) = \int_0^T \left[F(t,w(t)) + \frac{\mu}{\lambda}G(t,w(t))\right] dt \geq \int_0^T F(t,\sigma)dt + \frac{\mu}{\lambda}G_\sigma.$$

Therefore, we have

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r)} \int_0^T F(t,u(t))dt}{r} \\ &\leq \frac{\int_0^T \sup_{|\xi|\leq\theta} F(t,\xi)dt}{\frac{1-LTk^p}{pk^p}\theta^p}\end{aligned} \tag{3.5}$$

and

$$\frac{\Psi(w)}{\Phi(w)} = \frac{\int_0^T F(t,\sigma)dt + \frac{\mu}{\lambda}G_\sigma}{\Phi(w)} \geq \frac{\int_0^T F(t,\sigma)dt + \frac{\mu}{\lambda}G_\sigma}{\frac{1+LTk^p}{p}\sigma^p}. \tag{3.6}$$

Since $\mu < \delta_{\lambda,g}$, one has

$$\mu < \frac{(1-LTk^p)\theta^p - \lambda pk^p \int_0^T \sup_{|\xi|\leq\theta} F(t,\xi)dt}{pk^pG^\theta},$$

this means

$$\frac{\int_0^T \sup_{|\xi|\leq\theta} F(t,\xi)dt + \frac{\mu}{\lambda}G^\theta}{\frac{1-LTk^p}{pk^p}\theta^p} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{\frac{1+LTk^p}{p}\sigma^p - \lambda\int_0^T F(t,\sigma)dt}{G_\sigma},$$

this means

$$\frac{\displaystyle\int_0^T F(t,\sigma)dt + \frac{\mu}{\lambda} G_\sigma}{\dfrac{1+LTk^p}{p}\sigma^p} > \frac{1}{\lambda}.$$

Then,

$$\frac{\displaystyle\int_0^T \sup_{|\xi|\le\theta} F(t,\xi)dt + \frac{\mu}{\lambda} G^\theta}{\dfrac{1-LTk^p}{pk^p}\theta^p} < \frac{1}{\lambda} < \frac{\displaystyle\int_0^T F(t,\sigma)dt + \frac{\mu}{\lambda} G_\sigma}{\dfrac{1+LTk^p}{p}\sigma^p}. \tag{3.7}$$

Hence, from (3.5)-(3.7), the condition (a_1) of Theorem 2.1 is fulfilled. To this end, since $\mu < \overline{\delta}_{\lambda,g}$, we can fix $l > 0$ such that

$$\limsup_{t\to+\infty} \frac{\max_{t\in[0,T]} G(t,x)}{|x|^p} < l,$$

and $\mu l < \dfrac{1-LTk^p}{pk^pT}$. Therefore, there exists a function $h \in L^1[0,T]$ such that

$$G(t,x) \le l|x|^p + h(t)$$

for every $(t,x) \in [0,T]\times\mathbb{R}$. Fix

$$0 < \epsilon < \frac{1-LTk^p}{\lambda Tk^p p} - \frac{\mu l}{\lambda},$$

from (A_2) there is a function $h_\epsilon \in L^1[0,T]$ such that

$$F(t,x) < \epsilon|x|^p + h_\epsilon(t) \text{ for all } (t,x) \in [0,T]\times\mathbb{R}.$$

Taking (2.3) into account, it follows that, for each $u \in X$,

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &\ge \frac{1-LTk^p}{p}\|u\|_{\alpha,p}^p - \lambda\int_0^T [F(t,u(t)) + G(t,u(t))]\,dt \\
&\ge \frac{1-LTk^p}{p}\|u\|_{\alpha,p}^p - \lambda\epsilon\int_0^T u^p(t)dt - \lambda\|h_\epsilon\|_1 \\
&- \mu l\int_0^T u^p(t)dt - \mu\|h\|_1 \\
&\ge \frac{1-LTk^p}{p}\|u\|_{\alpha,p}^p - \lambda\epsilon T\|u\|_\infty^p \\
&- \lambda\|h_\epsilon\|_1 - \mu lT\|u\|_\infty^p - \mu\|h\|_1 \\
&= \left(\frac{1-LTk^p}{p} - \lambda\epsilon Tk^p - \mu lTk^p\right)\|u\|_{\alpha,p}^p - \lambda\|h_\epsilon\|_1 - \mu\|h\|_1
\end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty.$$

Therefore, (a_1) and (a_2) of Theorem 2.1 are fulfilled. By using relations (3.5) and (3.7) one also has

$$\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[.$$

Finally, Theorem 2.1 assures the existence of three critical points for the functional $\Phi - \lambda\Psi$ (with $\overline{v} = w$), which are solutions of the problem $(P^{f,g}_{\lambda,\mu})$ and we have the conclusion. $\square$

Now, we present a variant of Theorem 3.1 in which no asymptotic condition on the nonlinear term is requested.

For our goal, let us fix positive constants θ_1, θ_2 and σ, such that

$$\frac{\frac{3(1+LTk^p)}{2}\sigma^p}{\displaystyle\int_0^T F(t,\sigma)dt} < \min\left\{\frac{\frac{1-LTk^p}{k^p}\theta_1^p}{\displaystyle\int_0^T \sup_{|\xi|\le\theta_1} F(t,\xi)dt}, \frac{\frac{1-LTk^p}{k^p}\theta_2^p}{2\displaystyle\int_0^T \sup_{|\xi|\le\theta_2} F(t,\xi)dt}\right\}$$

and take

$$\Lambda' := \left(\frac{\frac{3(1+LTk^p)}{2p}\sigma^p}{\displaystyle\int_0^T F(t,\sigma)dt}, \min\left\{\frac{\frac{1-LTk^p}{pk^p}\theta_1^p}{\displaystyle\int_0^T \sup_{|\xi|\le\theta_1} F(t,\xi)dt}, \frac{\frac{1-LTk^p}{pk^p}\theta_2^p}{2\displaystyle\int_0^T \sup_{|\xi|\le\theta_2} F(t,\xi)dt}\right\}\right)$$

and

$$\delta'_{\lambda,g} := \frac{1}{pk^p}\min\left\{\frac{(1-LTk^p)\theta_1^p - \lambda pk^p\displaystyle\int_0^T \sup_{|\xi|\le\theta_1} F(t,\xi)dt}{G^{\theta_1}}, \frac{(1-LTk^p)\theta_2^p - 2\lambda pk^p\displaystyle\int_0^T \sup_{|\xi|\le\theta_2} F(t,\xi)dt}{2G^{\theta_2}}\right\}. \tag{3.8}$$

Theorem 3.2. *Assume that there exist positive constants θ_1, θ_2 and σ with $\theta_1 < k\sigma$ and $\max\left\{\sigma, \sqrt[p]{\frac{1+LTk^p}{1-LTk^p}}k\sigma\right\} < \theta_2$ such that*

(B$_1$) *$f(t,x) \ge 0$ for each $(t,x) \in [0,T]\times[-\theta_2,\theta_2]$;*

(B$_2$)

$$\max\left\{\frac{\displaystyle\int_0^T F(t,\theta_1)dt}{\theta_1^p}, \frac{2\displaystyle\int_0^T F(t,\theta_2)dt}{\theta_2^p}\right\} < \frac{1-LTk^p}{k^p(1+LTk^p)}\frac{\displaystyle\int_0^T F(t,\sigma)dt}{\sigma^p}.$$

Then, for every $\lambda \in \Lambda'$ and for every non-negative continuous function $g : [0,T]\times\mathbb{R}\to\mathbb{R}$, there exists $\delta'_{\lambda,g} > 0$ given by (3.8) such that, for each $\mu \in [0, \delta'_{\lambda,g})$, the problem $(P^{f,g}_{\lambda,\mu})$ possesses at least three non-negative solutions u_1, u_2 and u_3 such that

$$u_i(t) < \theta_2, \ \forall t \in [0,T] \ (i = 1,2,3).$$

Proof. We consider the auxiliary problem
$(P^{\hat{f},g}_{\lambda,\mu})$

$$\begin{cases} D^{\alpha}_{-T}\Phi_p({}^cD^{\alpha}_{0+}u(t)) + |u(t)|^{p-2}u(t) = \lambda\hat{f}(t,u(t)) + \mu g(t,u(t)), \quad t \neq t_j, \quad t \in (0,T), \\ \Delta(D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0+}u))(t_j) = I_j(u(t_j)) \\ u(0) = u(T) = 0 \end{cases}$$

where $\hat{f} : [0,T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function defined by

$$\hat{f}(t,\xi) = \begin{cases} f(t,0), & \text{if } \xi < -\theta_2, \\ f(t,\xi), & \text{if } -\theta_2 \leq \xi \leq \theta_2, \\ f(t,\theta_2), & \text{if } \xi > \theta_3. \end{cases}$$

If a solution of the problem $(P^{\hat{f},g}_{\lambda,\mu})$ satisfies the condition $-\theta_2 \leq u(t) \leq \theta_2$ for every $t \in [0,T]$, then, clearly it turns to be also a solution of $(P^{f,g}_{\lambda,\mu})$. Therefore, for our goal, it is enough to show that our conclusion holds for $(P^{f,g}_{\lambda,\mu})$. Fix λ, g and μ as in the conclusion and take Φ and Ψ as in the proof of Theorem 3.1. We observe that the regularity the assumptions of Theorem 2.2 on Φ and Ψ are fulfilled. Then, our aim is to verify (b_1) and (b_2). To this end, choose w as given in (3), as well as

$$r_1 = \frac{1 - LTk^p}{pk^p}\theta_1^p \text{ and } r_2 = \frac{1 - LTk^p}{pk^p}\theta_2^p.$$

In view of the conditions $\theta_1 < \dfrac{k\sigma}{\sqrt[p]{2}}$ and $\sqrt[p]{\dfrac{2(1 + LTk^p)}{1 - LTk^p}}k\sigma < \theta_2$, we have $2r_1 < \Phi(w) < \dfrac{r_2}{2}$. Since $\mu < \delta'_{\lambda,g}$ and $G_\sigma = 0$, one has

$$\begin{aligned} \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{r_1} &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\displaystyle\int_0^T \left[F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\right]dt}{r_1} \\ &\leq \frac{\displaystyle\int_0^T \max_{|x|\leq\theta_1} F(t,x)dt + \frac{\mu}{\lambda}G^{\theta_1}}{\dfrac{1 - LTk^p}{pk^p}\theta_1^p} < \frac{1}{\lambda} \\ &< \frac{2}{3}\frac{\displaystyle\int_0^T F(t,\sigma)dt + \frac{\mu}{\lambda}G_\sigma}{\dfrac{1 + LTk^p}{p}\sigma^p} \\ &\leq \frac{2}{3}\frac{\Psi(w)}{\Phi(w)} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
2\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2} =& 2\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\int_0^T\left[F(t,u(t))+\frac{\mu}{\lambda}G(t,u(t))\right]dt}{r_2}\\
\leq& 2\frac{\int_0^T\max_{|x|\leq\theta_2}F(t,x)dt+\frac{\mu}{\lambda}G^{\theta_2}}{\frac{1-LTk^p}{pk^p}\theta_2^p}<\frac{1}{\lambda}\\
<&\frac{2}{3}\frac{\int_0^T F(t,\sigma)dt+\frac{\mu}{\lambda}G_\sigma}{\frac{1+LTk^p}{p}\sigma^p}\\
\leq&\frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.
\end{aligned}$$

Finally, we verify that $\Phi-\lambda\Psi$ satisfies the second assumption of Theorem 2.2. Let u_1 and u_2 be two local minima for $\Phi-\lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi-\lambda\Psi$, and so, they are solutions for the problem ($P^{f,g}_{\lambda,\mu}$). Since f and g are non-negative then the solutions ensured are non-negative. Indeed, let u_* be a non-trivial solution of the problem ($P^{f,g}_{\lambda,\mu}$), then u_* is non-negative. Arguing by a contradiction, assume that the set $\mathcal{A}=\{t\in]0,T];\ u_*(t)<0\}$. Put $\bar{v}(t)=\min\{u_*(t),0\}$ for $t\in[0,T]$. Clearly, $\bar{v}\in \mathrm{E}_0^{\alpha,p}$ and one has

$$\begin{aligned}
&\int_0^T|{}^cD^\alpha_{0+}u_*(t)|^{p-2}({}^cD^\alpha_{0+}u_*(t))({}^cD^\alpha_{0+}\bar{v}(t))dt+\int_0^T|u_*(t)|^{p-2}u_*(t)\bar{v}(t)dt\\
&+\sum_{j=1}^m I_j(u_*(t_j))\bar{v}(t_j)-\lambda\int_0^T f(t,u_*(t))\bar{v}(t)dt-\mu\int_0^T g(t,u_*(t))\bar{v}(t)dt=0
\end{aligned}$$

and by choosing $\bar{v}=u_*$ and since f and g are non-negative, we have

$$\begin{aligned}
0\leq 2C_1\|u_*\|^p_{\mathrm{E}_0^{\alpha,p}(\mathcal{A})}\leq&\int_0^T|{}^cD^\alpha_{0+}u_*(t)|^p+\int_0^T|u_*(t)|^p+\sum_{j=1}^m I_j(u_*(t_j))u_*(t_j)\\
=&\lambda\int_{\mathrm{E}_0^{\alpha,p}(\mathcal{A})}f(t,u_*(t))u_*(t)dt+\mu\int_{\mathrm{E}_0^{\alpha,p}(\mathcal{A})}g(t,u_*(t))u_*(t)dt\leq 0.
\end{aligned}$$

Hence, since $LTk^p<1$,

$$\|u_*\|^p_{\mathrm{E}_0^{\alpha,p}(\mathcal{A})}\leq 0$$

which contradicts with this fact that u_* is a non-trivial solution. Hence, u_* is positive. Then, we observe $u_1(t)\geq 0$ and $u_2(t)\geq 0$ for every $t\in[0,T]$. Thus, it follows that $su_1+(1-s)u_2\geq 0$ for all $s\in[0,1]$, and that $(\lambda f+\mu g)(k,su_1+(1-s)u_2)\geq 0$, and consequently, $\Psi(su_1+(1-s)u_2)\geq 0$, for every $s\in[0,1]$. From Theorem 2.2, for every

$$\lambda\in\left]\frac{3}{2}\frac{\Phi(w)}{\Psi(w)},\ \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)},\ \frac{r_2/2}{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points, which are solutions of the problem $(P^{f,g}_{\lambda,\mu})$ and the conclusion is achieved. $\square$

Remark 3.3. If in Theorems 3.1 and 3.2, either $f(t,0) \neq 0$ for some $t \in [0,T]$ or $g(t,0) \neq 0$ for some $t \in [0,T]$, or both hold true, then the ensured solutions are obviously non-trivial.

Remark 3.4. If we consider the case in which the function f has separated variables of the problem $(P^{f,g}_{\lambda,\mu})$,

$(P^{f,g}_{\lambda,\mu})$

$$\begin{cases} D^{\alpha}_{-T}\Phi_p({}^cD^{\alpha}_{0+}u(t)) + |u(t)|^{p-2}u(t) = \lambda f_1(t)f_2(u(t)) + \mu g(t,u(t)), \quad t \neq t_j, \quad t \in (0,T), \\ \Delta(D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0+}u))(t_j) = I_j(u(t_j)) \\ u(0) = u(T) = 0 \end{cases}$$

where $f_1 \in C([1,T])$ and $f_2 \in C(\mathbb{R})$ are two non-negative functions, putting $\tilde{F}(t) = \displaystyle\int_0^t f_2(\xi)d\xi$ for all $t \in \mathbb{R}$, in Theorem 3.1 the assumptions (A_1) and (A_2) can be written as

(A$_3$) $\quad \dfrac{\tilde{F}(\theta)}{\theta_1^p} < \dfrac{1 - LTk^p}{k^p\,(1 + LTk^p)}\dfrac{\tilde{F}(\sigma)}{\sigma^p}$;

(A$_4$) $\quad \limsup_{x\to+\infty} \frac{\tilde{F}(x)}{|x|^p} \leq 0.$

respectively, as well as

$$\lambda \in \Lambda := \left(\frac{\dfrac{1+LTk^p}{p}\sigma^p}{\tilde{F}(\sigma)\displaystyle\int_0^t f(t)dt}, \frac{\dfrac{1-LTk^p}{pk^p}\theta^p}{\tilde{F}(\theta)\displaystyle\int_0^t f(t)dt} \right)$$

and

$$\delta_{\lambda,g} := \min\left\{ \frac{(1-LTk^p)\theta^p - \lambda pk^p\tilde{F}(\theta)\displaystyle\int_0^t f(t)dt}{pk^pG^\theta}, \frac{\dfrac{1+LTk^p}{p}\sigma^p - \lambda\tilde{F}(\sigma)\displaystyle\int_0^t f(t)dt}{G_\sigma} \right\}.$$

In this case, in Theorem 3.2 the assumption (B_2) assumes the form

(B$_3$)

$$\max\left\{ \frac{\tilde{F}(\theta_1)}{\theta_1^p}, \frac{2\tilde{F}(\theta_2)}{\theta_2^p} \right\} < \frac{\alpha^- p}{3T^p p\alpha^+}\frac{\tilde{F}(\sigma)}{\sigma^p}$$

as well as

$$\Lambda' := \left(\frac{\dfrac{3(1+LTk^p)}{2p}\sigma^p}{\tilde{F}(\sigma)\displaystyle\int_0^t f(t)dt}, \min\left\{ \frac{\dfrac{1-LTk^p}{pk^p}\theta_1^p}{\tilde{F}(\theta_1)\displaystyle\int_0^t f(t)dt}, \frac{\dfrac{1-LTk^p}{pk^p}\theta_2^p}{2\tilde{F}(\theta_2)\displaystyle\int_0^t f(t)dt} \right\} \right)$$

$$\delta'_{\lambda,g} := \frac{1}{pk^p} \min\left\{ \frac{(1-LTk^p)\theta_1^p - \lambda pk^p \tilde{F}(\theta_1)\displaystyle\int_0^t f(t)dt}{G^{\theta_1}}, \right.$$
$$\left. \frac{(1-LTk^p)\theta_2^p - 2\lambda pk^p \tilde{F}(\theta_2)\displaystyle\int_0^t f(t)dt}{2G^{\theta_2}} \right\}.$$

We now present the following example to illustrate Theorem 3.1.

Example 3.5. We consider the following problem

$$\begin{cases} D_{-1}^{\alpha}\Phi_3({}^cD_{0+}^{\alpha}u(t)) + |u(t)|u(t) = \lambda f(u(t)) + \mu g(u(t)), & t \neq \frac{1}{2}, \quad t \in (0,1), \\ \Delta(D_{-1}^{\alpha-1}\Phi_3({}^cD_{0+}^{\alpha}u))(\frac{1}{2}) = I_1(u(\frac{1}{2})) \\ u(0) = u(1) = 0 \end{cases} \tag{3.10}$$

where $\alpha = \dfrac{5}{6}$, $I_1(\zeta) = \dfrac{\Gamma^3(\frac{5}{6})}{2}(\dfrac{3}{4})^2\sin(\zeta)$ for every $\zeta \in \mathbb{R}$ and

$$f(t) = \begin{cases} 7t^6, & \text{if } t < 1, \\ 7, & \text{if } t = 1, \\ \frac{7}{t}, & \text{if } t > 1. \end{cases}$$

By the expression of f, we have

$$F(t) = \begin{cases} t^7, & \text{if } t < 1, \\ 7t - 6, & \text{if } t = 1, \\ 1 + 7\ln(t), & \text{if } t > 1. \end{cases}$$

Direct calculations give $k = \dfrac{1}{\Gamma(\frac{5}{6})(\frac{3}{4})^{\frac{2}{3}}}$. Taking $\theta = \dfrac{1}{10}$ and $\sigma = 1$, we clearly see that all assumptions of Theorem 3.1 are satisfied. Then, for every

$$\lambda \in \left(\frac{1}{2}, \frac{3 \times 10^4 \times \Gamma^3(\frac{5}{6})}{32} \right)$$

and for every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\bar{\delta}_{\lambda,g} > 0$ such that, for each $\mu \in [0, \bar{\delta}_{\lambda,g})$, the problem (3.10) possesses at least three solutions u_1, u_2 and u_3.

We present a special case of Theorem 3.1 as follows.

Theorem 3.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(x) := \displaystyle\int_0^x f(\xi)d\xi$ for all $x \in \mathbb{R}$. Assume that*

$$\liminf_{x\to 0} \frac{F(x)}{x^p} = \limsup_{x\to+\infty} \frac{F(x)}{x^p} = 0.$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and for every continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{x\to+\infty} \frac{\displaystyle\int_0^x g(s)ds}{|x|^p} < +\infty,$$

there exists $\delta > 0$ *such that, for each* $\mu \in [0, \delta)$, *the problem*

$$\begin{cases} D^{\alpha}_{-T}\Phi_p({}^cD^{\alpha}_{0^+}u(t)) + |u(t)|^{p-2}u(t) = \lambda f(u(t)) + \mu g(u(t)), \quad t \neq t_j, \quad t \in (0,T), \\ \Delta(D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0^+}u))(t_j) = I_j(u(t_j)) \\ u(0) = u(T) = 0 \end{cases}$$

admits at least three distinct solutions.

Proof. Fix $\lambda > \lambda^* := \dfrac{\frac{1+LTk^p}{p}\sigma^p}{TF(\sigma)}$ for some $\sigma > 0$. From the condition

$$\liminf_{\xi\to 0} \frac{F(\xi)}{\xi^p} = 0,$$

there is a sequence $\{\theta_n\} \subset]0, +\infty[$ such that $\lim\limits_{n\to+\infty} \theta_n = 0$ and

$$\lim_{n\to+\infty} \frac{\sup\limits_{|\xi|\le\theta_n} F(\xi)}{\theta_n^p} = 0.$$

Indeed, one has

$$\lim_{n\to\infty} \frac{\sup\limits_{|\xi|\le\theta_n} F(\xi)}{\theta_n^p} = \lim_{n\to\infty} \frac{F(\xi_{\theta_n})}{\xi^p_{\theta_n}} \frac{\xi^p_{\theta_n}}{\theta_n^p} = 0$$

where $F(\xi_{\theta_n}) = \sup\limits_{|\xi|\le\theta_n} F(\xi)$. Hence, there exists $\overline{\theta} > 0$ such that

$$\frac{\sup\limits_{|\xi|\le\overline{\theta}} F(\xi)}{\overline{\theta}^p} < \min\left\{ \frac{1-LTk^p}{k^p\,(1+LTk^p)} \frac{F(\sigma)}{\sigma^p}, \; \frac{1-LTk^p}{\lambda Tk^p p} \right\}$$

and

$$\overline{\theta} < \sqrt[p]{2}k\sigma.$$

Applying Theorem 3.1 we have the conclusion. □

We end this paper by giving the following result a simple consequence of Theorem 3.2.

Theorem 3.7. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a non-negative continuous function such that*

$$\lim_{x\to 0^+} \frac{f(x)}{x^3} = 0$$

and

$$\int_0^{10} f(\xi)d\xi < \frac{16 \times 10^4\Gamma^4(\frac{3}{4})}{3^5} \int_0^1 f(\xi)d\xi.$$

Then, for every $\lambda \in \left(\dfrac{9}{16\displaystyle\int_0^1 f(\xi)d\xi}, \dfrac{10^4\Gamma^4(\frac{3}{4})}{27\displaystyle\int_0^{10} f(\xi)d\xi} \right)$ *and for every non-negative continuous function* $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ *satisfying the condition*

$$\limsup_{x\to+\infty} \frac{\max_{|t|\in[0,1]} g(t,x)}{x^4} < +\infty,$$

there exists $\delta > 0$ such that, for each $\mu \in [0, \delta[$, the problem

$$\begin{cases} D_{-1}^{\frac{3}{4}}\Phi_4({}^cD_{0^+}^{\frac{3}{4}}u(t)) + |u(t)|^2u(t) = \lambda f(u(t)) + \mu g(u(t)), \quad t \neq t_j, \quad t \in (0,1), \\ \Delta(D_{-1}^{-\frac{1}{4}}\Phi_4({}^cD_{0^+}^{\frac{3}{4}}u))(t_j) = I_j(u(t_j)) \\ u(0) = u(1) = 0 \end{cases}$$

admits at least three distinct non-negative solutions.

Proof. Our aim is to employ Theorem 3.2 by choosing $\alpha = \dfrac{3}{4}$, $T = 1$, $p = 4$, $I(\xi) = \dfrac{\Gamma^4(\frac{3}{4})}{2}(\dfrac{2}{3})^3\sin(\xi)$ for all $\xi \in \mathbb{R}$, $\sigma = 1$ and $\theta_2 = 10$. Therefore, since $k = \dfrac{1}{\Gamma(\frac{3}{4})(\frac{2}{3})^{\frac{3}{4}}}$, we see that

$$\frac{\dfrac{3(1+LTk^p)}{2p}\sigma^p}{\displaystyle\int_0^T F(t,\sigma)dt} = \frac{9}{16\displaystyle\int_0^1 f(\xi)d\xi}$$

and

$$\frac{\dfrac{1-LTk^p}{pk^p}\theta_2^p}{2\displaystyle\int_0^T \sup_{|\xi|\leq\theta_2} F(t,\xi)dt} = \frac{10^4\Gamma^4(\frac{3}{4})}{27\displaystyle\int_0^{10} f(\xi)d\xi}.$$

Moreover, since $\lim_{t\to 0^+}\dfrac{f(x)}{x^3} = 0$, one has

$$\lim_{x\to 0^+}\frac{\displaystyle\int_0^x f(\xi)d\xi}{x^4} = 0.$$

Then, there exists a positive constant $\theta_1 < \dfrac{1}{\Gamma(\frac{3}{4})(\frac{2}{3})^{\frac{3}{4}}}$ such that

$$\frac{\displaystyle\int_0^{\theta_1} f(\xi)d\xi}{\theta_1^4} < \frac{8\Gamma^4(\frac{3}{4})\displaystyle\int_0^1 f(\xi)d\xi}{3^4}$$

and

$$\frac{\theta_1^4}{\displaystyle\int_0^{\theta_1} f(\xi)d\xi} > \frac{2\times 10^4}{3\displaystyle\int_0^{10} f(\xi)d\xi}.$$

Finally, we easily observe that all assumptions of Theorem 3.2 are satisfied, and it follows the result. □

3.1. **References.**

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