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Research Paper

MODIFIED HOUSEHOLDER ITERATIVE SCHEME REQUIRING NO FUNCTION DERIVATIVE FOR SOLVING NONLINEAR EQUATIONS

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Abstract. The Householder iterative scheme (HIS) for determining solution of equations that are nonlinear have existed for over fifty decades and have enjoyed several modifications in literature. However, in most HIS modifications, they usually require function derivative evaluation in their implementation. Obtaining derivative of some functions is difficult and in some cases, it is not achievable.To circumvent this setback, the divided difference operator was utilised to approximate function derivatives that appear in the scheme. This resulted to the development of a new variant of the HIS with high precision and require no function derivative. The theoretical convergence of the new scheme was established using Taylor's expansion approach. From the computational results obtained when the new scheme was tested on some non-linear problems in literature, it performed better than the Householder scheme.

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Keywords: Nonlinear equation, Iterative scheme, Householder iterative scheme, Derivative free.

1. Introduction

The problem of obtaining the exact solution (θ) of nonlinear equations (NLE) ($h(x) = 0$) that are found in diverse fields of science and engineering have attracted the interest of many researchers. Because of the absence of analytic techniques for solving some NLE, the numerical approach is resorted. Some of the famous classical numerical iterative scheme (IS) for obtaining the solution (θ) of NLE are the Newthon-Raphson and Householder iterative scheme(HIS); see [1]. Some interesting techniques used in the development of diverse IS for determining θ of NLE can be found in the literature [2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14] and some references that appeared in them.

In the implementation of the HIS to solve NLE, one has to evaluate the derivative of function up to the second order. This is expensive, considering the fact that obtaining derivatives of some functions can be a daunting task. Furthermore, some functions do not have derivative or second derivative at some points. This is one major setback of the HIS.

Recently, some authors have applied diverse techniques to modify the HIS by eliminating the second derivative that appear in it; see [7, 8, 9, 10, 11, 12]. For instance, in Noor and Gupta [11], the divided difference operator was applied to approximate the second function derivative in the HIS with the quotient of difference of function derivatives at two separate points. In the work of Nadeem et al. [8], they utilised the divided difference approach

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together with polynomial approximation technique to annihilate the second derivative in the HIS. However, not so much has been done toward the modification of the HIS to a scheme that requires no function derivative. For this reason, this article put forward a modified HIS that has the advantage of high accuracy and require no need of carrying out function derivative evaluation.
The structure of this article is arranged such that Section 2 presents the development of the modified HIS, Section 3 provides its convergence analysis, Section 4 was dedicated to numerical implementation of the scheme and concluding remarks were made in Section 5.

2. The Iterative Scheme

We begin the Iterative Scheme (IS) development by acknowledging the HIS [1] presented as:

$$x_{k+1}=x_k-\frac{h\left(x_k\right)}{h'\left(x_k\right)}-\frac{h^2\left(x_k\right)h''\left(x_k\right)}{2h'^3\left(x_k\right)},\quad k=0,1,2,\cdots. \tag{2.1}$$

The sequence of approximations of the solution θ generated by the HIS (2.1) converges with order three. However, some of the setbacks of the HIS includes, the scheme requiring function derivative up to second derivative and high k number of iteration required to achieve convergence when implemented to solving NLE. To this end, we take steps to annihilate these setbacks, by putting forward a variant of the HIS. To achieve this, the HIS in (2.1) is rewritten as:

$$x_{k+1}=x_k-\frac{h\left(x_k\right)}{h'\left(x_k\right)}\left[1+\Gamma\left(x_k\right)\right]. \tag{2.2}$$

where $\Gamma\left(x_k\right)=\frac{h(x_k)h''(x_k)}{2h'^2(x_k)}$.
By estimating the function derivatives $(h'(\cdot))$ using divided difference operators $h[\cdot,\cdot]$, such that

$$h\left[x_k,\beta\right]=\frac{h(\beta)-h(x_k)}{\beta-x_k},\quad \beta=x_k+\delta h\left(x_k\right)^m,\quad \delta\in\Re-\{0\},\quad m\geq 2, \tag{2.3}$$

and the approximation of the quantity $\Gamma\left(x_k\right)$ with $\frac{1}{h(x_k)}\left[h\left(x_k-\frac{h(x_k)^2}{h[x_k,\beta]\left[h(x_k)-h\left(x_k-\frac{h(x_k)}{h[x_k,\beta]}\right)\right]}\right)\right]$, a variant form of the HIS can be obtained as:

$$x_{k+1}=x_k-\frac{h\left(x_k\right)^2}{h\left[x_k,\beta\right]\left[h(x_k)-h\left(x_k-\frac{h(x_k)}{h[x_k,\beta]}\right)\right]}-\frac{h\left(x_k\right)}{h\left[x_k,\beta\right]\left[h(x_k)-h\left(x_k-\frac{h(x_k)}{h[x_k,\beta]}\right)\right]}\Lambda(x_k) \tag{2.4}$$

where $\Lambda(x_k)=h\left(x_k-\frac{h(x_k)^2}{h[x_k,\beta]\left[h(x_k)-h\left(x_k-\frac{h(x_k)}{h[x_k,\beta]}\right)\right]}\right)$.
The iterative scheme (2.4) is a modification of the HIS and for this reason it is denoted as the MHIS. Before some key features of the MHIS are stated, the following definition is first acknowledged.

Definition 2.1. For an equation $\ell_{k+1}=\lambda\ell_k^{\phi}+O(\ell_k^{\phi+1})$ derived from an IS through the Taylor series expansion of $h(\cdot)$ and $h'(\cdot)$, such that $\ell_k=x_k-\theta$ is its kth iteration error, then ℓ_{k+1} is called the IS error equation, λ error constant and ϕ convergence order (CO). Furthermore, the IS Efficiency index (E_{eff}) is calculated as $E_{eff} = \phi^{\frac{1}{\tau}}$, where τ is number of functions requires to be evaluated in one iteration cycle of the IS.

Remark 2.2. It is paramount to note that, in one complete iteration cycle, the HIS requires three different functions evaluation including second derivative evaluation and because of this, its efficiency index EI is 1.4422. On the other hand, the MHIS requires four separate functions evaluations with no function derivative evaluation to achieve CO four with EI as 1.4142 but with very high convergence precision. The high convergence precision, often will compensates its low efficiency. This can be observed in the computational implementation of the MHIS in Section 3.

Next, the convergence criteria for the sequence of approximations produced or generated by the MHIS is considered in the next section.

3. The MHIS convergence test

This section provides the theoretical convergence of the MHIS. To effectively do this, the MATHEMATICA 9.0 software was utilised in obtaining Taylor's expansions of functions that appeared in the proof. We begin by establishing the criteria for convergence of sequence of approximations produced by the MHIS when implemented to obtain NLE solutions.

Theorem 3.1. *Let the scalar function $h : I \subset \Re \to \Re$ be differentiable sufficiently in I and assumed that it has a simple solution $\theta \in I$. Suppose x_0 an initial guess is close to θ, then the sequence of approximations obtained by the MHIS converge to θ with CO four.*

Proof. Suppose $c_k = \frac{h^{(k)}(\theta)}{k! h'(\theta)}, k \geq 2$ and set x=x_k in the Taylor series expansion of $h(x)$. Then,

$$h(x_k) = h'(\theta)\left[\ell_k + c_2\ell_k^2 + c_3\ell_k^3 + c_4\ell_k^4 + O\left(\ell_k^5\right)\right], \tag{3.1}$$

and

$$h(x_k + \delta h(x_k)^3) = h'(\theta)\left[\ell_k + c_2\ell_k^2 + (\delta + c_3)\ell_k^3 + (5\delta c_2 + c_4)\ell_k^4 + O\left(\ell_k^5\right)\right]. \tag{3.2}$$

Using (3.1) and (3.2), the next expansion is obtained.

$$h[x_k, \beta] = \left[1 + 2c_2\ell_k + 3c_2\ell_k^2 + O\left(\ell_k^5\right)\right]. \tag{3.3}$$

From (3.1) and (3.3), the Taylor's expansion of $h\left(x_k - \frac{h(x_k)}{h[x_k,\beta]}\right)$ is derived as:

$$h\left(x_k - \frac{h(\theta)}{h[x_k,\beta]}\right) = h'(x_k)\left[c_2\ell_k^2 + (-2c_2^2 + 2c_3)\ell_k^3 + (\delta c_2 + 5c_2^3 - 7c_2c_3 + 3c_4)\ell_k^4 + O\left(\ell_k^5\right)\right]. \tag{3.4}$$

Applying the expansions in (3.1), (3.3) and (3.4), the following two next expressions are derived.

$$\frac{h(x_k)^2}{h[x_k,\beta]\left[h(x_k) - h\left(x_k - \frac{h(x_k)}{h[x_k,\beta]}\right)\right]} = \ell_k - c_2^2\ell_k^2 + \left(3c_2^3 - 3c_2c_3\right)\ell_k^4 + O\left(\ell_k^5\right), \tag{3.5}$$

and

$$\frac{h\left(x_k\right)}{h\left[x_k,\beta\right]\left[h(x_k)-h\left(x_k-\frac{h(x_k)}{h[x_k,\beta]}\right)\right]}=1-c_2\ell_k-c_2\ell_k + \left(3c_2^3-3c_2c_3\right)\ell_k^4+O\left(\ell_k^5\right). \tag{3.6}$$

Using (3.5),

$$\begin{aligned}\Lambda(x_k)&=h\left(x_k-\frac{h\left(x_k\right)^2}{h\left[x_k,\beta\right]\left[h(x_k)-h\left(x_k-\frac{h(x_k)}{h[x_k,\beta]}\right)\right]}\right)\\&=c_2^2\ell_k^3+\left(-3c_2^3+3c_2c_3\right)\ell_k^4+O\left(\ell_k^5\right).\end{aligned} \tag{3.7}$$

The use of the expansions in (3.5), (3.6) and (3.7) enabled obtaining the next expression.

$$\begin{aligned}&\frac{h\left(x_k\right)^2}{h\left[x_k,\beta\right]\left[h(x_k)+h\left(x_k-\frac{h(x_k)}{h[x_k,\beta]}\right)\right]}-\frac{h\left(x_k\right)}{h\left[x_k,\beta\right]\left[h(x_k)-h\left(x_k-\frac{h(x_k)}{h[x_k,\beta]}\right)\right]}\Lambda(x_k)\\&=\ell_k-c_2^3\ell_k^4+O\left(\ell_k^5\right).\end{aligned} \tag{3.8}$$

By substituting the expansion in (3.8) into (2.3), the next equation is derived.

$$x_{k+1}=\ell_k+\theta-\left[\ell_k-c_2^3\ell_k^4+O\left(\ell_k^5\right)\right]. \tag{3.9}$$

Consequently, (3.9) is reduced to

$$x_{k+1}=\theta+c_2^3\ell_k^4+O\left(\ell_k^5\right). \tag{3.10}$$

□

From Definition 2.1, $\phi=4$. This indicates that the CO of the IS in (2.3) is four. This brings the proof to end.

4. **The MHIS implementation**

This part of the manuscript illustrates how efficient the MHIS is; when used to determine the solution of NLE. To this end, six NLE recently used in literature [4, 5, 6] were obtained and also used in testing the developed MHIS with $\delta=0.001$. These includes the following:

Example 4.1. $h(x)=-2+(x-1)^3=0, \quad \theta=2.2599\cdots$, see[5].

Example 4.2. $h_2(x)=1-x^2+sin^2(x)=0, \quad \theta=1.4044\cdots$, see[5, 12].

Example 4.3. $h_3(x)=x^3+4x^2-10=0, \quad \theta=1.3652\cdots$, see[12].

Example 4.4. $h_4(x)=-.75e^{-0.05x}+1=0, \quad \theta=-5.753\cdots$, see[6].

Example 4.5. $h_5(x)=5-5e^{-x}-x=0, \quad \theta=4.9651\cdots$, see[4].

Example 4.6. $h_6(x)=cos(x)-x=0, \quad \theta=0.7390\cdots$, see[12].

Computation programs written in Maple 2017.0 software environment for the developed MHIS and the HIS were utilized to obtain solutions θ of the test equations (Example 4.1-4.6). To achieve better approximation of θ and reduce loss of approximation digits, 3000 digits was utilized in Maple 2017.0 program execution and the stopping criterion adopted is $|h(x_k)|<10^{-1000}$. The computation output obtained from each scheme are presented in Table 4.1 for comparison. For each test equation, we stated the absolute value of $h(x_k)$ (denoted as $|h(x_k)|$) when $1 \leq k \leq 7$. The presentation $a.b_{-\alpha}$ in Table 4.1 denotes $a.b$ decimal values with exponent α, where $a, b, \alpha \in \Re$.

The numerical results obtained in Table 4.1 indicates that MHIS solve all the test equations with overwhelming precision as compared to HIS. Although, in one iteration cycle, the HIS requires the evaluation of one function less than the number of functions evaluation required in MHIS, the MHIS requires no function derivatives. This setback of the MHIS is compensated by its level of accuracy and the non requirement of function derivative. For instance, in Table 4.1 it will require more than one additional iteration by HIS to achieve near accuracy of the MHIS in all the problems solved.

Scheme	$h(x_1)$	$h(x_2)$	$h(x_3)$	$h(x_4)$	$h(x_5)$	$h(x_6)$	$h(x_7)$
				Example 4.1	$x_0 = 3.0$		
HIS	6.0_{-1}	5.9_{-3}	9.4_{-09}	3.8_{-26}	2.6_{-078}	8.1_{-0235}	2.4_{-0704}
MHIS	2.3_{-1}	1.1_{-6}	6.4_{-23}	8.3_{-92}	2.2_{-367}	1.2_{-1469}	-
				Example 4.2	$x_0 = 2.0$		
HIS	1.7_{-1}	6.9_{-4}	5.9_{-11}	3.9_{-032}	1.1_{-095}	2.5_{-0286}	3.0_{-858}
MHIS	3.7_{-2}	3.3_{-6}	2.6_{-31}	1.4_{-124}	1.1_{-497}	4.0_{-1990}	-
				Example 4.3	$x_0 = 1.0$		
HIS	6.8_{-1}	5.6_{-04}	2.7_{-13}	3.0_{-41}	4.1_{-125}	1.1_{-0378}	2.0_{-1131}
MHIS	4.5_{-2}	1.6_{-10}	2.9_{-44}	2.8_{-179}	2.6_{-719}	1.8_{-2879}	-
				Example 4.4	$x_0 = 1.0$		
HIS	2.7_{-3}	2.0_{-11}	8.2_{-36}	5.7_{-109}	1.9_{-0328}	7.1_{-987}	-
MHIS	2.9_{-4}	9.4_{-18}	9.8_{-72}	1.1_{-287}	2.2_{-1151}	–	-
				Example 4.5	$x_0 = 0.5$		
HIS	6.0_{-1}	1.4_{-2}	9.0_{-08}	2.7_{-023}	6.7_{-070}	1.1_{-0209}	4.4_{-629}
MHIS	3.3_{-2}	4.1_{-9}	1.1_{-36}	4.9_{-147}	2.2_{-588}	8.5_{-2354}	-
				Example 4.6	$x_0 = 3.1$		
HIS				Diverged			
MHIS	2.2_{-2}	4.8_{-10}	1.3_{-40}	6.1_{163}	3.2_{-652}	2.4_{-2609}	-

TABLE 1. Computational results comparison for HIS and MHIS.

Conclusion

The iterative scheme put forward in this manuscript is effective for solving NLE and have the advantages of producing more accurate approximations compared to the scheme from which it was developed and does not requires function derivative evaluation. However, the method is not optimal as conjectured by Kung and Truab in [15]. This may be considered for further study.

REFERENCES

[1] A. S. Householder, *The numerical treatment of a single nonlinear equation.* McGraw-Hill, New York, 1970.

[2] A. Cordero; J. L. Hueso; E. Martinez and J. R. Torregrosa, Steffensen type methods for solving nonlinear equations. *Journal of Computational ana Applied Mathematics*, **236**:3058-3064, 2012.

[3] F. Soleymani, Optimal fourth-order iterative method free from derivative. *Miskole Mathematical Notes*, **12**(2):255-264, 2011.

[4] O. Ogbereyivwe and V. Ojo-Orobosa, Family of optimal two-step fourth order iterative method and its extension for solving nonlinear equations. *Journal of Interdisciplinary Mathematics*, **24**, 2021, 1347-1365.

[5] O. Ogbereyivwe and O. Izevbizua, A three-parameter class of power series based iterative method for approximation of nonlinear equations solution. *Iran. J. of Numer. Anal. and Optim.*, **13**(2), 2023, 157-169.

[6] O. Ogbereyivwe and S. A. Ogumeyo , Extension of the Double Newton's Method Convergence Order via the Bi-variate Power Series Weight Function for Solving Nonlinear Models. *Mathematical Analysis and its Contemporary Applications* , **4**(3), 2022, 35-47.

[7] M. Tanveer, M. Ahamd, A. Ali, W. Nazeer, K. Rehman, Modified Householder's method (MHHM) for solving nonlinear functions with convergence of order six. *Sci. Int.* , **28**, 2016, 83-87.

[8] Nadeem G. A., Aslam W. and F. Ali, An optimal fourth-order second derivative free iterative method for nonlinear scientific equations, *Kuwait J. Sci.*, **50**(2A), 2023,1-15.

[9] W. A. Khan, M. A. Noor, A. Kang, A. Rauf , Higher-order iterative methods by using Householder's method for solving nonlinear equation . *Math. Sci. Lett.* , **2**(2), 2013, 107-120.

[10] K. I. Noor, M. A. Noor, S. Mommani, Modified Householder iterative method for solving nonlinear equation . *Appl. Math. Comput.* , **190**, 2007, 1534-1539.

[11] M. A. Noor and V. Gupta, Modified Householder iterative method free from second derivatives for nonlinear equations, *Appl. Math, and Comput.* , 190(2007), 1701-1706.

[12] M. A. Hafiz, S. M. H. Al-Goria, Solving nonlinear equations using a new tenth-and seventh-order methods free from second derivatives. *Int. J. of Diff. Eq. and Appl.* , **12**(4), 2013, 169-183.

[13] F. Mirzaee and A. Hamzeh, Several-step to obtain derivative-free iterative methods for solving nonlinear equations. *Advances in Mathematical Modeling* , **2** (2), 21-31

[14] F. Mirzaee and A. Hamzeh, A sixth order method for solving nonlinear equations. *International Journal of Mathematical Modelling and Computation*, **4** (1), 2014, 55-60.

[15] H. Kung and J. F. Traub, Optimal order of one-point and multi-point iteration, *Applied Mathematics and Computation*, **21**(1974):643-651.

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