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Research Paper

LEGENDRE SPECTRAL ELEMENT AND BACKWARD EULER METHODS FOR SOLVING A FAMILY OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we use the spectral element method for solving the stochastic partial differential equation. For spatial discretization, we use the Legendre spectral element method, and we obtain the semi-discrete form. To solve the problem, we need to obtain the complete discrete form and we use the backward Euler method to this aim. The Weiner process is approximated by Fourier series and we obtain the fully discrete scheme of the problem. Error and convergence analysis are presented and, with a numerical example, we demonstrate the efficiency of the proposed method.

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Keywords: Stochastic partial differential equation, spectral element method, Legendre polynomials.

1. Introduction

1.1. **problem definition.** We consider the stochastic partial differential equation (SPDE) below

$$\begin{aligned} &dy = Aydt + \ell\,(y)\,dW\,(t) \qquad in\ \Omega, \quad 0 < t < T, \\ &y\,(x,0) = y_0 \in \mathbb{H}, \qquad\qquad x \in \Omega, \\ &y\,(x,t) = g, \qquad\qquad\qquad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}$. $\mathbb{H}$ is a separable Hilbert space of function defined on Ω. W is a Wiener process defined on the filtered probability space $\left(\aleph_W, \mathcal{F}_W, \{\mathcal{F}_t\}_{t=0}^{\infty}, \mathbf{P}_W\right)$ with the mean of zero and the following covariance function

$$\mathbb{E}\left(W\,(r,x)\,W\,(k,y)\right) = \min\{r,k\}\,Q\,(x,y)\,, \qquad x,y \in \Omega, \quad r,k > 0.$$

A is a linear, self-adjoint, positive definite and not necessarily bounded operator with a compact inverse and g is a known function.
SPDEs are widely studied today by researchers. These equations are used in physics, engineering, and economics. The existence, uniqueness and stability of the solution of SPDEs have been well studied [1, 2, 3, 4]. Many numerical methods have been proposed for the approximation of the solution of SPDEs. Cao and Yin [5] have used the spectral Galerkin method to approximate the solution of stochastic wave equations. They have also used a

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spectral method to solve nonlinear elliptic SPDEs in [6]. Liu in [7] formulated the solution of the stochastic equation Ginzburg-Landau based on the spectral method and examined the convergence in the spaces of Sobolev. Dehghan and Shirzadi in [8, 9] used a meshless simulation method based on radial basis functions for solving the stochastic advection-diffusion equations and elliptic SPDEs. Collocation methods are one of the most important methods for numerical solution of SPDEs [10, 11]. Finite element method is another numerical method for solving SPDEs that has been studied a lot. In [12] the authors have used finite element and finite difference methods for solving elliptic and parabolic SPDEs. Error analysis and convergence are presented for both methods and it is shown that the numerical results of the two methods are similar, but the finite element method has more accurate and efficient than finite difference methods. Babuska et al. in [13] investigated the finite element method and Monte Carlo method for solving SPDEs, and calculated the a-priori error. Barth and Lang in [14] used finite element method to simulate SPDEs and have shown that the implementation of this simulation is convergent. Convergence and error analysis of finite element method have been well studied in [15, 16]. Som of recent article consider FEM and spectral Galerkin method for SPDEs with non-globally Lipschitz coefficient, for example: [17, 18, 19, 20].

1.2. **A summary of the spectral element method.** A spectral element method (SEM) combine the advantages and disadvantages of Galerkin spectral methods with those of finite element methods by a simple application of the spectral method per element. One of the advantages of this method is the high accuracy and stable solving algorithm with a small number of elements under a wide range of conditions [21].

FEM was proposed for the first time in 1943 by Richard Courant [22]. He solved the Poisson equation based on minimizing piecewise linear approximations on finite subdomains.

Spectral method is a conventional method for solving partial differential equations, which was first introduced by the Navier for elastic sheet problems in 1825.

In 1984, Patera with the division of domains, applied a spectral method to a greater number of subdomains. He proposed the SEM by combining the spectral method and the FEM [23]. Patera in his innovative method, used the Chebyshev polynomials as the interpolation basis function. Legendre's spectral element method (LSEM) was developed by Maday and Patera [24]. The use of the Lagrangian interpolation conjugate with the Gauss-Legendre-Lobatto quadrature leads to a matrix of mass with a diameter structure [25]. The diagonal mass matrix is a very important property of the LSEM and is different from the Chebyshev spectral element method [26]. Chen et al., in [27], used the Legendre spectral element method to solve the constrained optimal control problem. An alternating direction implicit (ADI) Legendre spectral element method for the two-dimensional Schrodinger equation is developed in [28], and the optimal H^1 error estimate for the linear case is given. The aim of [29] is the Lagrange-Galerkin spectral element method for solving the two-dimensional shallow water equations. Authors of [30] have considered the numerical approximation of the acoustic wave equation by the spectral element method based on Gauss-Lobatto-Legendre quadrature formulas, and finite difference Newmark's explicit time advancing schemes. A modified set of basis functions for use with spectral element methods is presented in [31] for solving a mixed elliptic boundary value problem. These basis functions are constructed so that the axial conditions along a plane or axis of symmetry are satisfied identically. A numerical spectral element method for the computation of fluid flows governed by the incompressible Euler equations in a complex geometry is presented in [32]. Zhuang and Chen used this method to solve the biharmonic equations [33]. In [34], authors used the spectral element method with least-square formulation for parabolic interface problems. Ai et al., used fully diagonalized Legendre spectral element methods using Sobolev orthogonal/biorthogonal basis functions for solving second order elliptic boundary value problems [35]. A Legendre spectral element formulation of an improved time-splitting method is developed for the natural convection heat transfer problem in a square cavity by Wang and Qin [36]. Lotfi and Alipanah in [37], study the Legendre spectral element method for solving the sine-Gordon equation in one dimension. The stability and convergence analysis of the method is also done.

1.3. **The main aim of this article.** The main contribution of this article is to introduce an efficient numerical method for SPDEs. The proposed method is based on a backward Euler method in temporal direction and the Legendre spectral element method in spatial direction for obtaining a semi-discrete and full discrete form, respectively. The rest of article is organized as follows. In section 2, the Legendre polynomials and the associated SEM are given. In section 3, we first obtain the semi-discrete form of the Eq. (1.1) using the backward Euler method. Winer process approximated by the Fourier series and eventually the full discrete form of the problem is obtained using the LSEM and its matrices form is calculated. In Section 4, we present stability and convergence theorems and, in Section 5, we show the efficiency of the method by solving a numerical example.

2. Description of the method

2.1. **Legendre polynomials.** The Nth-degree Legendre polynomial $\mathcal{L}_N(\theta)$, is a solution of the second-order differential equation

$$\left(\left(\theta^2-1\right)\mathcal{L}'_N(\theta)\right)' - N(N+1)\mathcal{L}_N(\theta) = 0$$

In the normalized form of $\mathcal{L}_N(\theta)$ we have $\mathcal{L}_N(1)=1$, which can be calculated as follows

$$\mathcal{L}_N(\theta) = 2^{-N}\sum_{i=0}^{\left[\frac{N}{2}\right]}(-1)^i\begin{pmatrix} N \\ i \end{pmatrix}\begin{pmatrix} 2N-2i \\ N \end{pmatrix}\theta^{N-2i}$$

where $[x]$ denotes the integer part of x. For each pair of Legendre polynomial of degrees N and M, the following orthogonality property applies

$$\int_{-1}^{1}\mathcal{L}_N(\theta)\mathcal{L}_M(\theta)\,d\theta = \frac{2}{2N+1}\delta_{NM},$$

where δ_{NM} is Kronecker's delta. The Nth-degree Lobatto polynomial, $\mathcal{LO}_N$, derives from the $(N+1)$-degree Legendre polynomial, $\mathcal{L}_{N+1}$, as

$$\mathcal{LO}_N(\theta) = \mathcal{L}'_{N+1}(\theta).$$

Legendre and Lobatto polynomials can be calculated using the recursive relations [38]

$$\mathcal{L}_{N+1}(\theta) = \tfrac{2N+1}{N+1}\theta\mathcal{L}_N(\theta) - \tfrac{N}{N+1}\mathcal{L}_{N-1}(\theta),$$

$$\mathcal{LO}_{N-1}(\theta) = \tfrac{N(N+1)}{2N+1}\tfrac{(\mathcal{L}_{N-1}(\theta)-\mathcal{L}_{N+1}(\theta))}{1-\theta^2}.$$

2.2. **Legendre spectral element method.** In the Legendre spectral element method, we first divide the domain Ω into N_e non-overlapping subdomains Ω_e,

$$\bar{\Omega} = \bigcup_{e=1}^{N_e} \bar{\Omega}_e, \quad \bigcap_{e=1}^{N_e} \Omega_e = \phi.$$

Basis functions are considered as the Lagrangian interpolation polynomials defined at Gauss-Lobatto integration points on each element. If $N_e = 1$ we obtain a spectral Galerkin method of order $N-1$. If $N = 1$ or $N = 2$ a standard Galerkin FEM is obtained based on linear and quadratic elements respectively. Convergence is either obtained by increasing the degree of the polynomials or by increasing the number of elements N_e.
Now on each element Ω_e we define the approximate solution of order N as

$$y^e(x,t) = \sum_{j=0}^{N} y_j^e(t)\,\varphi_j(x), \quad 1 \le e \le N_e, \tag{2.1}$$

where φ_j is the j^{th} Lagrange polynomial of order N on the Gauss-Legendre-Lobatto points $\{\theta_i\}_{i=0}^{N}$ [39]

$$\varphi_j(\theta) = \frac{1}{N(N+1)\,\mathcal{L}_N(\theta_j)} \frac{\left(\theta^2-1\right)\mathcal{LO}_{N-1}(\theta)}{\theta-\theta_j}, \;\; 0 \le j \le N, \;\; -1 \le \theta \le 1.$$

To convert the $[-1,1]$ to eth element and its inverse, we use the following mapping functions

$$x(\theta) = \frac{(x_e - x_{e-1})\,\theta}{2} + \frac{x_e + x_{e-1}}{2}, \qquad -1 \le \theta \le 1,$$
$$\theta(x) = \frac{2x - (x_e + x_{e-1})}{x_e - x_{e-1}}, \qquad x_{e-1} \le x \le x_e,$$

where x_e and x_{e-1} are the endpoints of e^{th} element. The stiffness [40] and mass matrices [41] on each element are calculated as follows

$$S_{ij}^e = \int_{x_{e-1}}^{x_e} \varphi'_i(x)\,\varphi'_j(x)\,dx = \frac{2}{h_e}\int_{-1}^{1} \varphi'_i(\theta)\,\varphi'_j(\xi)\,d\theta,$$
$$M_{ij}^e = \int_{x_{e-1}}^{x_e} \varphi_i(x)\,\varphi_j(x)\,dx = \frac{h_e}{2}\int_{-1}^{1} \varphi_i(\theta)\,\varphi_j(\theta)\,d\theta,$$

where

$$h_e = x_e - x_{e-1}.$$

By using the Gauss quadrature we obtain [38]

$$S_{ij}^e = \frac{2}{h_e}\sum_{k=0}^{N} d_{ik} d_{jk} w_k,$$

$$M_{ij}^e = \frac{h_e}{2}\delta_{ij} w_i,$$

where

$$w_k = \frac{2}{N(N+1)\left[\mathcal{L}_N(t_k)\right]^2}, \quad 0 \le k \le N,$$

and

$$d_{ik} = \frac{\mathcal{L}_N(\theta_k)}{\mathcal{L}_N(\theta_i)} \frac{1}{\theta_k - \theta_i}, \quad i \ne k,$$

$$d_{ii}=\frac{\mathcal{LO}_{N-1}\left(\theta_i\right)}{2\mathcal{L}_N\left(\theta_i\right)}.$$

3. DISCRETIZATION

3.1. **Temporal Discretization.** For the discretization of the time variable in Eq (1.1), we sue the backward Euler

$$0\leq t_0\leq t_1\leq\ldots\leq t_n=T,$$

so we arrive at

$$Y^n-Y^{n-1}=\varepsilon AY^n+\ell\left(Y^{n-1}\right)\delta W^n, \tag{3.1}$$

where

$$Y^n=Y\left(x,t_n\right),\qquad \varepsilon=t_n-t_{n-1},\qquad \delta W^n=W_{t_n}-W_{t_{n-1}},$$

3.2. **Approximation of the Weiner process.** Suppose $W\left(t\right)$ is a Wiener process with covariance Q. This process can be approximated by the Fourier series. Suppose that Q is linear, bounded, self–adjoint, positive definite on $\mathbb{H}$ with eigenvalues $\lambda_i>0$ and eigenfunctions ρ_i. η_i, $i=1,2,\ldots$ is a sequence of real-valued independently and identically distributed Brownian motion, then [43]

$$W\left(t\right)=\sum_{i=1}^{\infty}\sqrt{\lambda_i}\rho_i\eta_i\left(t\right). \tag{3.2}$$

3.3. **Spatial Discretization.** First suppose that $\mathcal{H}^r\left(\Omega\right)$ and $\mathcal{H}_0^r\left(\Omega\right)$ are Sobolev spaces with the norm $\|\,.\,\|_r$ and semi-norm $|\,.\,|_r$. We define the spatial projection operator as follows

$$\Lambda_h:\mathcal{H}_0^1\left(\Omega\right)\to\mathcal{X}_0^N,$$

in which

$$\left(\nabla\left(y-\Lambda_h y,\nabla\upsilon\right)\right)=0,\qquad y\in\mathcal{H}_0^1\left(\Omega\right),\qquad\forall\upsilon\in\mathcal{X}_0^N,$$

where $\mathcal{X}_0^N$ is the spectral element approximation space

$$\mathcal{X}_0^N=\left\{\upsilon\in\mathcal{H}_0^1\left(\Omega\right):\ \upsilon\,|_{\Omega_e}\in P_N\left(\Omega\right),\quad e=1,2,\ldots,N_e\right\},$$

where P_N is a polynomial space of less than or equal to N. With these definitions, the discrete form of (3.1) will be as follows

$$\begin{aligned}&Y^n+\varepsilon\Lambda_h Y^n=Y^{n-1}+\int_{t_{n-1}}^{t_n}\Lambda_h\ell\left(Y^{n-1}\right)dW\left(s\right)\\&Y^0=\Lambda_h y_0.\end{aligned} \tag{3.3}$$

The LSEM approximation of Eq. (3.3) is obtained as follows.
For each element Ω_e, find $Y_e^n\in\mathcal{X}_0^N$, such that

$$\left(Y_e^n,\upsilon\right)+\left(\varepsilon\Lambda_h Y_e^n,\upsilon\right)=\left(Y_e^{n-1},\upsilon\right)+\left(\int_{t_{n-1}}^{t_n}\Lambda_h\ell\left(Y_e^{n-1}\right)dW\left(s\right),\upsilon\right). \tag{3.4}$$

Now, if we consider the test function υ to be the k^{th} Lagrange's function of order N and let

$$Y_e^n\left(t\right)=\sum_{j=0}^{N}\alpha_j^e\left(t_n\right)\varphi_j\left(x\right),\qquad 1\leq e\leq N_e,$$

and

$$Ay = \Delta y + f,$$

Eq. (3.4) becomes as follows

$$\begin{aligned}
&\sum_{j=0}^{N} \alpha_j^e (t_n) \left(\int_{\Omega_e} \varphi_j \varphi_k dx + \varepsilon \int_{\Omega_e} {\varphi'}_j {\varphi'}_k dx \right) \\
&\quad = \sum_{j=0}^{N} \alpha_j^e (t_{n-1}) \left(\int_{\Omega_e} \varphi_j \varphi_k dx \right) + \varepsilon \int_{\Omega_e} f(x, t_n) \varphi_k dx \\
&\quad\quad + \int_{\Omega_e} \left(\int_{t_{n-1}}^{t_n} \Lambda_h \ell \left(Y_e^{n-1} \right) dW(t_n) \right) \varphi_k dx.
\end{aligned} \tag{3.5}$$

The second integral on the left hand side, is obtained by integration by parts. The matrix form of the Eq. (3.5) will be as follows

$$M_e Y_e^n + \varepsilon S_e Y_e^n = M_e Y_e^{n-1} + \varepsilon F_e^n + \mathrm{B}_e^n. \tag{3.6}$$

The vector Y_e^n contains the approximation solution of the order N on the element Ω_e at the time t_n, M_e is a local diagonal mass matrix and S_e is a local stiffness matrix on the element Ω_e. F_e and B_e are local known vectors. In order to obtain an approximate solution on the general domain, we must assemble the local matrices M_e, S_e ,F_e and B_e and obtain the general matrices M, S, F and B [38]. The algorithm for assembling local matrices is shown in Table 1. So the Eq. (3.6) will be as follows

$$(M + \varepsilon S) Y^n = M Y^{n-1} + \varepsilon F^n + \mathrm{B}^n, \tag{3.7}$$

In which Y^n is the vector of the approximate solution on the general domain Ω at the time t_n.

TABLE 1. Algorithm for assembling local matrices

Algorithm for assembling local matrices S^e, M^e , B^e and F^e
For $e = 1, ..., N_e$ Run over the elements
Compute local matrices S^e, M^e , B^e and F^e
For $i = 1, ..., N_e$ Run over the spectral element nodes
$i_1 = c(e, i)$, c is connectivity matrix
For $j = 1, ..., N_e$ Run over the spectral element nodes
$i_2 = c(e, j)$
$M_{i_1,i_2} = M_{i_1,i_2} + M^e_{i_1,i_2}$
$S_{i_1,i_2} = S_{i_1,i_2} + S^e_{i_1,i_2}$
$F_{i_1} = F_{i_1} + F^e_{i_1,}$
$B_{i_1} = B_{i_1} + B^e_{i_1}$
End For
End For
End For

4. Stability and convergence analysis

4.1. Stability of (3.7). Equation (3.7) can be written as follows

$$\left(I+\varepsilon M^{-1}S\right)Y^{n}=Y^{n-1}+M^{-1}\left(\varepsilon F^{n}+\mathrm{B}^{n}\right).$$

Since $M^{-1}\left(\varepsilon F^{n}+\mathrm{B}^{n}\right)$ is known vector at each step and does not play a role in the stability analysis, we need to consider the following equation

$$Y^{n}=\left(I+\varepsilon M^{-1}S\right)^{-1}Y^{n-1} \tag{4.1}$$

Theorem 4.1. *Eq.* (4.1) *always stable.*

Proof. We must show

$$\left\|\left(I+\varepsilon M^{-1}S\right)^{-1}\right\|_{2}^{2}=\rho\left(\left(I+\varepsilon M^{-1}S\right)^{-1}\right)\leq 1.$$

If μ_i is the eigenvalues of the diagonal matrix M and λ_i is the eigenvalues of the matrix S, then $\frac{\lambda_i}{\mu_i}$ is the eigenvalues of the matrix $M^{-1}S$.
We must show

$$\left|\frac{\mu_i}{\mu_i+\varepsilon\lambda_i}\right|\leq 1.$$

From above equation we obtain

$$\mu_i\leq\mu_i+\varepsilon\lambda, \tag{4.2}$$

or

$$\varepsilon\lambda_i\geq 0.$$

According to [42], we know that

$$CN^{-1}h\leq\lambda_i,$$

where C is a positive constant. Also we know $\varepsilon>0$, then Eq (4.2) is always true. □

4.2. Convergence of spectral element method.

Lemma 4.2. *Let $y\in\mathbb{H}^r$, $(r\geq 1)$, then* [44]

$$\|y-\Lambda_h y\|\leq Ch^{\min\{N+1,r\}}N^{1-r}\|y\|_r.$$

Definition 4.3. The mild solution of SPDE (1.1) is defined as follows [45]

$$y\left(t_n\right)=\mathsf{E}\left(t_n\right)y_0+\int_0^{t_n}\mathsf{E}\left(t_n-s\right)\ell\left(y\left(s\right)\right)dW\left(s\right).$$

where $\mathsf{E}\left(t_n\right)=\exp\left(t_nA\right)$ is the analytic semigroup generated by A.

Lemma 4.4. *[necessary inequalities]*
If $\mathsf{K}_{\varepsilon h}=1+\varepsilon\mathrm{A}_h$, The following inequality is classical and one can easily prove it by using the spectral decomposition of A (positive constant C is independent of n) [1]

$$\left\|\mathsf{K}_{\varepsilon h}^{-n}-\exp\left(t_nA_h\right)\right\|\leq\frac{C}{n},\quad n\geq 1. \tag{4.3}$$

For the analytic semigroup $\mathsf{E}\left(t\right)$ the following properties hold true [43]

$$\int_0^t s^u\left|D_t^k\mathsf{E}\left(s\right)\right|_v^2\vartheta ds\leq C\left|\vartheta\right|_{2k+v-u-1}^2,\quad u,v\in\mathbb{R},\quad t,u,k\geq 0, \tag{4.4}$$

$$\left\|A^{-v}\left(I-\mathsf{E}\left(t\right)\right)\right\| \leq Ct^{v}, \qquad 0\leq v\leq 1, \quad t\geq 0. \tag{4.5}$$

Lemma 4.5. *If* $\Theta_n = \mathsf{K}_{\varepsilon h}^{-n}\Lambda_h - \mathsf{E}\left(t_n\right)$, *we have* [43]

$$\left(\varepsilon\sum_{j=1}^{n}\left\|\Theta_n f\right\|^2\right)^{\frac{1}{2}} \leq C\left(\varepsilon^{\frac{\beta}{2}}+h^{\beta}\right)|f|_{\beta-1}, \quad 0\leq\beta\leq 1$$

According to the relation (3.3)

$$Y^n = \mathsf{K}_{\varepsilon h}^{-1}\left(Y^{n-1}+\int\limits_{t_{n-1}}^{t_n}\Lambda_h\ell\left(Y^{n-1}\right)dW\left(s\right)\right),$$

Then

$$Y^n - y\left(t_n\right) = \mathsf{K}_{\varepsilon h}^{-n}\Lambda_h y_0 - \mathsf{E}\left(t_n\right)y_0$$

$$+\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left(\mathsf{K}_{\varepsilon h}^{-n-j}\Lambda_h\ell\left(Y^{j-1}\right)-\mathsf{E}\left(t_j-s\right)\ell\left(y\left(s\right)\right)\right)dW\left(s\right)$$

Defining $\Theta_n = \mathsf{K}_{\varepsilon h}^{-n}\Lambda_h - \mathsf{E}\left(t_n\right)$, so we have (For simplicity, we assume that $\ell\left(.\right)=I$)

$$Y^n - y\left(t_n\right) = \Theta_n y_0$$

$$+\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left(\mathsf{K}_{\varepsilon h}^{-n-j}\Lambda_h - \mathsf{E}\left(t_n-t_j\right)+\mathsf{E}\left(t_n-t_j\right)-\mathsf{E}\left(t_j-s\right)\right)dW\left(s\right)$$

$$=\Theta_n y_0 + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathsf{K}_{\varepsilon h}^{-n-j}\Lambda_h - \mathsf{E}\left(t_n-t_j\right)dW\left(s\right)$$

$$+\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left[\mathsf{E}\left(t_n-t_j\right)-\mathsf{E}\left(t_j-s\right)\right]dW\left(s\right),$$

Thus

$$Y^n - y\left(t_n\right) \leq \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.$$

By Eq. (4.3) of lemma 4.4, I_1 can be easily estimated by

$$\begin{aligned}\left\|\mathrm{I}_1\right\| &= \left\|\left(\mathsf{E}_{\varepsilon h}^{-n}-\exp\left(t_nA_h\right)+\exp\left(t_nA_h\right)-\exp\left(t_nA\right)\right)y_0\right\| \\ &\leq \left\|\Theta_n y_0\right\| + \left\|\left(\exp\left(t_nA_h\right)-\exp\left(t_nA\right)\right)y_0\right\| \qquad (4.6) \\ &\leq C\tfrac{\|y_0\|}{n} + \left\|\left(\exp\left(t_nA_h\right)-\exp\left(t_nA\right)\right)y_0\right\|\end{aligned}$$

Using lemma 4.2,

$$\left\|\mathrm{I}_1\right\| \leq C\left(\frac{\|y_0\|}{n}+h^{\min\{N+1,r\}}N^{1-r}\|y_0\|_r\right).$$

For I_2, by the isometry property $\mathbb{E}\left\|\int_0^t\psi\left(s\right)dW\left(s\right)\right\|^2 = \int_0^t\left\|\mathbb{E}\psi\left(s\right)\right\|^2ds$, we have

$$\left\|\mathrm{I}_2\right\|^2 = \left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\Lambda_h\Theta_{n-j}dW\left(s\right)\right\|^2 = \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\|\Lambda_h\Theta_{n-j}\right\|^2ds.$$

Using an orthonormal basis $\{\nu_k\}_{k=1}^{\infty}$ in $\mathbb{H}$, we obtain

$$\mathrm{I}_2^2=\sum_{k=1}^{\infty}\left(\varepsilon\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\|\Theta_{n-j}\sqrt{Q}\nu_k\right\|^2 ds\right).$$

Setting $f=\sqrt{Q}\nu_k$ in lemma 4.5, we have

$$\|\mathrm{I}_2\|^2\le C\sum_{k=1}^{\infty}\left(\varepsilon^{\beta}+h^{2\beta}\right)\left|\left(\sqrt{Q}\nu_k\right)\right|_{\beta-1}^2$$

$$=C\sum_{k=1}^{\infty}\left(\varepsilon^{\beta}+h^{2\beta}\right)\left\|\sqrt{A^{\beta-1}Q}\nu_k\right\|^2=C\left(\varepsilon^{\beta}+h^{2\beta}\right)\left\|\sqrt{A^{\beta-1}}\right\|^2.$$

For I_3, we have,

$$\|\mathrm{I}_3\|^2=\left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left(\mathsf{E}\left(t_n-t_j\right)-\mathsf{E}\left(t_n-s\right)\right)dW\left(s\right)\right\|^2$$

$$=\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left\|\left(\mathsf{E}\left(t_n-t_j\right)-\mathsf{E}\left(t_n-s\right)\right)\right\|^2 ds$$

$$=\sum_{k=1}^{\infty}\left(\sum_{j=1}^{n}\left\|\int_{t_{j-1}}^{t_j}\sqrt{A^{-\beta}}\left(\mathsf{E}\left(s-t_j\right)-\mathrm{I}\right)\sqrt{A^{\beta}}\mathsf{E}\left(t_n-s\right)\sqrt{Q}\nu_k\right\|^2 ds\right).$$

Using (4.4) and (4.5) of lemma 4.4 with $\vartheta=\sqrt{A^{\beta-1}Q}\,\nu_k$, we obtain

$$\|\mathrm{I}_3\|^2\le C\varepsilon^{\beta}\sum_{k=1}^{\infty}\int_0^{t_n}\left\|\sqrt{A}\mathsf{E}\left(t_n-s\right)\sqrt{A^{\beta-1}Q}\nu_k\right\|^2 ds$$

$$\le C\varepsilon^{\beta}\sum_{k=1}^{\infty}\left\|\sqrt{A^{\beta-1}Q}\nu_k\right\|^2=C\varepsilon^{\beta}\left\|\sqrt{A^{\beta-1}}\right\|^2.$$

Corollary 4.6. *With before assumptions and if $y\left(t_n\right)$ be the mild solution and Y^n be the solution of the (3.3), we have the following error estimate*

$$\left\|Y^n-y\left(t_n\right)\right\|\le C\left(\frac{\|y_0\|}{n}+h^{\min\{N+1,r\}}N^{1-r}\|y_0\|_r+\left(2\varepsilon^{\beta}+h^{2\beta}\right)\left\|\sqrt{A^{\beta-1}}\right\|^2\right).$$

5. Numerical results

In this section, we solve the example of SPDE with the numerical method presented in the previous sections.

Example 5.1. We consider the following one-dimensional SPDE [12, 46, 43]

$$\frac{\partial y}{\partial t}\left(t,x\right)-\frac{\partial^2 y}{\partial x^2}\left(t,x\right)+by\left(t,x\right)=\frac{\partial^2 W}{\partial t\partial x}\left(t,x\right)+f\left(t,x\right),\quad t>0,$$

with Dirichlet boundary conditions and the initial values

$$y\left(t,0\right)=y\left(t,1\right)=0,\qquad t\ge 0,$$

$$y\left(0,x\right)=10x^2(1-x)^2,\quad 0\le x\le 1.$$

TABLE 2. Discrete error-norms with different values of N, N_e and ε in time $t = 1$.

		$\varepsilon = 0.1$		$\varepsilon = 0.01$	
N	N_e	L^{∞}_{error}	RMS_{error}	L^{∞}_{error}	RMS_{error}
1	40	4.4093e-02	2.6930e-02	1.0073e-02	9.4629e-03
2	30	1.2518e-02	8.9776e-03	3.0531e-03	1.3130e-03
3	30	5.2023e-03	1.9553e-04	8.6104e-04	3.4071e-04
4	20	3.9778e-04	1.9421e-04	4.9109e-05	2.4771e-05
5	20	3.5862e-04	1.3858e-04	3.5507e-05	1.0154e-05
6	20	1.5050e-04	9.3304e-05	2.8967e-05	1.0065e-05
7	20	3.7733e-05	1.2734e-05	6.4761e-06	4.5359e-06

$\frac{\partial^2 W}{\partial t \partial x}(t,x)$ denotes the mixed second order derivative of the Brownian sheet, $b = 0.5$, and

$$f(t,x) = 10(1+b)x^2(1-x)^2\exp(t) - 10\left(2 - 12x + 12x^2\right)\exp(t).$$

In the absence of noise, the exact solution would be $y(t,x) = 10\exp(t)(1-x)^2$. We use this exact solution to check the expected error

$$\mathbf{e}_i^n = \left|\mathbb{E}\left(Y\left(t_n, x_i\right)\right) - y\left(t_n, x_i\right)\right|,$$

where $y(t_n, x_i)$ and $\mathbb{E}(Y(t_n, x_i))$ are, the exact and expectation of numerical solutions at point x_i and time t_n, respectively. We approximate the $\mathbb{E}(Y(t_n, x_i))$ by sample mean from $s := 500$ simulated sample paths, i.e.,

$$\mathbb{E}\left(Y\left(t_n, x_i\right)\right) \approx \frac{1}{s}\sum_{k=1}^{s} Y\left(t_n, x_i, w_k\right).$$

The following discrete error-norms are defined

$$L^{\infty}_{error}(t_n) = \max_{i=1,\ldots,N_g} \left|\mathbf{e}_i^n\right|,$$

$$RMS_{error}(t_n) = \left(\frac{1}{N_g}\sum_{i=1}^{N_g}\left|\mathbf{e}_i^n\right|^2\right)^{\frac{1}{2}},$$

Where N_g is the global number of interpolation points and $N_g = N_e \times N + 1$. We approximate the stochastic integral $\int\limits_{t_{n-1}}^{t_n} \Lambda_h \ell\left(Y_e^{n-1}\right) dW(t_n)$ by (For simplicity, we assume that $\ell\left(Y_e^{n-1}\right) = I$)

$$\int\limits_{t_{n-1}}^{t_n} \Lambda_h dW \approx \sum_{i=1}^{N_g} \sqrt{\lambda_i}\eta_i\left(\beta_i(t) - \beta_{i-1}(t)\right)$$

where $\eta_i = \sqrt{2}\sin(i\pi x)$, $\lambda_i = 1$ and $\beta_i(t)$ is a independent Brownian motions . The estimate errors for different values of N, N_e and ε in time $t = 1$ are shown in Table 2. Also, Figure 1 shows the approximate mean solution for $t = 1, 2, 3, 4$ with $N = 4$, $N_e = 20$ and $\varepsilon = 0.1$. In the following figures we have used a logarithmic scale for the both axes. In figure 2 we show the RMS_{error} as a function of the degree of the polynomials N, with $N_e = 30$. Figure 3 where we report the quantity RMS_{error} for N_e ranging from 1 to 10, fixed $N = 4$.

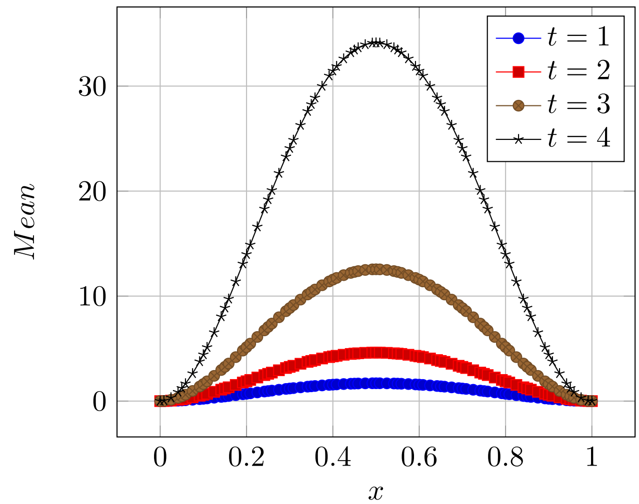

FIGURE 1. Approximate mean solution for $t = 1, 2, 3, 4$ with $N = 4$, $N_e = 20$ and $\varepsilon = 0.1$.

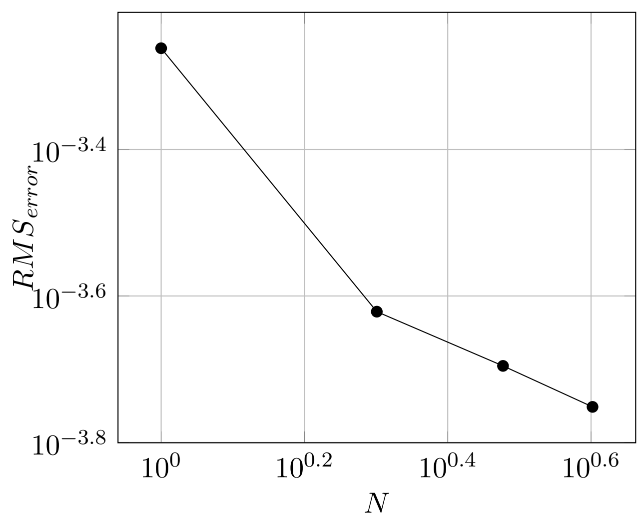

FIGURE 2. The RMS^{error} as a function of N: $N_e = 30$.

6. CONCLUSION

The spectral methods are useful tools for solving ordinary and partial differential equations. Also, the incorporation of the finite element method with the spectral polynomials i.e. the use of the spectral polynomials as a new shape function in the finite element method is very efficient for obtaining a numerical algorithm with high accuracy. In this article, we constructed a Legendre spectral element method for the solution of the SPDE. We used the Legendre spectral element method for discretizing the spatial space. Also we used a backward Euler scheme for discretizing the temporal space. We presented theorems on the stability and convergence. Finally, using the test problem, we demonstrated that the algorithm is efficient for obtaining approximation solutions of SPDEs.

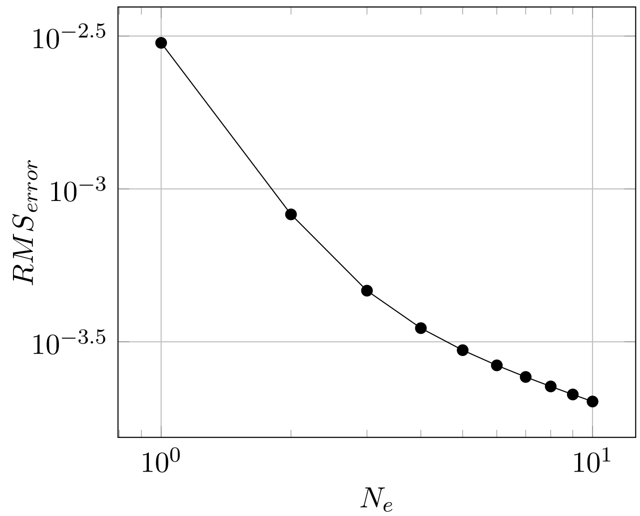

Figure 3. The RMS^{error} as a function of N_e: $N = 4$.

Availability of data and material

The results and numerical data obtained in this paper have been fully tested. These results are obtained using MATLAB R2017a(win64) software and Windows 8 operating system on a intel(R) Core(TM) i7 CPU , 1.73 GHz processor with 4 GB RAM. The authors declare that all data and material in the paper are available and veritable.

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Competing interests

Author declare that i have no competing interests.

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