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Research Paper

NEW INEQUALITIES INVOLVING OPERATOR MEANS FOR SECTOR MATRICES

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ABSTRACT. The main goal of this paper is to discuss the famous inequalities from positive definite matrices to sector matrices in a more general setting. This includes the mean-convex inequality and Callebaut inequalitiy. Afterward, several inequalities involved positive linear map, are presented for sector matrices. For instance, we show that if $A, B \in \mathcal{S}_\alpha$ are two sector matrices, then for all $\sigma \geq \sharp$ we have

$$\mathcal{R}(\Phi^{-1}(A\sigma B)) \leq \sec^2 \alpha \ \mathcal{R}(\Phi(A^{-1})\sharp\Phi(B^{-1})).$$

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1. Introduction and Background

Let $\mathbb{M}_n$ denote the set of $n \times n$ complex matrices. For Hermitian matrices $A, B \in \mathbb{M}_n$, we write that $A \geqslant 0$ if A is positive semidefinite, i.e. if $\langle Ax, x\rangle \geqslant 0$ for all vectors $x \in \mathbb{C}^n$. We also write $A > 0$ if A is positive definite, i.e. if $\langle Ax, x\rangle > 0$ for all vectors $x \in \mathbb{C}^n$, and $A \geqslant B$ if $A - B \geqslant 0$.

A matrix $A \in \mathbb{M}_n$ is called accretive if in its Cartesian (or Toeplitz) decomposition, $A = \mathcal{R}A + i\mathcal{I}A$, $\mathcal{R}A$ is positive, where $\mathcal{R}A = \frac{A+A^*}{2}$, $\mathcal{I}A = \frac{A-A^*}{2}$.
The numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ \|x\| = 1\}.$$

Let $W(A) \subset \mathcal{S}_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$, where $\mathcal{S}_\alpha$ denote the sector region in the complex plane as follows:

$$\mathcal{S}_\alpha = \{z \in \mathbb{C} : \mathcal{R}z > 0, |\mathcal{I}z| \leq (\mathcal{R}z)\tan\alpha\}.$$

In this case, we will write $A \in \mathcal{S}_\alpha$. Since $0 \notin \mathcal{S}_\alpha$, then all $A \in \mathcal{S}_\alpha$ are invertible.
A linear map $\Phi : B(H) \to B(H)$ is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. If $\Phi(I) = I$, where I denoted the identity operator, then we say that Φ is unital.
An operator mean σ in the sense of Kubo-Ando is defined by an operator monotone function $f : (0, \infty) \to (0, \infty)$ with $f(1) = 1$ as

$$A\sigma B = A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2},$$

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for positive invertible operators A and B. The function f is called the representing function of σ. Recently, Bedraniet al. in [3] proved that this definition can be used for accretive operators too.

Some important operator means are as follows:

- Arithmetic mean: $A\nabla B = (A+B)/2$ and ν-weighted arithmetic mean: $A\nabla_\nu B = \nu A + (1-\nu)B$. $(0<\nu<1)$
- Geometric mean: $A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ and ν-weighted geometric mean: $A\sharp_\nu B = A^{1/2}(A^{-1/2}BA^{-1/2})^{\nu}A^{1/2}$. $(0<\nu<1)$
- Hienz mean: $H_\nu(A,B) = (A\sharp_\nu B + B\sharp_\nu A)/2$. $(0<\nu<1)$
- Heron mean: $F_\nu(A,B) = \nu(A\nabla B) + (1-\nu)A\sharp B$. $(0<\nu<1)$
- Logarithmic mean: $L(A,B) = \int_0^1 A\sharp_t B\ dt$.

For two operator means σ_1, σ_2, we say that $\sigma_1 \leq \sigma_2$ if $A\sigma_1 B \leq A\sigma_2 B$ for all positive operators A, B. Later, Bedrani et al. [3] defined the following mean of two accretive matrices $A, B \in \mathbb{M}_n$:

Definition 1.1. Let $A, B \in \mathbb{M}_n$ be two accretive matrices, $f \in m$, and let ν_f be the probability measure characterizing σ_f. We define the matrix mean σ_f of A and B by

$$A\sigma_f B = \int_0^1 A!_t B d\nu_f(t) \tag{1.1}$$

Recently Ghazanfari and malekinejad [7] define the Heron mean of sector matrices (in particular, positive definite matrices) to be follows:

$$\mathcal{F}_\nu(A,B) = \nu(A\nabla B) + (1-\nu)A\sharp B$$

where $\nu \in [0,1]$. And derived the following inequalities regarding Heron mean for sector matrices:

$$(a)\ 0 \leq \mathcal{F}_\nu(\mathcal{R}A, \mathcal{R}B) \leq \mathcal{R}\mathcal{F}_\nu(A,B) \leq \sec^2\alpha\ \mathcal{F}_\nu(\mathcal{R}A, \mathcal{R}B), \tag{1.2}$$

$$\begin{aligned}(b)\ 0 &\leq \cos^{2\nu}\alpha\ \mathcal{R}A\sharp\mathcal{R}B \leq \cos^{2\nu}\alpha\ \mathcal{R}(A\sharp B) \leq \mathcal{R}\mathcal{F}_\nu(A,B)\\ &\leq \sec^2\alpha\ (1-\nu\sin^2\alpha\ )\mathcal{R}(A\nabla B),\end{aligned} \tag{1.3}$$

and if A, B are invertible, then

$$\begin{aligned}\cos^2\alpha\ \mathcal{R}^{-1}(\mathcal{F}_\nu(A,B)) &\leq \mathcal{R}(\mathcal{F}_\nu^{-1}(A,B))\\ &\leq \sec^2\alpha\ \mathcal{R}\mathcal{F}_\nu(A^{-1},B^{-1})\end{aligned} \tag{1.4}$$

where $\nu \in [0,1]$.

In the same paper, malekinejad et al. [11] proved that if $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset \mathcal{S}_\alpha$ and $\sigma_1 \leq \sigma_2$, then

$$\mathcal{R}(A\sigma_1 B) \leq \sec^2\alpha\ \mathcal{R}(A\sigma_2 B). \tag{1.5}$$

In 1956, Callebaut [5] gave the following refinement of the Cauchy-Schwarz inequality: Given a real number s, non-proportional sequences of positive real numbers $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$, the function $f(r,s) = \left(\sum_{i=1}^n a_i^{s+r}b_i^{s-r}\right)\left(\sum_{i=1}^n a_i^{s-r}b_i^{s+r}\right)$ is increasing in $0 \leq |r| \leq 1$. If $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$ are proportional, then this expression is independent of r.

Thus one can obtain many well-ordered inequalities lying between the left and the right sides of the Cauchy-Schwarz inequality. In particular, if $0 \le t \le s \le \frac{1}{2}$ or $\frac{1}{2} \le s \le t \le 1$, then

$$\begin{aligned}\left(\sum_{j=1}^{m} a_j^{1/2} b_j^{1/2}\right)^2 \le \left(\sum_{j=1}^{m} a_j^{s} b_j^{1-s}\right)\left(\sum_{j=1}^{m} a_j^{1-s} b_j^{s}\right) &\le \left(\sum_{j=1}^{m} a_j^{t} b_j^{1-s}\right)\left(\sum_{j=1}^{m} a_j^{1-t} b_j^{s}\right)\\ &\le \left(\sum_{j=1}^{m} a_j\right)\left(\sum_{j=1}^{m} b_j\right) \end{aligned} \tag{1.6}$$

for all positive real numbers a_j, b_j $(1 \le j \le m)$. This triple inequality is well-known as the Callebaut inequality.
At the end of this section, we present the Lemmas we need to prove the main theorems.

Lemma 1.2. *([9], [10]) If $A \in \mathcal{S}_\alpha$, then*

$$\mathcal{R}(A^{-1}) \le \mathcal{R}(A)^{-1} \le \sec^2\alpha\ \mathcal{R}(A^{-1}).$$

The famous Choi's inequality says:

Lemma 1.3. *Let $A \in B(H)$ be positive. Then, for every positive unital linear map Φ,*

$$\Phi^{-1}(A) \le \Phi(A^{-1}).$$

Ando [1] showed the following property of a positive linear map in conection with the operator mean :

Lemma 1.4. *Let $A, B \in \mathbb{M}_n$ be positive definite. Then*

$$\Phi(A\sigma B) \le \Phi(A)\sigma\Phi(B).$$

In particular for the weighted geometric mean, we have :
Let $A, B \in \mathbb{M}_n$ be positive definite, then, for every positive linear map Φ,

$$\Phi(A\sharp_\nu B) \le \Phi(A)\sharp_\nu\Phi(B),$$

where $\nu \in [0, 1]$.

Lemma 1.5. [3] *Let $A, B \in \mathbb{M}_n$ be accretive matrices. Then*

$$\mathcal{R}A\sigma\mathcal{R}B \le \mathcal{R}(A\sigma B) \le \sec^2\alpha\ (\mathcal{R}A\sigma\mathcal{R}B).$$

2. Main Results

Let Φ be a positive linear map. If $A \in \mathbb{M}_n$ with $W(A) \subset \mathcal{S}_\alpha$. Then $\Phi(\mathcal{R}(A)) = \mathcal{R}(\Phi(A))$ and $W(\Phi(A)) \subset \mathcal{S}_\alpha$. In particular, if $A \in \mathbb{M}_n$ be accretive, then so is $\Phi(A)$. (see [15, Lemma 1]).
We begin with a Callebaut inequality for sector matrices . Whose positive matrix version states of Callebaut inequality is as follows [13]:

$$\sum_{i=1}^{n}(A_i\sharp B_i) \le \left(\sum_{i=1}^{n} A_i\sigma B_i\right)\sharp\left(\sum_{i=1}^{n} A_i\sigma^{\perp} B_i\right) \le \left(\sum_{i=1}^{n} A_i\right)\sharp\left(\sum_{i=1}^{n} B_i\right). \tag{2.1}$$

Theorem 2.1. *Let $A_i, B_i \in \mathbb{M}_n$ be such that $W(A_i), W(B_i) \subset \mathcal{S}_\alpha$, for $j = 1, 2, \ldots, n$. Then for any unital positive linear map Φ and every operator means σ, it holds*

$$\mathcal{R}\left(\sum_{i=1}^{n}\Phi(A_i\sharp B_i)\right) \leq \sec^2\alpha\left(\sum_{i=1}^{n}\mathcal{R}(\Phi(A_i))\sigma\mathcal{R}(\Phi(B_i))\right)\sharp\left(\sum_{i=1}^{n}\mathcal{R}(\Phi(A_i))\sigma^{\perp}\mathcal{R}(\Phi(B_i))\right)$$
$$\leq \sec^2\alpha\ \mathcal{R}\left(\sum_{i=1}^{n}\Phi(A_i)\sharp\sum_{i=1}^{n}\Phi(B_i)\right).$$

Proof. By (2.14), Lemmas 1.4 and 1.5 we have

$$\begin{aligned}
&\mathcal{R}\left(\sum_{i=1}^{n}\Phi(A_i\sharp B_i)\right) = \sum_{i=1}^{n}\mathcal{R}\left(\Phi(A_i\sharp B_i)\right)\\
&= \sum_{i=1}^{n}\Phi\left(\mathcal{R}(A_i\sharp B_i)\right) \leq \sec^2\alpha\sum_{i=1}^{n}\Phi(\mathcal{R}A_i\sharp\mathcal{R}B_i)\\
&\leq \sec^2\alpha\sum_{i=1}^{n}\Phi(\mathcal{R}A_i)\sharp\Phi(\mathcal{R}B_i) = \sec^2\alpha\sum_{i=1}^{n}\mathcal{R}\left(\Phi(A_i)\right)\sharp\mathcal{R}\left(\Phi(B_i)\right)\\
&\leq \sec^2\alpha\left(\sum_{i=1}^{n}\mathcal{R}(\Phi(A_i))\sigma\mathcal{R}(\Phi(B_i))\right)\sharp\left(\sum_{i=1}^{n}\mathcal{R}(\Phi(A_i))\sigma^{\perp}\mathcal{R}(\Phi(B_i))\right)\\
&\leq \sec^2\alpha\left(\sum_{i=1}^{n}\mathcal{R}(\Phi(A_i))\sharp\sum_{i=1}^{n}\mathcal{R}(\Phi(B_i))\right) \leq \sec^2\alpha\ \mathcal{R}\left(\sum_{i=1}^{n}\Phi(A_i)\sharp\sum_{i=1}^{n}\Phi(B_i)\right).
\end{aligned}$$

□

A mean-convex inequality can be stated as follows [14].

$$(\nu(A\sigma C) + (1-\nu)(B\sigma D)) \leq ((\nu A + (1-\nu)B)\sigma(\nu C + (1-\nu)D)). \tag{2.2}$$

Where $A, B \in \mathbb{M}_n$ be positive matrices. The following Theorem is extension of inequality (2.2) and [12, Theorem 2.4].

Theorem 2.2. *Let $A, B, C, D \in \mathcal{S}_\alpha$. Then for any unital positive linear map Φ and every operator means σ, it holds*

$$\mathcal{R}\left(\Phi\left(\nu(A\sigma C) + (1-\nu)(B\sigma D)\right)\right) \leq \sec^2\alpha\ \mathcal{R}\left(\Phi(\nu A + (1-\nu)B)\sigma\Phi(\nu C + (1-\nu)D)\right).$$

Proof. Using the inequality (2.2), Lemmas 1.4 and 1.5 , we obtain

$$\begin{aligned}
\mathcal{R}\left(\Phi\left(\nu(A\sigma C) + (1-\nu)(B\sigma D)\right)\right) &= \Phi\left(\mathcal{R}\left(\nu(A\sigma C) + (1-\nu)(B\sigma D)\right)\right)\\
&= \Phi\left(\nu\mathcal{R}(A\sigma C) + (1-\nu)\mathcal{R}(B\sigma D)\right)\\
&\leq \sec^2\alpha\ \Phi\left(\nu(\mathcal{R}A\sigma\mathcal{R}C) + (1-\nu)(\mathcal{R}B\sigma\mathcal{R}D)\right)\\
&\leq \sec^2\alpha\ \Phi\left((\nu\mathcal{R}A + (1-\nu)\mathcal{R}B)\sigma(\nu\mathcal{R}C + (1-\nu)\mathcal{R}D)\right)\\
&= \sec^2\alpha\ \Phi\left(\mathcal{R}(\nu\ A + (1-\nu)B)\sigma\mathcal{R}(\nu C + (1-\nu)D)\right)\\
&\leq \sec^2\alpha\ \Phi\left(\mathcal{R}(\nu\ A + (1-\nu)B)\right)\sigma\Phi\left(\mathcal{R}(\nu C + (1-\nu)D)\right)\\
&= \sec^2\alpha\ \mathcal{R}\left(\Phi(\nu\ A + (1-\nu)B)\right)\sigma\mathcal{R}\left(\Phi(\nu C + (1-\nu)D)\right)\\
&\leq \sec^2\alpha\ \mathcal{R}\left(\Phi(\nu A + (1-\nu)B)\sigma\Phi(\nu C + (1-\nu)D)\right).
\end{aligned}$$

□

The following proposition is a extension of second inequality of (1.3) if setting $\Phi(X) = X$ and $\sigma = F_\nu$ for every $X \in \mathbb{M}_n$. Also The following proposition is a extension of inequality (1.5).

Proposition 2.3. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset \mathcal{S}_\alpha$. Then for any unital positive linear map Φ and every two operator means $\sigma_1 \le \sigma_2$, it holds*

$$\mathcal{R}\left(\Phi(A\sigma_1 B)\right) \le \sec^2\alpha\ \mathcal{R}(\Phi(A\sigma_2 B)).$$

Proof. By inequality (1.5), we have

$$\begin{aligned}\mathcal{R}\left(\Phi(A\sigma_1 B)\right) &= \Phi(\mathcal{R}(A\sigma_1 B))\\ &\le \Phi(\sec^2\alpha\ \mathcal{R}(A\sigma_2 B))\\ &= \sec^2\alpha\ \Phi(\mathcal{R}(A\sigma_2 B))\\ &= \sec^2\alpha\ \mathcal{R}(\Phi(A\sigma_2 B)).\end{aligned}$$

□

Next we give a relation between the inverse of the mean of sector matrices and the geometric mean of inverse sector matrices involving positive linear maps.

Theorem 2.4. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset \mathcal{S}_\alpha$. Then for any unital positive linear map Φ and $\sigma \ge \sharp$, it holds*

$$\mathcal{R}\left(\Phi^{-1}(A\sigma B)\right) \le \sec^2\alpha\ \mathcal{R}(\Phi(A^{-1})\sharp\Phi(B^{-1})).$$

Proof. We have

$$\begin{aligned}\mathcal{R}\left(\Phi^{-1}(A\sigma B)\right) &\le \mathcal{R}^{-1}(\Phi(A\sigma B)) && \text{(by Lemma 1.2)}\\ &= \Phi^{-1}(\mathcal{R}(A\sigma B))\\ &\le \Phi^{-1}(\mathcal{R}A\sigma\mathcal{R}B) && \text{(by Lemma 1.5)}\\ &\le \Phi^{-1}(\mathcal{R}A\sharp\mathcal{R}B)\\ &\le \Phi((\mathcal{R}A\sharp\mathcal{R}B)^{-1}) && \text{(by Lemma 1.3)}\\ &= \Phi((\mathcal{R}A)^{-1}\sharp(\mathcal{R}B)^{-1})\\ &\le \Phi(\sec^2\alpha\ \mathcal{R}A^{-1}\sharp\sec^2\alpha\ \mathcal{R}B^{-1})) && \text{(by Lemma 1.2)}\\ &= \sec^2\alpha\ \Phi(\mathcal{R}A^{-1}\sharp\mathcal{R}B^{-1})\\ &\le \sec^2\alpha\ (\Phi(\mathcal{R}A^{-1})\sharp\Phi(\mathcal{R}B^{-1})) && \text{(by Lemma 1.4)}\\ &= \sec^2\alpha\ (\mathcal{R}\Phi(A^{-1})\sharp\mathcal{R}\Phi(B^{-1}))\\ &\le \sec^2\alpha\ \mathcal{R}(\Phi(A^{-1})\sharp\Phi(B^{-1})) && \text{(by Lemma 1.5).}\end{aligned}$$

□

Remark 2.5. Yang and Lu [16] proved that if $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset \mathcal{S}_\alpha$, then for any positive linear map Φ, it holds

$$\Phi(\mathcal{R}H_\nu(A^{-1}, B^{-1})) \le \sec^2\alpha\ \mathcal{R}H_\nu(\Phi(A^{-1}), \Phi(B^{-1})), \tag{2.3}$$

and

$$\mathcal{R}^{-1}(\Phi(A\sharp B)) \leq \sec^2\alpha\ \Phi(\mathcal{R}H_\nu(A^{-1}, B^{-1})), \tag{2.4}$$

where $\nu \in [0,1]$. Therefore by (2.3) and (2.4), we have

$$\begin{aligned}\mathcal{R}^{-1}(\Phi(A\sharp B)) &\leq \sec^2\alpha\ \Phi(\mathcal{R}H_\nu(A^{-1}, B^{-1}))\\ &\leq \sec^4\alpha\ \mathcal{R}H_\nu(\Phi(A^{-1}), \Phi(B^{-1})).\end{aligned}$$

Then by Lemma 1.2,

$$\mathcal{R}(\Phi^{-1}(A\sharp B)) \leq \sec^4\alpha\ \mathcal{R}H_\nu(\Phi(A^{-1}), \Phi(B^{-1})). \tag{2.5}$$

Therefore, we presented a relation between the Heinz mean of inverse sector matrices and the inverse of the geometric mean of sector matrices involving positive linear maps. On the other hand, if putting $\sigma = H_\nu$ in Theorem 2.4, we obtain:

$$\mathcal{R}\left(\Phi^{-1}(H_\nu(A,B))\right) \leq \sec^2\alpha\ \mathcal{R}(\Phi(A^{-1})\sharp\Phi(B^{-1})).$$

The above inequality is a relationship between the geometric mean of inverse sector matrices and the inverse of the Heinz mean of sector matrices involving positive linear maps.

Let us derive operator inequalities involving operator means by making use of the previous theorems.
The following lemma, helps us to present the norm version of Theorems 2.3 and 2.4. The operator norm $\|A\|$ of $A \in \mathbb{M}_n$ is defined by

$$\|A\| = \sup\{\langle Ax, y\rangle : x.y \in \mathbb{C}^n, \|x\| = \|y\| = 1\}.$$

Recall that a norm $|||\cdot|||$ on $\mathbb{M}_n$ is unitarily invariant if $|||UAV||| = |||A|||$ for any $A \in \mathbb{M}_n$ and for all unitary matrices $U, V \in \mathbb{M}_n$.

Lemma 2.6. *([2],[17]) Let $A \in \mathcal{S}_\alpha$. Then*

$$|||\mathcal{R}A||| \leq |||A||| \leq \sec\alpha|||\mathcal{R}A|||$$

for any unitarily invariant norm $|||\cdot|||$ on $B(H)$.

Proposition 2.7. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset \mathcal{S}_\alpha$. Then for any unital positive linear map Φ, every two operator means $\sigma_1 \leq \sigma_2$ and unitarily invariant norm $\|.\|$, it holds*

$$\|\ \Phi(A\sigma_1 B)\ \| \leq \sec^3\alpha\ \ \|\ \Phi(A\sigma_2 B)\ \|\ .$$

Proof. Notice that

$$\begin{aligned}\|\ \Phi(A\sigma_1 B)\ \| &\leq \sec\alpha\ \ \|\ \mathcal{R}\Phi(A\sigma_1 B)\ \| && \text{(by Lemma 2.6)}\\ &\leq \sec^3\alpha\ \ \|\ \mathcal{R}\Phi(A\sigma_2 B)\ \| && \text{(by Theorem 2.3)}\\ &\leq \sec^3\alpha\ \ \|\ \Phi(A\sigma_2 B)\ \| && \text{(by Lemma 2.6).}\end{aligned}$$

□

In addition, using the following few useful lemma, the singular value of Theorems 2.3 and Theorem 2.4, can be obtained.

Lemma 2.8. *([2],[6]) Let $A \in \mathcal{S}_\alpha$. Then*

$$\lambda_j(\mathcal{R}A) \leq s_j(A) \leq \sec^2\alpha\ \lambda_j(\mathcal{R}A),\ \ j = 1, 2, \ldots, n,$$

for any unitarily invariant norm $|||\cdot|||$ on $B(H)$.

Proposition 2.9. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset \mathcal{S}_\alpha$. Then for any unital positive linear map Φ and every two operator means $\sigma_1 \leq \sigma_2$, it holds*

$$s_j\left(\Phi(A\sigma_1 B)\right) \leq \sec^4\alpha \; s_j\left(\Phi(A\sigma_2 B)\right),$$

where $j = 1, 2, \ldots, n$.

Proof. For every two operator means $\sigma_1 \leq \sigma_2$, we have

$$\begin{aligned} s_j\left(\Phi(A\sigma_1 B)\right) &\leq \sec^2\alpha \; \lambda_j\left(\mathcal{R}(\Phi(A\sigma_1 B))\right) && \text{(by Lemma 2.8)} \\ &\leq \sec^4\alpha \; \lambda_j\left(\mathcal{R}(\Phi(A\sigma_2 B))\right) && \text{(by Theorem 2.3)} \\ &\leq \sec^4\alpha \; s_j\left(\Phi(A\sigma_2 B)\right) && \text{(by Lemma 2.8).} \end{aligned}$$

□

By applying the following lemma, we get the determinant version of Theorems 2.3 and 2.4.

Lemma 2.10. *([8], [9]) If $A \in \mathbb{M}_n$ has positive definite real part, then*

$$\det(\mathcal{R}A) \leq |\det A| \leq \sec^n\alpha \;\; \det(\mathcal{R}A)$$

Proposition 2.11. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset \mathcal{S}_\alpha$. Then for any unital positive linear map Φ and every two operator means $\sigma_1 \leq \sigma_2$, it holds*

$$|\det\left(\Phi(A\sigma_1 B)\right)| \leq \sec^{3n}\alpha \;\; |\det\left(\Phi(A\sigma_2 B)\right)|.$$

Proof. Compute

$$\begin{aligned} |\det\left(\Phi(A\sigma_1 B)\right)| &\leq \sec^n\alpha \;\; \det\left(\mathcal{R}\left(\Phi(A\sigma_1 B)\right)\right) && \text{(by Lemma 2.10)} \\ &\leq \sec^{3n}\alpha \;\; \det\left(\mathcal{R}\left(\Phi(A\sigma_2 B)\right)\right) && \text{(by Theorem 2.3)} \\ &\leq \sec^{3n}\alpha \;\; |\det\left(\Phi\left(A\sigma_2 B\right)\right)| && \text{(by Lemma 2.10).} \end{aligned}$$

□

The numerical radius $\omega(A)$ of $A \in \mathbb{M}_n$ is defined by

$$\omega(A) = \sup\{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}.$$

When $A \in \mathcal{S}_0$, we have $\omega(A) = \|A\|$ therefore

$$\omega(\mathcal{R}A) = \|\mathcal{R}A\|. \tag{2.6}$$

Bedrani et al. [4] showed if $A \in \mathcal{S}_\alpha$, then

$$\cos\alpha\|A\| \leq \|\mathcal{R}A\| = \omega(\mathcal{R}A) \leq \omega(A) \leq \|A\|. \tag{2.7}$$

Proposition 2.12. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset \mathcal{S}_\alpha$. Then for any unital positive linear map Φ and every two operator means $\sigma_1 \leq \sigma_2$, it holds*

$$\omega\left(\Phi(A\sigma_1 B)\right) \leq \sec^3\alpha \; \omega\left(\Phi(A\sigma_2 B)\right).$$

Proof. Compute

$$\begin{aligned} \omega\left(\Phi(A\sigma_1 B)\right) &\leq \| \; \Phi(A\sigma_1 B) \; \| && \text{(by (2.7))} \\ &\leq \sec\alpha \;\; \| \; \mathcal{R}\left(\Phi(A\sigma_1 B)\right) \; \| && \text{(by Lemma 2.6)} \\ &\leq \sec^3\alpha \;\; \| \; \mathcal{R}\left(\Phi(A\sigma_2 B)\right) \; \| && \text{(by Theorem 2.3)} \\ &= \sec^3\alpha \; \omega\left(\mathcal{R}(\Phi(A\sigma_2 B))\right) && \text{(by (2.6))} \\ &\leq \sec^3\alpha \; \omega\left(\Phi(A\sigma_2 B)\right) && \text{(by (2.7)).} \end{aligned}$$

□

Utilizing Theorem 2.4, similar to the proof of Propositions 2.7, 2.9, 2.11 and 2.12, one can conclude the following observations.

Proposition 2.13. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset \mathcal{S}_\alpha$. Then for any unital positive linear map Φ and every operator means $\sigma \geq \sharp$, it holds*

$$\begin{aligned}
&|\det \Phi^{-1}(A\sigma B))| \leq \sec^{3n}\alpha \ \ |\det\left(\Phi(A^{-1})\sharp\Phi(B^{-1})\right)| \\
&\| \Phi^{-1}(A\sigma B) \| \leq \sec^{3}\alpha \ \ \| \left(\Phi(A^{-1})\sharp\Phi(B^{-1})\right) \| \\
&s_j\left(\Phi^{-1}(A\sigma B)\right) \leq \sec^{4}\alpha \ s_j\left(\Phi(A^{-1})\sharp\Phi(B^{-1})\right) \\
&\omega\left(\Phi^{-1}(A\sigma B)\right) \leq \sec^{3}\alpha \ \omega\left(\Phi(A^{-1})\sharp\Phi(B^{-1})\right).
\end{aligned}$$

Note that if A is accretive-dissipative ($\mathcal{R}A > 0,\ \mathcal{I}A > 0$), then $W(e^{\frac{-i\pi}{4}}A) \subset \mathcal{S}_{\frac{\pi}{4}}$. Moreover, by (1.1),

$$\begin{aligned}
(e^{\frac{-i\pi}{4}}A)\sigma_f(e^{\frac{-i\pi}{4}}B) &= e^{\frac{-i\pi}{4}}\int_0^1 A!_t B \ d\nu_f(t) \\
&= e^{\frac{-i\pi}{4}}(A\sigma_f B).
\end{aligned}$$

One readily finds that following inequalities from Propositions 2.7, 2.9 and 2.12 by specifying α to be equal to $\frac{\pi}{4}$.

Corollary 2.14. *Let $A, B \in \mathbb{M}_n$ be accretive-dissipative. Then for any unital positive linear map Φ and every two operator means $\sigma_1 \leq \sigma_2$, it holds*

$$\begin{aligned}
&\| \Phi(A\sigma_1 B) \| \leq 2\sqrt{2} \, \| \Phi(A\sigma_2 B) \| \\
&\omega\left(\Phi(A\sigma_1 B)\right) \leq 2\sqrt{2} \ \omega\left(\Phi(A\sigma_2 B)\right) \\
&s_j\left(\Phi(A\sigma_1 B)\right) \leq 4 s_j\left(\Phi(A\sigma_2 B)\right),
\end{aligned}$$

where $j = 1, 2, \ldots, n$.

Similar to Corollary 2.14, the following results can be presented by Proposition 2.13.

Corollary 2.15. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset \mathcal{S}_\alpha$. Then for any unital positive linear map Φ and every operator means $\sigma \geq \sharp$, it holds*

$$\begin{aligned}
&\| \Phi^{-1}(A\sigma B) \| \leq 2\sqrt{2} \ \ \| \left(\Phi(A^{-1})\sharp\Phi(B^{-1})\right) \| \\
&\omega\left(\Phi^{-1}(A\sigma B)\right) \leq 2\sqrt{2} \ \omega\left(\Phi(A^{-1})\sharp\Phi(B^{-1})\right) \\
&s_j\left(\Phi^{-1}(A\sigma B)\right) \leq 4 \ s_j\left(\Phi(A^{-1})\sharp\Phi(B^{-1})\right),
\end{aligned}$$

where $j = 1, 2, \ldots, n$.

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