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Research Paper

EQUIVALENCE RELATIONS ON BEST COAPPROXIMATION AND WORST COAPPROXIMATION

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Abstract. A kind of approximation, called best coapproximation was introduced and discussed in normed linear spaces by C. Franchetti and M. Furi in 1972. Subsequently, this study was taken up by several researchers in different abstract spaces.
In this paper, we define relations on best coapproximation and worst coapproximation. We show that these relations are equivalence relation. We obtain cosets sets of best coapproximation and worst approximation. We obtain some results on these sets, compactness and weakly compactness and define coqproximinal and coqremotal.

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1. Introduction and Background

As a counter part to best approximation, a kind of approximation called best coapproximation was introduced in normed linear spaces by C. Franchettiand M. Furi [3] to study some characteristic properties of real Hilbert spaces.
Subsequently, this theory has been developed to a large extent in normed linear spaces and in Hilbert spaces by C. Franchetti and M. Furi, H. Mazaheri, P.L.Papini and I. Singer, Geetha S. Rao and by many others (see e.g. [3, 5, 6, 12, 13] and references cited therein).
In a series of papers, G. Albinus, G.G. Lorentz, T.D. Narang, G. Pantelidis, K. Schnatz, A.I. Vasilev and others (see e.g. [1,4, 7, 11, 14, 16, 19] and references cited therein) have tried to extend various results on best approximation available in normed linear spaces to metric linear spaces. The situation in case of best coapproximation is somewhat different.Whereas some attempts have been made to discuss best coapproximation in metric linear spaces (see e.g. [9, 10]) but still in these spaces this theory is less developed as compared to the theory of best approximation. The present paperis also a step in this direction.
The paper mainly deals with some results on the existence and uniqueness of best coapproximation in quotient spaces when theunderlying spaces are metric linear spaces. We also show how coproximinalityis transmitted to and from quotient spaces. The results proved in the paper extend and generalize various known results on the subject.

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Let $(X, \|.\|)$ be a normed linear space, W a non-empty subset of X. A point $y_0 \in W$ is said to be a best coapproximation point for $x \in X$, if

$$\|y - y_0\| \leq \|x - y\|,$$

Suppose $g \in W$, we set coset best coapproximation

$$R_g = \{x \in X : \|y - g\| \leq \|x - y\| \ for\ y \in W)\},$$

Let $(X, \|.\|)$ be a normed linear space, W a non-empty subset of X, $x \in X$ and $0 \in W$. We set

$$R_W(x) = \{g_0 \in W : \ ||y - g_0|| \leq \|x - y\| \ for\ all\ y \in W\}.$$

The set W is coproximinal if for all $x \in X$, $R_W(x)$ is non-empty. The set W is cochebyshev if for all $x \in X$, $R_W(x)$ is singelton.

Let X be a normed linear space and W a non-empty subset of X. A point $q(x) \in W$ is said to be a cofarthest point for $x \in X$ if

$$\|y - q(x)\| \geq \|x - y\|,$$

we set

$$G_g = \{x \in X : \ \|y - g\| \geq \|x - y\| \ for\ all\ y \in W\},$$

Let $(X, \|.\|)$ be a normed linear space, W a non-empty subset of X, $x \in X$ and $0 \in W$. We have

$$G_W(x) = \{g_0 \in W : \ ||y - g_0|| \geq \|x - y\| \ for\ all\ y \in W\}.$$

The set W is coremotal if for all $x \in X$, $G_W(x)$ is non-empty. The set W is uniquely coremotal if for all $x \in X$, $G_W(x)$ is singelton.

Definition 1.1. Let X be a normed linear space, W a subset of X.

i) W is called qcoproximinal if for every $x \in X$, the set $(x - W) \cap R_0$ is a non-empty compact subset of X.

ii) W is called qcoremotal if for every $x \in X$, the set $(x - W) \cap G_0$ is a non-empty compact subset of X.

2. Equivalance relations on coapproximate sets

In this section we define two equivalance relations on coapproiximte sets. We obtains some results on these relations.

Definition 2.1. Let $(X, \|.\|)$ be a normed space, W a coproximinal subset in X and $x, y \in X$. We define two relations on X, with

(i)

$$x \rhd_1 y \Leftrightarrow \forall w \in W : \ \|x - w\| = \|y - w\|.$$

(ii)

$$x \rhd_2 y \Leftrightarrow for\ some\ g \in W : \ g \in R_W(x) \cap R_W(y).$$

We denoted the equivalence class of $x \in X$ under ralation $\rhd_1(\rhd_2)$ by $[x]_1([x]_2)$

Theorem 2.2. *Let $(X, \|.\|)$ be a normed space, W a proximinal subset in X. The relations $\rhd_1$ is equivalance relation.*

Proof. These relation is reflexive and symmetric. We show that trnsitivity relation $\rhd_1$. For all elements $a, b, c \in X$, if $a \ \rhd_1 b$ and $b \ \rhd_1 c$, then $\forall w \in W : \ \|a - w\| = \|b - w\| = \|c - w\|$. It follows that $a \rhd_1 c$. □

Theorem 2.3. *Let $(X, \|.\|)$ be a normed space, W a proximinal subset in X. The relations $\rhd_2$ is equivalance relation.*

Proof. These relation is reflexive and symmetric. We show that trnsitivity relation $\rhd_1$. For all elements $a, b, c \in X$, if $a \ \rhd_2 b$ and $b \ \rhd_2 c$, then $R_W(a) = R_W(b) = R_W(c)$. It follows that $a \rhd_2 c$. □

Theorem 2.4. *Let $(X, \|.\|)$ be a normed space, W a cproximinal subset in X. Then for every $x \in X$, there exists a $g \in W$ such that $[x]_1 \subseteq R_g$.*

Proof. Suppose $x \in X$, since W is coproximinal, there exists a $g \in R_W(x)$. Now

$$\begin{aligned} y \in [x]_1 \quad &\Longleftrightarrow \quad x \rhd_1 y \\ &\Longleftrightarrow \quad \forall w \in W: \ \|x - w\| = \|y - w\|. \end{aligned}$$

Also $\forall w \in W: \ \|w - g\| \leq \|w - x\|$. Therefore

$$[x]_1 \subseteq R_g.$$

□

Theorem 2.5. *Let $(X, \|.\|)$ be a normed space, W a coproximinal subset in X. Then for every $x \in X$, there exists a $g \in W$ such that $[x]_2 \subseteq R_g$.*

Proof. Suppose $x \in X$,Since W is coproximinal, there exists $g \in W$ such that $g \in R_W(x)$. Now

$$\begin{aligned} y \in [x]_2 \quad &\Rightarrow \quad x \rhd_2 y \\ &\Rightarrow \quad R_W(x) = R_W(y) \\ &\Rightarrow \quad g \in R_W(y) \\ &\Rightarrow \quad y \in R_g. \end{aligned}$$

□

Example 2.6. Suppose $X = \mathbb{R}^2$ with the norm $\|(x, y)\| = \sqrt{x^2 + y^2}$ and $W = \{(x, 0): \ x \in \mathbb{R}\}$. W is coproximinal, becasue if $(x, y) \in R^2$, we set $g_0 = (0, y)$. It is clear that $g_0 \in R_W((x, y))$. Also

$$\{(a, 1): \ a \in R\} \subseteq [(0, 1)]_1$$

Theorem 2.7. *Let $(X, ||.||)$ be a normed linear space and W a cochebyshev subspace of X, $x, y \in X$ and $g_0 \in W$. If $g_0 = R_W(x)$ and $y \in [g_0, x)$, then $g_0 = R_W(y)$. (where $[g_0, x) = \{\lambda g_0 + (1 - \lambda)x: \ \lambda \geq 0\}$.)*

Proof. Since $g_0 = R_W(x)$ and $y \in [g_0, x)$, for some $\lambda > 0$, $y = \lambda g_0 + (1 - \lambda)x$ and for all $w \in W$, we have $\|g_0 - w\| \leq \|x - w\|$.For all $w \in W$

$$\begin{aligned} \|y - w\| \quad &= \quad = \|\lambda g_0 + (1 - \lambda)x - \lambda w - (1 - \lambda)w\| \\ &= \quad \|(1 - \lambda)(x - w)\| + \lambda\|g_0 - w\| \\ &\geq \quad (1 - \lambda)\|w - g_0\| + \lambda\|g_0 - w\| \\ &\geq \quad \|w - g_0\| \end{aligned}$$

Therefore $g_0 = R_W(y)$. . □

Definition 2.8. ([14],[17]) For any two elements x and y in normed linear space X, x is said to be orthogonal to y in the sense of Birkhorff-James, written as $x \perp y$, if $\|x+\lambda y\| \geq \|x\|$ for every real scaler λ.

Let W is closed subspace of a normed space X. From [2],

$$g_0 \in R_W(x) \iff W \perp x - g_0.$$

Let W is closed subspace of a normed space X. We set

$$^{\perp}W = \{x \in X : \ W \perp x\}.$$

Theorem 2.9. *Let $(X, \|.\|)$ be a normed linear space and W a coproximinal subspace of X, $^{\perp}W$ a convex set and $x, y \in X$. If $x \rhd_2 y$, then $x - y \in^{\perp} W$.*

Proof. Suppose $x \rhd_2 y$, then there exists a $g_0 \in R_W(x) \cap R_W(y)$. Then $x - g_0, y - g_0 \in^{\perp} W$, therefore $x - y = 2\frac{x-g_0+y-g_0}{2} \in^{\perp} W$. □

Theorem 2.10. *Let $(X, \|.\|)$ be a normed linear space. If W is a subset of X. Then*

i) for every $g \in W$, R_g and G_g are closed set;

ii) thel set W is cochebyshev if and only if for every $g_1, g_2 \in W$ and $g_1 \neq g_2$, we have $R_{g_1} \cap R_{g_2} = \emptyset$;

iii) the set W is uniquely coremotal if and only if for every $g_1, g_2 \in W$ and $g_1 \neq g_2$, we have $G_{g_1} \cap G_{g_2} = \emptyset$;

iv) if W is cochebyshev, $R_g \cap R_{-g} \neq \emptyset$ then $g = 0$;

v) if W is uniquely coremotal $G_g \cap G_{-g} \neq \emptyset$ then $g = 0$.

Proof. (i) Suppose $g \in W$ and $\{x_n\} \subset R_g$ and $x_n \to x$ as $n \to \infty$. Then $\forall w \in W$, $\|w - g\| \geq \|x_n - w\|$. Therefore $\forall w \in W$, $\|w - g\| \geq \|x - w\|$, it folows that $x \in R_g$. Similary G_g is closed.

(ii) suppose W is cochebyshev, $g_1, g_2 \in W$, $g_1 \neq g_2$ and $x \in R_{g_1} \cap R_{g_2}$. Then $g_1, g_2 \in R_W(x)$ and $g_1 = g_2$, that is a contraction.
On converse, if $g_1, g_2 \in W$ and $g_1 \neq g_2$ and $R_{g_1} \cap R_{g_2} = \emptyset$. Suppose for some $x \in X$, there exists $h_1, h_2 \in R_W(x)$ and $h_1 \neq h_2$. Then $x \in R_{h_1} \cap R_{h_2}$. That is a contraction, it follows that W is cochebyshev.

iii) The prove is simillar to (ii).

iv) There exists a $x \in X$ such that $x \in R_g^{\epsilon} \cap R_{-g}$. Therefore $g, -g \in R_W(x)$ and $g = -g$. It follows that for all $g = 0$.

v) There exists a $x \in X$ such that $x \in G_g \cap G_{-g}$. Therefore $g, -g \in G_W(x)$ and $g = -g$. It follows that for all $g = 0$. □

3. Equivalance relations on worst coapproximate

In this section we define two equivalance relations on worst coapproiximte. We obtains some results on these relations.

Definition 3.1. Let $(X, \|.\|)$ be a normed space, W a coremotal subset in X and $x, y \in X$. We define two relations on X, with
(i)

$$x \lhd_1 y \Leftrightarrow \forall w \in W : \ \|x - w\| = \|y - w\|.$$

(ii)

$$x \lhd_2 y \Leftrightarrow for\ some\ g \in W : \ g \in G_W(x) \cap G_W(y).$$

We denoted the equivalence class of $x \in X$ under ralation $\lhd_1 (\lhd_2)$ by ${}_1[x]({}_2[x])$

Theorem 3.2. *Let $(X, \|.\|)$ be a normed space, W a proximinal subset in X. The relations $\lhd_1$ is equivalance relation.*

Proof. These relation is reflexive and symmetric. We show that trnsitivity relation $\lhd_1$. For all elements $a, b, c \in X$, if $a \ \lhd_1 b$ and $b \ \lhd_1 c$, then $\forall w \in W: \ \|a - w\| = \|b - w\| = \|c - w\|$. It follows that $a \lhd_1 c$. □

Theorem 3.3. *Let $(X, \|.\|)$ be a normed space, W a proximinal subset in X. The relations $\lhd_2$ is equivalance relation.*

Proof. These relation is reflexive and symmetric. We show that trnsitivity relation $\lhd_1$. For all elements $a, b, c \in X$, if $a \ \lhd_2 b$ and $b \ \lhd_2 c$, then $R_W(a) = R_W(b) = R_W(c)$. It follows that $a \lhd_2 c$. □

Theorem 3.4. *Let $(X, \|.\|)$ be a normed space, W a cproximinal subset in X. Then for every $x \in X$, there exists a $g \in W$ such that $[x]_1 \subseteq P_g$.*

Proof. Suppose $x \in X$, since W is coproximinal, there exists a $g \in R_W(x)$. Now

$$\begin{aligned} y \in [x]_1 \quad &\Longleftrightarrow \quad x \rhd_1 y \\ &\Longleftrightarrow \quad \forall w \in W: \ \|x - w\| = \|y - w\|. \end{aligned}$$

Also $\forall w \in W: \ \|w - g\| \leq \|w - x\|$. Therefore

$$[x]_1 \subseteq R_g.$$

□

Theorem 3.5. *Let $(X, \|.\|)$ be a normed space, W a coproximinal subset in X. Then for every $x \in X$, there exists a $g \in W$ such that ${}_2[x] \subseteq R_g$.*

Proof. Suppose $x \in X$,Since W is coproximinal, there exists $g \in W$ such that $g \in R_W(x)$. Now

$$\begin{aligned} y \in_2 [x] \quad &\Rightarrow \quad x \lhd_2 y \\ &\Rightarrow \quad R_W(x) = R_W(y) \\ &\Rightarrow \quad g \in R_W(y) \\ &\Rightarrow \quad y \in R_g. \end{aligned}$$

□

Example 3.6. Suppose $X = \mathbb{R}^2$ with the norm $\|(x, y)\| = \sqrt{x^2 + y^2}$ and $W = \{(x, 0): \ x \in \mathbb{R}\}$. W is coproximinal, becasue if $(x, y) \in R^2$, we set $g_0 = (0, y)$. It is clear that $g_0 \in R_W((x, y))$. Also

$$\{(a, 1): \ a \in R\} \subseteq [(0, 1)]_1$$

Theorem 3.7. *Let $(X, ||.||)$ be a normed linear space and W a cochebyshev subspace of X, $x, y \in X$ and $g_0 \in W$. If $g_0 = R_W(x)$ and $y \in [g_0, x)$, then $g_0 = R_W(y)$. (where $[g_0, x) = \{\lambda g_0 + (1 - \lambda)x: \ \lambda \geq 0\}$.)*

Proof. Since $g_0 = R_W(x)$ and $y \in [g_0, x)$, for some $\lambda > 0$, $y = \lambda g_0 + (1-\lambda)x$ and for all $w \in W$, we have $\|g_0 - w\| \leq \|x - w\|$.For all $w \in W$

$$\begin{aligned}\|y - w\| &= = \|\lambda g_0 + (1-\lambda)x - \lambda w - (1-\lambda)w\| \\ &= \|(1-\lambda)(x-w)\| + \lambda\|g_0 - w\| \\ &\geq (1-\lambda)\|w - g_0\| + \lambda\|g_0 - w\| \\ &\geq \|w - g_0\|\end{aligned}$$

Therefore $g_0 = R_W(y)$. . □

4. Qcoproximinal and Qcoremotal

In this section we are bring some propeties of qcoproximinal and qcoemotal.

Theorem 4.1. *Let X be a Banach space, W a coproximinal subset of X and for every $g \in W$, $W - g = W$. If for all $g \in W$, R_0 is compact. Then $R_W(x)$ is compact and W is qcoproximinal.*

Proof. Suppose $g \in W$ and R_g is compact. Since $W - g = W$ we have $R_g = -g + R_0$, becasuse

$$\begin{aligned}x \in R_g &\iff \|g - w\| \leq \|w - x\| \; \forall w \in W \\ &\iff \|w - g\|\| \leq \|w - g + x + g \\ &iff \quad \|w\| \leq \|x + g - w\| \\ &\iff x + g \in R_0 \\ &\iff x \in -g + R_0\end{aligned}$$

Since R_g is compact, then R_0 is compact and for all $x \in X$, $(x - W) \cap R_0$ is compact. Therefore W is quasi proximinal. If $\{g_n\}_{n\geq 1}$ is a sequence in $P_W(x)$, then for all $n \geq 1$, $x \in P_{g_n}$. Then $\{x - g_n\}_{n\geq 1}$ is a sequence in P_0. Since P_0 is compact, there exists a convergence subsequence $\{x - g_{n_k}\}_{k\geq 1}$ in P_0^ϵ, therefore there exists a convergence subsequence $\{g_{n_k}\}_{k\geq 1}$ in $R_W(x)$. Therefore $R_W(x)$ is compact. □

Theorem 4.2. *Let X be a Banach space, W a coremotal subset of X and for every $g \in W$, $W - g = W$. If for every $g \in W$, F_g is compact. Then W is qcoremotal.*

Proof. Suppose $g \in W$ and F_g is compact. Since $g - W = W$ we have $G_g = -g + G_0$, becasuse

$$\begin{aligned}x \in G_g &\iff \|g - w\| \geq \|w - x\| \; \forall w \in W \\ &\iff \|w - g\|\| \geq \|w - g + x + g \\ &iff \quad \|w\| \geq \|x + g - w\| \\ &\iff x + g \in G_0 \\ &\iff x \in -g + G_0\end{aligned}$$

Since G_g is compact, then G_0 is compact and for all $x \in X$, $(x - W) \cap G_0$ is compact. Therefore W is qcoremotal. if $\{g_n\}_{n\geq 1}$ is a sequence in $G_W(x)$, then for all $n \geq 1$, $x \in G_{g_n}$. Then $\{x - g_n\}_{n\geq 1}$ is a sequence in G_0. Since G_0 is compact, there exists a convergence subsequence $\{x - g_{n_k}\}_{k\geq 1}$ in G_0, therefore there exists a convergence subsequence $\{g_{n_k}\}_{k\geq 1}$ in $G_W(x)$. Therefore $G_W(x)$ is compact. □

Theorem 4.3. *Let X be a Banach space, W a coproximinall subset of X and for every $x \in X$ the set $R_W(x)$ is compact, Then W is qcoproximinal.*

Proof. for every $x \in X$ the set $R_W(x)$ is compact, $x \in X$ and $\{x_n\} \subset (x - W) \cap R_0$. Then $\{x_n - x\} \subset W$ and for all $w \in W$, we have $\|w\| \leq \|x_n - w\|$. We set $g_n = x_n - x$, we have

$$\|g_n\| \leq \|x_n - g_n\|.$$

Since $R_W(x)$ is compact. There exists a subsequence $\{g_{n_k}\}$ such that for $l_0 \in W$,

$$g_{n_k} \to l_0 \quad k \to \infty.$$

It follows that

$$x_{n_k} \to x + l_0 \quad k \to \infty.$$

□

Theorem 4.4. *Let X be a Banach space, W a coremotall subset of X and for every $x \in X$ the set $G_W(x)$ is compact, Then W is qcoremotal.*

Proof. or every $x \in X$ the set $G_W(x)$ is compact, $x \in X$ and $\{x_n\} \subset (x - W) \cap G_0$. Then $\{x_n - x\} \subset W$ and for all $w \in W$, we have $\|w\| \leq \|x_n - w\|$. We set $g_n = x_n - x$, we have

$$\|g_n\| \leq \|x_n - g_n\|.$$

Since $G_W(x)$ is compact. There exists a subsequence $\{g_{n_k}\}$ such that for $l_0 \in W$,

$$g_{n_k} \to l_0 \quad k \to \infty.$$

It follows that

$$x_{n_k} \to x + l_0 \quad k \to \infty.$$

□

Theorem 4.5. *Let X be a Banach space and W a coproximinal hyperplane subspace of X. Then the following statements are equivalent:*

i) W is qcoproximinal,

ii) for every $g \in W$ and for every sequence $\{x_n\}_{n \geq 1}$ with $x_n \in R_g$ has a convergent subsequence.

Proof. *i)* $\to$ *ii)*. Since $codimen(W) = 1$, there exists a $y_0 \in X$ such that $X = y_0 + W$. Also $(y_0 - W) \cap R_0$ is nonempty and compact.
For $g \in W$, if the sequence $\{x_n\}_{n \geq 1} \subseteq R_g$. Then here exists a sequence $\{g_n\}$ in W such that $x_n - g = y_0 + g_n$. It is clear $\{y_0 + g_n\}$ is a sequence in $(y_0 - W) \cap R_0$. Therefore $\{x_n - g\}$ and $\{x_n\}$ has a convergent subsequence.

ii) $\to$ *i)*. If for every $g \in W$ and sequence $\{x_n\}_{n \geq 1}$ and $x_n \in R_g$ has a convergent subsequence. Therefore R_g is compact, from Theorem 2.4,, W is qcoproximinal. □

Theorem 4.6. *Let X be a Banach space and W a coremotal hyperplane subset of X. Then the following statements are equivalent:*

i) W is qcoremotal,

ii) for every $g \in W$ and for every sequence $\{x_n\}_{n \geq 1}$ with $x_n \in G_g$ has a convergent subsequence.

Proof. $i) \to ii)$. Since $codimen(W) = 1$, there exists a $y_0 \in X$ such that $X = y_0 + W$. Also $(y_0 - W) \cap G_0$ is nonempty and compact.
For $g \in W$, if the sequence $\{x_n\}_{n\geq 1} \subseteq G_g$. Then here exists a sequence $\{g_n\}$ in W such that $x_n - g = y_0 + g_n$. It is clear $\{y_0 + g_n\}$ is a sequence in $(y_0 - W) \cap G_0$. Therefore $\{x_n - g\}$ and $\{x_n\}$ has a convergent subsequence.

$ii) \to i)$. If for every $g \in W$ and sequence $\{x_n\}_{n\geq 1}$ and $x_n \in G_g$ has a convergent subsequence. □

Theorem 4.7. *Let X be a Banach space and W a coproximinal subspace of X. Then the following statements are equivalent:*

i) W is qcoproximinal.

ii) for every $g \in W$, for every subspace of X of form $W_x = W + span\{x\}$ and for every sequence $\{x_n\}_{n\geq 1} \subset W_x$ with $x_n \in P_g^{W_x}$ has a convergent subsequence.

Proof. $i) \to ii)$ If W is qcoroximinal in X. Then W, is qcoproximinal in every W_x $(x \in X\backslash W)$, Since $codim(W) = 1$ in every W_x. From Theorem 2.6, for every sequence $\{x_n\}_{n\geq 1} \subset W_x$ with $x_n \in P_n^{W_x}$ has a convergent subsequence.
$ii) \to i)$ Assume that we have (ii), $codim(W) = 1$ in every W_x. Also W is coproximinal in W_x and $X = \cup_{x \in X\backslash W} W_x$. It follows that W is qcoproximinal in X. □

Theorem 4.8. *Let X be a Banach space, W a coproximinal subspace of X, $x \in X$, $g \in W$. Then the following statements are equivalent:*

i) $x \in R_g$,

ii) for $w \in W$, $\|w - g_0\| \leq \|x - w\|_{W^\perp}$,

where

$$\|x - g_0\|_{W^\perp} = \sup\{|f(x - g_0)| : \ \|f\| \leq 1, \ f \in W^\perp\}.$$

Proof. . $i) \to ii)$ Suppose $x \in R_g$, from Lemma 1.1, for all $w \in W$, there exists a $f^w \in X^*$ such that $\|f^w\| = 1$, $f^w(x) = f^w(g)$ and $f^w(g) \geq \|g\|$. Therefore $|f(x - g_0)| \geq f(x - g_0) \geq \|x - g_0\| - \epsilon$ and $\|x - g_0$
$_{W^\perp} \geq \|x - g_0\| - \epsilon$.

$ii) \to i)$ Suppose $f \in W^\perp$, $\|f\| \leq 1$ and $h \in W$, then $|f(x - g_0)| = |f(x - h) \leq$
$f\|\|x - h\| \leq \|x - h\|$. Therefore

$$\|x - g_0\|_{W^\perp} \leq \|x - h\|.$$

We have $\|x - g_0\| \leq \|x - g_0\|_{W^\perp} \leq \|x - h\| + \epsilon$, $x \in P_g$. □

Theorem 4.9. *Let X be a Banach space, W a coproximinal subspace of X, $x \in X$, $g \in W$, $E \subseteq X$. Then the following statements are equivalent:*

i) for $g \in W$, $E \subseteq R_g$,

ii) for $x \in E$, $\|x - g_0\| \leq \|x - g_0\|_{W^\perp}$.

Proof. $i) \to ii)$ Suppose $g \in W$, if $x \in E$, then $x \in P_g$. From Theorem 2.2, $\|x - g_0\| \leq \|x - g_0\|_{W^\perp}$.

$ii) \to i)$ Suppose $x \in E$ then $\|x - g_0\| \leq \|x - g_0\|_{W^\perp}$, from Theorem 2.2, $x \in P_{g_0}$.. Then from Lemma 1.1, there exits a $f \in X*$ such that $\|f\| = 1$, $f|_W = 0$ and $f(x - g_0) \geq \|x - g - 0\|$. It follows that for $g \in W$

$$f(x - g) = f(x) = f(x - g_0) \geq \|x - g_0\|_{W^\perp}.$$

From Lemma 1.1, $x \in R_g$. □

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