Mathematical Analysis
& Convex Optimization

Vol. 3 (2022), No. 2, 7–16
https:\\ maco.lu.ac.ir
DOI: 10.52547/maco.3.2.2

Research Paper

DETECTION OF A TIME-DEPENDENT FORCING TERM IN A ONE-DIMENSIONAL WAVE EQUATION WITH A DYNAMIC-TYPE BOUNDARY CONDITION

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Abstract. In the current paper, we study an inverse problem of identifying a time-dependent forcing term in the one-dimensional wave equation. We have the information of the wave displacement at two different instants of time and two sensor locations of space along with a dynamic type boundary condition. We prove the unique solvability of the problem under some regularity and consistency conditions. Then, an approximate solution of the given inverse problem based on employing the Ritz technique along with the collocation method is presented which converts the problem to a linear system of algebraic equations.

The method takes advantage of the Tikhonov regularization technique to solve the linear system of equations that is not well-conditioned in order to achieve stable solutions. Numerical findings are also included to support the claim that the presented method is reliable in finding accurate and stable solutions.

MSC(2010): 35L20; 35R30; 65M70.
Keywords: Inverse source problems, dynamic-type boundary condition, collocation method, Tikhonov Regularization.

1. Introduction and Background

Regarding the theory of wave propagations, we consider a string of unit length acting upon a time-dependent force function $f(t)$ which is modeled by the second-order one-dimensional hyperbolic equation

$$u_{tt}(x,t) - u_{xx}(x,t) = f(t), \quad in \quad D_T = \{(x,t),\ 0 \le x \le 1,\ 0 \le t \le T\}, \tag{1.1}$$

supplemented with the time terminals

$$u(x,0) = u_0(x), \quad u(x,T) = u_1(x), \quad x \in [0,1], \tag{1.2}$$

and the boundary conditions, corresponding to the flux tension of the string at two interior locations $0 \le y_1 < y_2 \le 1$, namely

$$u_x(y_1,t) = b_1(t), \quad u_x(y_2,t) = b_2(t), \quad t \in [0,T]. \tag{1.3}$$

The system of equations (1.1)-(1.3) models the vibration of a uniform string of unit length subject to the body forces. In more details, as in (1.3) we impose two boundary fluxes $b_1(t)$ and $b_2(t)$ while the string had reached the shapes described by $u_0(x)$ and $u_1(x)$ and we use only a uniform in space body force $f(t)$ exhibited in (1.1). Now, we are interested in discovering the possibility of reconstructing the body force $f(t)$ that needs to be exerted on the whole string so that its extremities are subject to fixed forces $b_1(t)$ and $b_2(t)$ and it reaches at the

Date: Received: May 14 ,2022, Accepted: December 1, 2022.

instants, $t=0$ and $t=T$, the given shapes $u_0(x)$ and $u_1(x)$, respectively. In this regard, we take advantage of the following extra information described in the form of dynamically boundary condition

$$\alpha_1 u_{xx}(1,t)+\alpha_2 u_x(1,t)+\alpha_3 u(1,t)=h(t), \quad t\in[0,T], \tag{1.4}$$

which characterizes the behavior of the string at a given boundary $x=1$ and $\alpha_i,\ i=\overline{1,3}$ are considered as the arbitrary real numbers. The following example shows that the mild instability may occur for the inverse problem (1.1)-(1.4). Given the following properties as

$$u_{0n}(x)=\frac{2+\alpha_4 x}{n^\theta},\ u_{1n}(x)=\frac{1+\alpha_4 x+\alpha_5 T^K+\cos(nT)}{n^\theta},\ \alpha_4,\alpha_5\in\mathbb{R},\ 0<\theta<K<2, \tag{1.5}$$

$$b_{1n}(x)=b_{2n}(x)=\frac{\alpha_4}{n^\theta},\ h_n(t)=\frac{\alpha_2\alpha_4+\alpha_3(1+\alpha_4+\alpha_5 t^K+\cos(nt))}{n^\theta}, \tag{1.6}$$

the corresponding exact solutions of (1.5)-(1.6) are

$$f_n(t)=\frac{K(K-1)\alpha_5 t^{K-2}}{n^\theta}-n^{2-\theta}\cos(nt),\ u_n(x,t)=\frac{1+\alpha_4 x+\alpha_5 t^K+\cos(nt)}{n^\theta}.$$

Instability in recovering the source term $f(t)$ is obvious because all the supplemented conditions (1.5)-(1.6) tend to zero as $n\longrightarrow\infty$ whilst f_n is unbounded and possible small errors in boundary conditions may lead to tremendous output errors.

The inverse source problems for the second-order hyperbolic equations have been well studied from numerical and analytical points of view [5, 6]. In [13], the authors considered the problem of approximating a time-dependent wave source, proved the uniqueness of the solution and provided a Holder stability estimate of the unknown force function and the initial condition in terms of extra measurement. In [7], the author presented a reconstruction method for multiple moving point/dipole wave sources from boundary measurements. In [4], the authors discussed the uniqueness to some inverse source problems for the wave equation in unbounded domains where dynamical boundary surface data of Dirichlet kind was taken into account. In [1], the authors studied the problem of recovering the space and time-dependent forcing term along with the temperature in hyperbolic systems from several integral observations. In [2], the authors studied the wave propagation in a homogeneous 2D or 3D membrane of the finite size and proved the uniqueness and stability of the solution with respect to the extra conditions.

The paper is organized as follows: We prove the uniqueness of the solution for the inverse problem (1.1)-(1.4) in Section 2. Then, we focus on the computational aspects and propose a numerical scheme for approximating the unknown functions in Section 3. Section 4 is devoted to the reports of numerical simulations.

2. **Uniqueness**

In the first place, by assuming that $u_0(x),\ u_1(x)\in C^1([0,1])$ and further the following consistency conditions hold

$$u_0'(y_1)=b_1(0),\ u_1'(y_1)=b_1(T),\ u_0'(y_2)=b_2(0),\ u_1'(y_2)=b_2(T), \tag{2.1}$$

$$h(0)=\alpha_1 u_0''(1)+\alpha_2 u_0'(1)+\alpha_3 u_0(1), \quad h(T)=\alpha_1 u_1''(1)+\alpha_2 u_1'(1)+\alpha_3 u_1(1), \tag{2.2}$$

we define the suitable transformation

$$v(x,t)=u_x(x,t), \tag{2.3}$$

to convert the main problem to the new one which contains only one unknown function, that is

$$v_{tt}(x,t) - v_{xx}(x,t) = 0, \quad in \quad D_T = \{(x,t),\ 0 \le x \le 1,\ 0 \le t \le T\}, \tag{2.4}$$

$$v(x,0) = u_0'(x), \quad v(x,T) = u_1'(x), \quad x \in [0,1], \tag{2.5}$$

$$v(y_1,t) = b_1(t), \quad v(y_2,t) = b_2(t), \quad t \in [0,T]. \tag{2.6}$$

Now, suppose that the system of equations (2.4)-(2.6) has the solutions $v_1(x,t)$ and $v_2(x,t)$ such that $v_1(x,t) \neq v_2(x,t)$. Tuus by defining $w(x,t) := v_2(x,t) - v_1(x,t)$ we achieve to

$$w_{tt}(x,t) - w_{xx}(x,t) = 0, \quad in \quad D_T = \{(x,t),\ 0 \le x \le 1,\ 0 \le t \le T\}, \tag{2.7}$$

$$w(x,0) = w(x,T) = 0, \quad x \in [0,1], \tag{2.8}$$

$$w(y_1,t) = w(y_2,t) = 0, \quad t \in [0,T]. \tag{2.9}$$

We employ the standard Fourier method and introduce the nontrivial solution to the system (2.7)-(2.9) as

$$w(x,t) = \sum_{i=1}^{\infty} g_n(x) \sin(\frac{n\pi t}{T}), \tag{2.10}$$

and substitute (2.10) in the equations (2.7) and (2.9) to get

$$\sum_{i=1}^{\infty} \Big(g_n''(x) + \lambda_n^2 g_n(x) \Big) \sin(\lambda_n t) = 0,\ \lambda_n = \frac{n\pi}{T} \Rightarrow g_n''(x) + \lambda_n^2 g_n(x) = 0,\ g_n(y_1) = g_n(y_2) = 0. \tag{2.11}$$

Paying attention to (2.11), we get

$$g_n(t) = \beta_{1n} \cos(\lambda_n t) + \beta_{2n} \sin(\lambda_n t), \tag{2.12}$$

which using $g_n(y_1) = g_n(y_2) = 0$ we arrive at

$$\begin{pmatrix} \cos(\lambda_n y_1) & \sin(\lambda_n y_1) \\ \cos(\lambda_n y_2) & \sin(\lambda_n y_2) \end{pmatrix} \beta = 0,\ \beta^{Tr} = [\beta_{1n}, \beta_{2n}]. \tag{2.13}$$

Obviously, if $\frac{y_1 - y_2}{T} \notin \mathbb{Q}$ then the system (2.13) possesses the unique trivial solution $\beta = 0$ which results in $w(x,t) = 0$ and therefore $v_1(x,t) - v_2(x,t) = 0$. Accordingly, the system of equations (2.4)-(2.6) has a unique solution, namely $v(x,t)$. From (2.3) we obtain

$$u(x,t) = \int_0^x v(y,t)dy + H(t), \tag{2.14}$$

where $H(t)$ is an arbitrary function. Next, utilizing equation (1.4) we get

$$\alpha_1 v_x(1,t) + \alpha_2 v(1,t) + \alpha_3 \int_0^1 v(y,t)dy + \alpha_3 H(t) = h(t), \tag{2.15}$$

where $H(t)$ is uniquely detected provided that $\alpha_3 \neq 0$ since $v(x,t)$ is the unique solution of (2.4)-(2.6). Hence from (2.14) and (2.15) one can conclude that $u(x,t) = \int_0^x v(y,t)dy + H(t)$ is the unique solution of the system (1.1)-(1.4). Finally, from (1.1) and (1.4) we uniquely derive the forcing function as

$$f(t) = u_{tt}(1,t) - \frac{1}{\alpha_1} \Big\{ h(t) - \alpha_2 u_x(1,t) - \alpha_3 u(1,t) \Big\}, \quad \alpha_1 \neq 0. \tag{2.16}$$

Thus, we can claim the following theorem.

Theorem 2.1. *Assume that* $\frac{y_1-y_2}{T} \notin \mathbb{Q}$ *and* $\alpha_1 \neq 0,\ \alpha_3 \neq 0$*, then the inverse problem given by equations (1.1)-(1.4) possesses a unique solution in* $C^{2,2}(D_T)$*.*

3. Solution technique

For solving the inverse problem (1.1)-(1.4), we take advantage of transformation (2.3) and equivalently approximate the unknown function $v(x,t)$ satisfying (2.4)-(2.6). By defining the auxiliary functions

$$B_1(x,t) := u_0'(x) + \frac{t}{T}(u_1'(x) - u_0'(x)), \tag{3.1}$$

$$B_2(x,t) := b_1(t) + \frac{x-y_1}{y_2-y_1}(b_2(t) - b_1(t)), \tag{3.2}$$

$$S(x,t) := B_1(x,t) + B_2(x,t) - \left(B_2(x,0) + \frac{t}{T}(B_2(x,T) - B_2(x,0))\right) \tag{3.3}$$

$$= u_0'(x) + \frac{t}{T}(u_1'(x) - u_0'(x)) + b_1(t) + \frac{x-y_1}{y_2-y_1}(b_2(t) - b_1(t)) - \Bigg\{ b_1(0) + \frac{x-y_1}{y_2-y_1}(b_2(0) - b_1(0))$$

$$+ \frac{t}{T}(b_1(T) + \frac{x-y_1}{y_2-y_1}(b_2(T) - b_1(T)) - b_1(0) - \frac{x-y_1}{y_2-y_1}(b_2(0) - b_1(0)))\Bigg\}, \tag{3.4}$$

the approximation of $v(x,t)$ is introduced as

$$\begin{aligned} v_{N,N'}(x,t) &:= t(t-T)(x-y_1)(x-y_2)\phi^T(x)C\psi(t) + S(x,t) \\ &= \sum_{i=0}^{N}\sum_{j=0}^{N'} c_{ij} t(t-T)(x-y_1)(x-y_2)\phi_i(x)\psi_j(t) + S(x,t), \end{aligned} \tag{3.5}$$

such that the unknown matrix C is given by

$$C = \begin{pmatrix} c_{00} & \cdots & c_{0N'} \\ \vdots & & \vdots \\ c_{N0} & \cdots & c_{NN'} \end{pmatrix}, \tag{3.6}$$

and $\phi_i(x)$ and $\psi_j(t)$ are the orthonormal Bernstein basis functions (OBBFs) defined over the intervals $[0,1]$ and $[0,T]$, respectively [11]. Here we mention that the properties of the OBBFs have been discussed in the published papers and the interested reader is referred to the related works such as [10, 11, 12] and references therein.

It is worthy to point out that the approximation $v_{N,N'}(x,t)$ proposed by equation (3.5) precisely satisfies the initial and boundary conditions (2.5)-(2.6). We call (3.5) as the approximate solution of the system of equations (2.4)-(2.6) if the following condition is also included

$$R_1(v(x,t)) := v_{tt}(x,t) - v_{xx}(x,t) = 0. \tag{3.7}$$

Hence, by taking (3.5) into account and employing the operational matrices of differentiation [8, 9, 10, 11, 12, 14, 15, 16] D_N and $D_{N'}$ which are corresponding to the bases $\phi_i(x)$ and $\psi_j(t)$, respectively, we compute the following approximations of $v_{tt}(x,t)$ and $v_{xx}(x,t)$

(3.8)
$$v_{tt}(x,t) \simeq (x-y_1)(x-y_2)\phi^T(x)C\Big(2\psi(t)+(4t-2T)D_{N'}\psi(t)+(t^2-tT)(D_{N'})^2\psi(t)\Big)+S_{tt}(x,t),$$

(3.9)
$$v_{xx}(x,t) \simeq (t^2-tT)\Big(2\phi^T(x)+(4x-2y_1-2y_2)\phi^T(x)D_N^T+(x-y_1)(x-y_2)\phi^T(x)(D_N^T)^2\Big)C\psi(t)+S_{xx}(x,t).$$

By substituting the approximations (3.8)-(3.9) in equation (3.7) we get
(3.10)
$$R_1(v_{N,N'}(x,t)) \simeq (x-y_1)(x-y_2)\phi^T(x)C\Big(2\psi(t)+(4t-2T)D_{N'}\psi(t)+(t^2-tT)(D_{N'})^2\psi(t)\Big)$$
$$+b_1''(t)+\frac{x-y_1}{y_2-y_1}\Big(b_2''(t)-b_1''(t)\Big)-u_0'''(x)-\frac{t}{T}\Big(u_1'''(x)-u_0'''(x)\Big)$$
$$-(t^2-tT)\Big(2\phi^T(x)+(4x-2y_1-2y_2)\phi^T(x)D_N^T+(x-y_1)(x-y_2)\phi^T(x)(D_N^T)^2\Big)C\psi(t)=0.$$
,

Next, by collocating [12] the residual function (3.10) at the points

$$x_i=\frac{i}{N+2},\ t_j=\frac{jT}{N'+2},\ i=\overline{1,N+1},\ j=\overline{1,N'+1}, \tag{3.11}$$

we achieve to the following system of algebraic equations

$$R_1(v_{N,N'}(x_i,t_j))=0,\ i=\overline{1,N+1},\ j=\overline{1,N'+1}, \tag{3.12}$$

and the result of which generically will be represented by

$$Ac=b,$$

where the vector c contains the unknowns $c_{ij},\ i=\overline{1,N},\ j=\overline{1,N'}$. The Tikhonov regularization method solves the following modified system

$$(A^{Tr}A+\lambda I)c=A^{Tr}b, \quad \lambda>0, \tag{3.13}$$

to get the vector $c=(A^{Tr}A+\lambda I)^{-1}A^{Tr}b$ and hereby the approximation $v_{N,N'}(x,t)$ is specified. In the next step, we substitute $v_{N,N'}(x,t)$ in equations (2.14) and (2.15) and obtain
(3.14)
$$\bar{u}(x,t)=\int_0^x v_{N,N'}(y,t)dy+\frac{h(t)-\alpha_1 v_{N,N'_x}(1,t)-\alpha_2 v_{N,N'}(1,t)-\alpha_3\int_0^1 v_{N,N'}(y,t)dy}{\alpha_3}.$$

Finally, utilizing equation (3.15) we get

$$\bar{f}(t)=\bar{u}_{tt}(1,t)-\frac{1}{\alpha_1}\Big\{h(t)-\alpha_2\bar{u}_x(1,t)-\alpha_3\bar{u}(1,t)\Big\}. \tag{3.15}$$

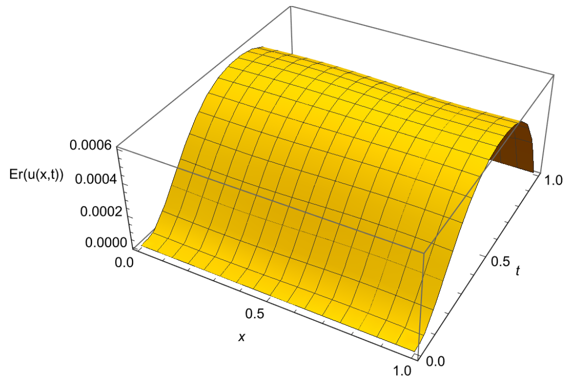

FIGURE 1. Graph of the absolute error corresponding to the approximate solutions for $u(x,t)$ obtained by employing the proposed method with $N = N' = 2$, in the presence of exact input data, discussed in Example 4.0.1.

4. NUMERICAL EXPERIMENTS

The following test example is solved to demonstrate the applicability of the proposed method. We consider

$$Er(u) = |u(x,t) - \bar{u}(x,t)|, \quad Er(f) = |f(t) - \bar{f}(t)|,$$

to represent the absolute value of the difference between the exact and approximate solutions for functions $u(x,t)$ and $f(t)$, respectively. The numerical implementation is carried out with Wolfram Mathematica software (version 12.3) in a personal computer. We use the command LinearSolve for solving the systems of linear algebraic equations. The regularization parameters λ are chosen by applying the L-Curve criterion [3].

4.0.1. *Example 1.* Consider the inverse problem

$$u_{tt}(x,t) - u_{xx}(x,t) = f(t), \quad in \quad [0,1] \times [0,1], \tag{4.1}$$

with the following initial and boundary conditions

$$u_0(x) = e^x, \; u_1(x) = e^{x+1} - \sin(2), \; 0 \leq x \leq 1, \tag{4.2}$$

$$u_x(\frac{1}{4}, t) = e^{0.25+t}, \; u_x(\frac{\pi}{4}, t) = e^{t+\frac{\pi}{4}}, \; 0 \leq t \leq 1, \tag{4.3}$$

$$u(1,t) + u_{xx}(1,t) + u_x(1,t) = 3e^{1+t} - \sin(2t), \; 0 \leq t \leq 1, \tag{4.4}$$

and the exact solutions of this problem are $u(x,t) = e^{t+x} - \sin(2t), \; f(t) = 4\sin(2t)$. We solve the problem by employing the numerical scheme presented in Section 3, with $m = m' = 2$ in the presence of the exact boundary data. Figures 1-2 show the good agreement between the exact and approximate solutions. Moreover the results shown in Table 1 indicate that the accuracy is improved by increasing the number of basis functions used in the approximations.

Observing the performance of the proposed method in the presence of the perturbed boundary data is of special interest. Thus we utilize the following rule

$$h^\sigma(t) = h(t) + \sigma \sin(\frac{t}{\sigma^2}), \quad \sigma = r \times 10^{-2}, \; r \in \mathbb{N}, \tag{4.5}$$

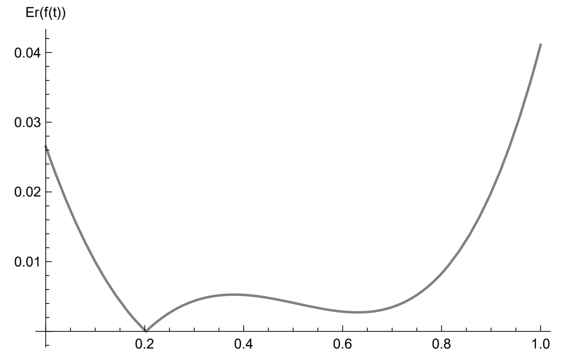

FIGURE 2. Graph of the absolute error corresponding to the approximate solutions for $f(t)$ obtained by employing the proposed method with $N = N' = 3$, in the presence of exact input data, discussed in Example 4.0.1.

TABLE 1. Results of the L^2-norm of errors corresponding to the approximations of unknown functions $f(t)$ and $u(x,t)$ in the presence of exact boundary data, discussed in Example 4.0.1.

(N, N')	$\|Er(f(t))\|_2$	$\|Er(u(x,t))\|_2$	λ
$(2,2)$	4×10^{-2}	6×10^{-3}	10^{-5}
$(3,3)$	2.5×10^{-3}	4×10^{-4}	10^{-5}
$(4,4)$	1.5×10^{-4}	2×10^{-6}	10^{-8}
$(5,5)$	8×10^{-6}	1.4×10^{-7}	10^{-7}

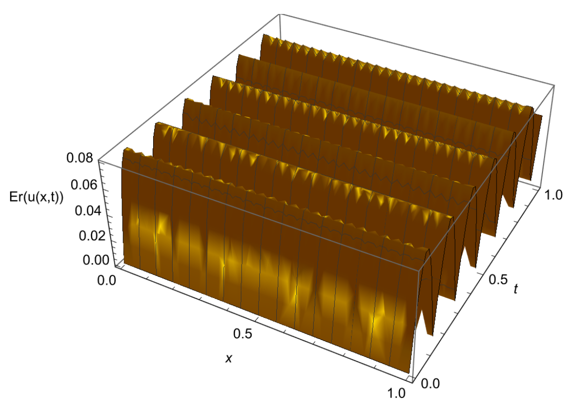

FIGURE 3. Graph of the absolute error corresponding to the approximate solutions for $u(x,t)$ obtained by employing the proposed method with $N = N' = 3$, in the presence of perturbed input data with $\sigma = 0.08$, discussed in Example 4.0.1.

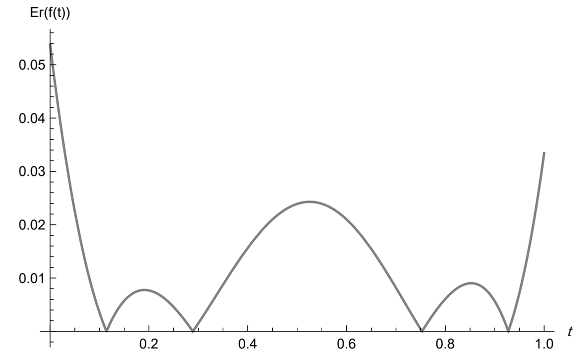

FIGURE 4. Graph of the absolute error corresponding to the approximate solutions for $f(t)$ obtained by employing the proposed method with $N = N' = 3$, in the presence of perturbed input data with $\sigma = 0.08$, discussed in Example 4.0.1.

to produce artificial errors. To obtain the stable results, we need to modify the approximations given by Section 3 via the mollification technique [11, 12]. Let $h^\sigma(t)$ be perturbation such that $\|h(t) - h^\sigma(t)\|_\infty \le \sigma$. We take into account the Gaussian mollifier $F_\delta(t) = \frac{\exp(-\frac{t^2}{\delta^2})}{\delta\sqrt{\pi}}$ where $\delta > 0$ is the radius of mollification. The mollification of the contaminated data $(h^\sigma(t))''$ is performed utilizing the convolution

$$\Big\{F_\delta * (h^\sigma)''\Big\}(t) := \int_{-\infty}^{+\infty} F_\delta(r)(h^\sigma)''(t-r)dr. \tag{4.6}$$

As the property of convolution we use

$$\Big\{F_\delta * (h^\sigma)''\Big\}(t) = \Big\{F_\delta^{''} * (h^\sigma)\Big\}(t), \tag{4.7}$$

and for a given $\delta > 0$ we calculate $\Big\{F_\delta^{''} * (h^\sigma)\Big\}(t)$ numerically using the mid-point integration rule, that is

$$\Big\{F_\delta^{''} * (h^\sigma)\Big\}(t) \simeq \frac{\pi}{m_\delta}\sum_{i=0}^{m_\delta - 1} Q(t, -\frac{\pi}{2} + \frac{\pi i}{m_\delta} + \frac{\pi}{2m_\delta}),\ Q(t,r) = F_\delta^{''}(t - \tan r)h^\sigma(\tan r)\sec^2 r. \tag{4.8}$$

Then, we consider the following

$$(h^\sigma)''(t) = \Big\{F_\delta^{''} * (h^\sigma)\Big\}(t) \simeq d_{\delta,\sigma}^T \psi(t) = \sum_{i=0}^{N'} d_i^{\delta,\sigma}\psi_i(t), \tag{4.9}$$

and consequently

$$h^\sigma(t) \simeq d_{\delta,\sigma}^T \int_0^t \int_0^y \psi(z)dzdy + th'(0) + h(0), \tag{4.10}$$

where considering the compatibility conditions (2.1)-(2.2) we have

$$h(0) = \alpha_1 u_0^{''}(1) + \alpha_2 u_0^{'}(1) + \alpha_3 u_0(1),\ h(T) = \alpha_1 u_1^{''}(1) + \alpha_2 u_1^{'}(1) + \alpha_3 u_1(1),$$

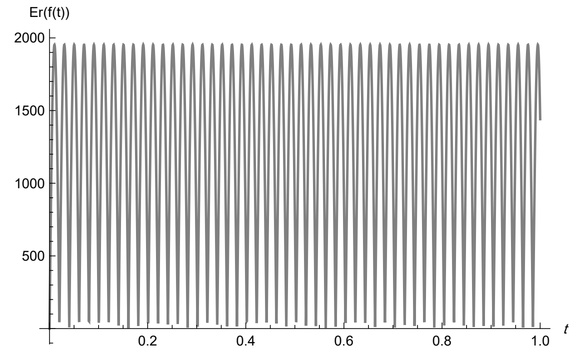

FIGURE 5. Graph of the absolute error corresponding to the approximate solution for $f(t)$ obtained by employing the proposed method with $N = N' = 3$ and without applying the mollification technique in the presence of perturbed input data when $\sigma = 0.08$, discussed in Example 4.0.1.

$$h'(0) = \frac{h(T) - d^T_{\delta,\sigma} \int_0^t \int_0^y \psi(z)dzdy - h(0)}{T}.$$

We call the strategy given by (4.7)-(4.10) admissible if for small ϵ values, and the appropriate given values δ and m_δ we acquire

$$\|d^T_{\delta,\sigma} \int_0^t \int_0^y \psi(z)dzdy + th'(0) + h(0) - h^\sigma(t)\|_\infty \le \epsilon. \tag{4.11}$$

If so, we consider the approximation (4.9) instead of $h''(t)$ while calculating $u_{tt}(1,t)$ for recovering $\bar{f}(t)$ in equation (3.15). In the numerical simulations, we take

$$\sigma = 0.08,\ \lambda = 10^{-6},\ \delta = 0.04,\ m_\delta = 500,\ \epsilon = \sigma,\ N = N' = 3,$$

and obtain the results shown in Figures 3-4. To show the impact of the mollification method on getting stable results we repeat the calculations for retrieving the wave sink $f(t)$ in the presence of perturbed input data without applying the scheme (4.7)-(4.10). The outcome is reported in Figure 5 implying that small perturbation with input data can generate huge errors with productions if the mollification is not employed.

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