Mathematical Analysis
& Convex Optimization

Vol. 3 (2022), No. 2, 17–24
https:\\ maco.lu.ac.ir
DOI: 10.52547/maco.3.2.3

Research Paper

TOPOLOGICAL AND BANACH SPACE INTERPRETATION FOR REAL SEQUENCES WHOSE CONSECUTIVE TERMS HAVE A BOUNDED DIFFERENCE

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Abstract. In this paper we give a topology-dynamical interpretation for the space of all integer sequences P_n whose consecutive difference $P_{n+1} - P_n$ is a bounded sequence. We also introduce a new concept *"Rigid Banach space"*. A rigid Banach space is a Banach space X such that for every continuous linear injection $j : X \to X$, $\overline{J(X)}$ is either isomorphic to X or it does not contain any isometric copy of X. We prove that ℓ_∞ is not a rigid Banach space. We also discuss about rigidity of Banach algebras.

MSC(2010): 46B03, 46B45, 11A41
Keywords: Rigid Banach space, sequence space, Prime numbers.

1. INTRODUCTION

An alternative representation of the classical Banach space ℓ_∞ of all bounded real sequences is the space

$$\hat{\ell}_\infty = \{(a_n) \in \mathbb{R}^{\mathbb{N}} \mid sup|a_{n+1} - a_n| < \infty\}$$

The space $\hat{\ell}_\infty$ is equipped with the norm $|(a_n)| = Max(|a_0|, sup_n|a_{n+1} - a_n|)$. This is the ℓ_∞ analogy of the space

$$bv = \{(a_n) \in \mathbb{R}^{\mathbb{N}} \mid \sum_{n=0}^{\infty} |a_n - a_{n+1}| < \infty\}$$

of all bounded variation sequence introduced in [4, Chapter IV]. In fact the space bv is an alternative representation of the classical Banach space ℓ^1 of all real sequence (a_n) with $\sum_{i=0}^{\infty} |a_i| < \infty$. Consider the difference operator Δ defined on the sequences space $\mathbb{R}^{\mathbb{N}}$ with

$$\Delta(x) = (x_n - x_{n-1})_{n=1}^{\infty}$$

With this operator Δ one can introduce $\hat{\ell}_\infty$ as the space of all sequence $x = (x_n) \in \mathbb{R}^\infty$ with $\Delta(x) \in \ell_\infty$. Moreover

$$\Delta : \hat{\ell}_\infty \to \ell_\infty \tag{1}$$

is a linear isometry whose inverse $\Delta^{-1} : \ell_\infty \to \hat{\ell}_\infty$ is defined by $\Delta^{-1}(x) = y$ where $y_n = x_1 + x_2 + \ldots + x_n$. The idea of difference sequences space and operator Δ is initiated by H.Kizmaz in [8] which generated a considerable amount of researches.

Date: Received: June 1, 2022, Accepted: November 16, 2022.

This alternative representation of ℓ_∞ is a motivation for consideration of all sequence of integers (P_n) which belongs to $\hat{\ell}_\infty$. In fact this perspective is also closely related to the prime gap type problems, those problems who concern the growth of $g_n = P_{n+1} - P_n$ where P_n is the n_{th} prime number. In this paper we give a topological interpretation for a sequence of integers to be belong to $\hat{\ell}_\infty$. In particular it is a classical fact that the sequence of prime numbers is not an element of $\hat{\ell}_\infty$, namely $P_{n+1} - P_n$ is not a bounded sequence. So this would be a motivation to find a topology_dynamical interpretation for some problems in the theory of prime gaps. For the problem of prime gaps see [7].

On the other hand in the construction of both $\hat{\ell}_\infty$ and bv one evidently observe that they contain ℓ_∞ and ℓ^1 respectively. These inclusions are continuous inclusions but are not embedding. So ℓ_∞ is considered as a non closed subspace of $\hat{\ell}_\infty$. A precise evidence is $(\ln(n)) \in \hat{\ell}_\infty$ which can be approximated by the eventually constant sequence $b_N \in \ell^\infty$ with $b_N = (ln(1), ln(2), \ldots, ln(N), ln(N), ln(N), \ldots)$. But there is an essential difference between the inclusion $\ell^1 \subset bv$ and $\ell_\infty \subset \hat{\ell}_\infty$. In the first inclusion $\ell^1 \subset bv$, according to the Cauchy criterion of convergent series, one conclude that ℓ^1 is a dense subspace of bv but in the second one ℓ_∞ is not a dense subspace of $\hat{\ell}^\infty$. For example the sequence $a_n = n$ belongs to $\hat{\ell}_\infty \setminus \overline{\ell_\infty}$ where $\overline{\ell_\infty}$ is the closure of ℓ_∞ in $\hat{\ell}_\infty$. So $\overline{\ell_\infty}$ is a proper Banach subspace of $\hat{\ell}_\infty$. So a natural question is that what is the structure of $\overline{\ell_\infty}$ as a Banach space and how is it sat in $\hat{\ell}_\infty$? This space has a well known structure: it is isomorphic to the space of all bounded sequence which are almost convergent to zero, see [3]. In the last section of the paper we prove that $\overline{\ell_\infty}$ is not a complemented subspace of $\hat{\ell}_\infty$. This implies that $\overline{\ell_\infty}$ is not isomorphic to ℓ_∞. So by some abuse of terminologies this violates the validity of a possible squeeze theorem in the context of Banach spaces: $\ell_\infty \subset \overline{\ell_\infty} \subset \hat{\ell}_\infty \simeq \ell_\infty$ but $\overline{\ell_\infty} \not\simeq \ell_\infty$. We observe more about $\overline{\ell_\infty}$: It also contains an isometric copy of ℓ_∞ This situation leads us to a new concept *rigid Banach space*. A Banach space X is a rigid Banach space if for every injective continuous operator $j : X \to X$ the space $\overline{j(X)}$ is either isomorphic to X or it does not contain any isometric copy of X. So we actually show that ℓ_∞ is not a rigid Banach space.

2. Topological interpretation

In this section we provide a topological interpretation for boundedness of an increasing sequence of positive integers. Let P_n be a strictly increasing sequence of positive integers with $P_1 > 1$

Put $P = \{P_1, P_2, \ldots\}$ and $\mathbb{Z}_0^+ = \{0, 1, 2, \ldots, \}$. Associated to the sequence P_n we define a mapping $f : \mathbb{Z}_0^+ \to \mathbb{Z}_0^+$ as follows:

$$f(x) = \begin{cases} 0 & x \in \{0,1\} \\ 1 & x \in P \\ x+1 & \text{otherwise} \end{cases} \tag{2}$$

Then with the above notations we have the following theorem:

Theorem 1. A necessary and sufficient condition for existence of a topology τ on $\mathbb{Z}_0^+$ such that $(\mathbb{Z}_0^+, \tau)$ is a compact space and $f : (\mathbb{Z}_0^+, \tau) \to \mathbb{Z}_0^+, \tau)$ is a continuous map is that $P_{n+1} - P_n$ is a bounded sequence.

To prove the theorem we need the following lemma which is an exercise mentioned in [11]:

Lemma 1. Let $f : X \to X$ be a continuous map on a compact topological space X. Then $f : \bigcap_{n=0}^{\infty} f^n(X) \to \bigcap_{n=0}^{\infty} f^n(X)$ is a surjective map where $f^n = \overbrace{f \circ f \circ \ldots \circ f}^{n \text{ times}}$

Proof of Lemma 1. Let $y \in \bigcap_{n=0}^{\infty} f^n(X)$ so there exists a sequence $x_n \in f^n(X)$ with $f(x_n) = y$. Without loss of generality we may assume that the set $\{x_1, x_2, \ldots, x_n, \ldots\}$ is an infinite set otherwise by passing from the sequence to the subsequence we may assume that $(x_n) = x$ is a constant sequence for some $x \in \bigcap_{n=0}^{\infty} f^n(X)$. This implies that $f(x) = y$ for $x \in \bigcap_{n=0}^{\infty} f^n(X)$. This completes the proof of the lemma in the case that $\{x_1, x_2, \ldots, x_n, \ldots\}$ is a finite set. So we assume that $\{x_1, x_2, \ldots, x_n, \ldots\}$ is an infinite set. Since X is a compact space there is an accumulation point x for sequence (x_n). Since f is a continuous map then $f(x)$ belongs to the closure of $\{f(x_1), f(x_2), \ldots, f(x_n), \ldots\}$. On the other hand $f(x_n) = y$ so $f(x) = y$. Since x is an accumulation point for $\{x_1, x_2, \ldots, x_n, \ldots\}$ and $x_i \in f^i(X)$ we conclude that $x \in \bigcap_{n=0}^{\infty} f^n(X)$. So $f(x) = y$ for some $x \in \bigcap_{n=0}^{\infty} f^n(X)$. This completes the proof of the lemma.

Proof of Theorem 1. Assume that $P_{n+1} - P_n$ is an unbounded sequence. To the contrary assume that there is a topology τ on $\mathbb{Z}_0^+$ such that $\mathbb{Z}_0^+$ is a compact space and f is a continuous map on it. Put $\lambda_n = P_{n+1} - P_n$. Then $f^{\lambda_n}(P_n + 1) = 1$ so $1 \in f^{\lambda_n}(\mathbb{Z}_0^+)$. Since λ_n is an unbounded sequence then $1 \in \bigcap_{n=0}^{\infty} f^n(X)$. So $\bigcap_{n=0}^{\infty} f^n(X) = \{0, 1\}$. Obviously the map $f : \{0, 1\} \to \{0, 1\}$ is not a surjective map. This contradicts to the above lemma. So one side of the theorem is proved. Now we prove the other side. Assume that $P_{n+1} - P_n$ is a bounded sequence. We define a metric d on $\mathbb{Z}_0^+$ which make it into a compact space and f is a continuous map. Let $M = Max_n(P_{n+1} - P_n)$. Let k be the smallest integer with

$$P_k - P_{k-1} = M \tag{3}$$

For every non zero $x \in \mathbb{Z}_0^+$ there exist a unique non negative integer $n(x)$ with $f^{n(x)}(x) = 1$. We define an equivalent relation on $\mathbb{Z}_0^+$ as follows: $x \sim y$ if $n(x) = n(y)$. So $P = \{P_1, P_2, \ldots, P_n, \ldots\}$ is an equivalent class since

$$n(P_m) = 1, \forall m \in \mathbb{N}$$

Each of the singletons $\{0\}, \{1\}$ is also an equivalent class. We define the following metric d on $\mathbb{Z}_0^+$:

$$d(x, y) = \begin{cases} 5 & x \nsim y \\ |\frac{1}{i} - \frac{1}{j}| & x = P_i, y = P_j, k \notin \{i, j\} \\ \frac{1}{i} & x = P_k, y = P_i, i \neq k \\ d(f^{n-1}(x), f^{n-1}(y)) & x \sim y, n = n(x) = n(y) \end{cases} \tag{4}$$

This is a metric on $\mathbb{Z}_0^+$. We check the triangle inequality for this metric the other two properties of the metric is obvious. The space $\mathbb{Z}_0^+$ is partitioned to disjoint equivalent classes. Note that the distance between every two non equivalent element is equal to 5 and the restriction of d to each equivalent class is less or equal to 1. So to prove the triangle inequality we only need to check it for three elements x, y, z liying in the same equivalent class. On the other hand every equivalent class maps isometrically via f^{n-1} into P. So it is sufficient to prove the triangle inequality for elements of P. The restriction of d to P satisfies the triangle inequality because (P, d) is isometric to $\{1, 1/2, 1/3, \ldots, 1/n, \ldots\} \bigcup \{0\}$. So d is a metric on $\mathbb{Z}_0^+$ and sequence P_n converges to point P_k. As we observed above the set P is homeomorphic to the compact set $S = \{0\} \bigcup \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$ with the standard topology inherited from the real line. There are finitely many equivalent classes of $(\mathbb{Z}_0^+, \sim)$. Each equivalent class is either a finite set or is homeomorphic to the space S described above. So $\mathbb{Z}_0^+$ being a finite union of compact sets is a compact space. To prove that f is a continuous map we have to check only that f is continuous at each accumulation point of $\mathbb{Z}_0^+$. The accumulation points of the compact set $\mathbb{Z}_0^+$ is a subset of $\{P_{k-1}+1, P_{k-1}+2, \ldots, P_{k-1}+M-1, P_k\}$, according to (3). It is obvious that f is continuous at P_k since the sequence P_n converges to P_k and $f(P) = \{1\}$. Notice that every sequence in $\mathbb{Z}_0^+$ which converges to P_k is eventually a subsequence of (P_n). The continuity of f at all other accumulation points of $\mathbb{Z}_0^+$ is a consequence of the fact that f is a local isometry at all points of $\mathbb{Z}_0^+ \setminus \{P_k\}$. This completes the proof of the theorem.

3. Rigid Banach spaces

The main but implicit motivation for this section is that the obvious inclusion of $(\ell_\infty, |\ |_\infty)$ in the space

$$\hat{\ell}_\infty = \{(a_n) \in \mathbb{R}^{\mathbb{N}} \mid sup|a_{n+1} - a_n| < \infty\}$$

with norm $|(a_n)| = Max(|a_0|, sup_n|a_{n+1} - a_n|)$ is a continuous linear injection whose image is not a closed subspace. Moreover we shall prove that the closure of its image is not isomorphic to ℓ_∞. So according to the fact that $\ell_\infty \sim \hat{\ell}_\infty$ we actually get a linear continuous injection $j : \ell_\infty \to \ell_\infty$ such that $\overline{j(X)}$ is not isomorphic to X. This situation naturally leads us to the following definition:

Definition 1. A Banach space X is a rigid Banach space if for every continuous linear injection $j : X \to X$ the space $\overline{j(X)}$ is either isomorphic to X or it does not contain any isometric copy of X.

Example . Every Hilbert space is a rigid Banach space. This is a consequence of the generalization of the Gram Schmidt processes in the infinite dimensional space. We thank Qiaochu Yuan for bringing this infinite dimensional version of the Gram Schmidt process to our attention.

So according to this definition we shall prove that the space ℓ_∞ is an example of a non rigid Banach space:

Theorem 2. The Banach space ℓ_∞ is not a rigid space.

We provide some necessary preliminaries for proof of this theorem. Its proof is essentially based on the usage of natural densities of subsets of $\mathbb{N}$ and also the following Theorems 3&4&5 in [3], [9] and [10]. We thank Colin McQuillan for bringing the reference [3] and theorems 4&5 to our attention. We first recall the definition of complemented subspace:

Definition 2. Let Y be a Banach subspace of a Banach space X. Then Y is a complemented subspace of X if there exists a Banach subspace $Z \subset X$ with $X = Y \oplus Z$.

The following theorem classifies all complemented subspace of ℓ_∞.

Theorem 3 (Joram Lindenstrauss [9])**.** A Banach subspace of ℓ_∞ is a complemented subspace if and only if it is isomorphic to ℓ_∞.

In what follows the space Σ_∞ below plays a crucial role :

Notation. The subspace Σ_∞ of ℓ_∞ is defined as $\Sigma_\infty = \{(a_n) \in \ell_\infty \mid b_n = \sum_{i=1}^n a_i \text{ belongs to} \ell_\infty\}$

The closure $\overline{\Sigma_\infty}$ of space Σ_∞ in ℓ_∞ is described in the following theorem:

Theorem 4 (Bennett, G.; Kalton, N. J. [3])**.** The closure $\overline{\Sigma_\infty}$ of Σ_∞ in ℓ_∞ is the space of all sequence which are almost convergent to zero.

Recall that a sequence (a_n) almost converges to 0 if $L((a_n)) = 0$ for every Banach limit functional $L : \ell_\infty \to \mathbb{R}$. From [3] we have the following equivalent formulation for almost convergence of a sequence to 0.

Theorem 5 (G.G. Lorentz [10])**.** A sequence $(x_n))$ almost converges to 0 if and only if

$$\lim_{p\to\infty} \frac{1}{p}(x_n + x_{n+1} + \ldots + x_{n+p-1}) = 0$$

uniformly in n.

Now we recall the definition of natural density. Assume that $A \subseteq \mathbb{N}$ is given. We define $A(n) = A \bigcap \{1, 2, 3, \ldots, n\}$, $a(n) = |A(n)|$. The lower and upper natural density of A is defined as follows:

$$\bar{d}(A) = \limsup \frac{a(n)}{n}$$

$$\underline{d}(A) = \liminf \frac{a(n)}{n}$$

Definition 3. We say $A \subset \mathbb{N}$ has natural density $d(A)$ if the above two quantities are equal. Namely $\bar{d}(A) = \underline{d}(A)$.

Definition 4. For a sequence $(a_n) \in \mathbb{R}^{\mathbb{N}}$, the support of (a_n) is the set $A = \{n \in \mathbb{N} \mid a_n \neq 0\}$.

The following lemma plays a crucial role to prove the main result of this section.

Lemma 2. Let $(a_n) \in \{0, 1\}^{\mathbb{N}}$ be the characteristic function of a set $A \subset \mathbb{N}$ with non zero upper density. Then $(a_n) \notin \overline{\Sigma_\infty}$.

Proof of Lemma 2. Let $A \subseteq \mathbb{N}$ has strictly positive upper density. Assume that (a_n) is the characteristic sequence of A with $\begin{cases} a_n = 1 & n \in A \\ a_n = 0 & n \notin A \end{cases}$. Since A has strictly positive upper density then the mean sequence $\frac{1}{n}\sum_{i=1}^{n} a_i$ does not converges to zero. So Theorem 4 and Theorem 5 implies that $(a_n) \notin \overline{\Sigma_\infty}$.

The main lemma which is necessary to proof of Theorem 2 is the following Lemma. We are grateful to Nik Weaver for his proof mentioned see [13]

Lemma 3. There are uncountably many subsets A_α of $\mathbb{N}$ such that each A_α has positive upper density. Moreover $A_\alpha \cap A_\beta$ is a finite set for $\alpha \neq \beta$.

Proof of Lemma 3. According to [12] there are uncountably many F_α of infinite subsets of $\mathbb{N}$ with mutual finite intersection. Then $A_\alpha = \bigsqcup_{n\in F_\alpha} [n!, (n+1)!)$ satisfies the required property as in the lemma.

Proof of the main Theorem 2. To prove that ℓ_∞ is not a rigid space we shall provide a linear continuous injection $j : \ell_\infty \to \ell_\infty$ such that $\overline{j(\ell_\infty)}$ is not isomorphic to ℓ_∞ but it contains an isometric copy of ℓ_∞. Put $j = \Delta \circ i$ where Δ is defined as in (1) and $i : \ell_\infty \to \hat{\ell_\infty}$

is the obvious inclusion explained in the introduction. Thus $\overline{j(\ell_\infty)} = \overline{\Sigma_\infty}$. So to complete the proof of theorem 2, according to Theorem 3 mentioned above proved in [9] we need to prove that $\overline{\Sigma_\infty}$ is not a complemented subspace in ℓ_∞.

The method of proof is quite similar to method of [12]. For a simplification of the proof see
https://math.stackexchange.com/q/2467426/143009. Now we prove that $\overline{\Sigma_\infty}$ is not complemented in ℓ_∞. To the contrary assume that

$$\ell_\infty = \overline{\Sigma_\infty} \oplus C \tag{5}$$

for some Banach subspace C of ℓ_∞. Then we claim that there exist a subset Z of $\mathbb{N}$ with positive upper natural density $\bar{d}(Z)$ with the property that every $(a_n) \in \ell_\infty$ with $Supp((a_n)) \subseteq Z$ must satisfies $(a_n) \in \overline{\Sigma_\infty}$. Assuming this claim we immediately get a contradiction hence we conclude the theorem. Because the characteristic function χ_Z associated to Z is a sequence which is supported in Z but does not almost converge to 0. So according to lemma 2 χ_Z is supported in Z but does not belong to $\overline{\Sigma_\infty}$. This is a contradiction so the proof of the theorem would be completed. So it remains to proof the above claim: We assume that (5) hold. We shall find a set $Z \subset \mathbb{N}$ with the property that all sequence (a_n) supported in Z must belongs to $\overline{\Sigma_\infty}$. To the contrary assume that there is no any subset Z with such a property. Let $P : \ell_\infty \to \overline{\Sigma_\infty}$ be the associated projection. Put $Q = I - P$. So Q can be represented in the form $Q = (Q_1, Q_2, \ldots, Q_n, \ldots)$ where each Q_i is a bounded linear functional on ℓ_∞. Notice that $\ker Q = \overline{\Sigma_\infty}$

So after a permutation of indices in $Q_1, Q_2, Q_3, \ldots, Q_n, \ldots$, we may assume that there are constant $k > 0$ and uncountably many sequences (a_n^α) supported in a subset A_α with $\bar{d}(A_\alpha) > 0,\ |A_\alpha \cap A_\beta| < \infty$ and $(a_n^\alpha) \notin \overline{\Sigma_\infty}$ hence $Q_1(a_n^\alpha) \geq k$ and $|a_n^\alpha|_\infty = 1$. This contradicts to boundedness of functional Q.(For a quite similar situation see [12]). This concludes that $\overline{\Sigma_\infty}$ as closure of an isomorphism copy of ℓ_∞ in itself is not complemented hence non isomorphic to ℓ_∞. On the other it is obvious that the space of all sequence $(a_n) \in \ell_\infty$ which almost converges to 0 contains an isometric copy of ℓ_∞. To see this define an isometric embedding $\phi : \ell_\infty \to \ell_\infty$ with $\phi((a_n) = a_{n!}$. This is an isometric embedding whose image is contained in the space of all bounded sequence which almost converges to 0. This shows that ℓ_∞ is not a rigid Banach space.

4. Conclusion and further questions

In the first section we introduce a topological interpretation for bounded ness of the difference sequence space $P_{n+1} - P_n$ where P_n is a strictly increasing sequence of positive integers. On the other hand the sequence of prime numbers does not satisfies this bounded property. So a natural question is that can one introduces some properties of distribution of prime numbers and the behaviour of the prime gaps in terms of certain topology-dynamical properties for certain maps associated to the sequence of primes. This leads us to the following precise question:

Question 1 With notations of formulation of theorem 1, let P be the set of prime numbers. So we have a non invertible dynamical system (Y, f) with Y non compact where $Y = \{0\} \cup \mathbb{N}$ and f is the function described in the theorem. How can one reformulate the twin prime conjecture in a topology-dynamical framework?

The first description is the following:

If (P_{n-1}, P_n) is a pair of twin primes then the integer x with $p_{n-1} < x < P_n$ satisfies the following property: The system (Y, f) has a unique fixed point. The orbits of all points goes to this fixed point in a finite time. There is a special subset P with $f^2(P) =$ The unique fixed point. Then x does not belong to the image of f but $f(x) \in P$. In fact we can extract some dynamical problems from the situation of twins primes: Namely we may study some dynamical system (Y, f), for some certain space Y with a unique fixed point q, with a unique element $z_1 \in f^{-1}(q)$ and an infinite subset P with $f^{-1}(z_1) = P$. Then ask the following question: under which conditions the cardinality (or measure of) $f^{-1}(P) \cap (f(Y))^c$ is infinite. Or under which condition the later is a non compact space. One may search for topological spaces which admit such a dynamical situation which naturally arise from the twin primes conjecture?

In the paper we introduced the concept of non rigid Banach space: A Banach space X which admit an injective continuous linear operator $J : X \to X$ such that $\overline{J(X)}$ is not isomorphic to X but it contains an isometric copy of X. Now one may consider the following generalization in the context of Banach algebras:

Question 2 A Banach algebra A is said to be a rigid Banach algebra if for for every injective continuous Banach algebraic morphism $j : A \to A$ we have $\overline{j(A)}$ either isomorphic to A or it does not contain an isometric Banach algebraic copy of A. How can one classify rigid Banach algebras? (or at least what are some non trivial examples or non examples). Notice that rigidity is a trivial subject in the context of C^* algebras since every C^* injective morphism is automatically an isomorphism but it is not the case for Banach algebras.

Question 3 What can be said about the tensor product of two rigid or non rigid Banach spaces or Banach algebras?

Question 4 What can be said about rigidity of separable Banach spaces?What is an example of a non rigid separable Banach space?

References

[1] E.Alekhno *in russian:Some special properties of Mazur functionals. II*, in russian:Analytical methods of analysis and differential equations. Vol. 2. Proceedings of the Institute of Mathematics of the National Academy of Sciences of Belarus, 2006. Pp. 17–23

[2] S. Banach, *Theorie des Oprations Lineaires*, Editions Jacques Gabay, Sceaux, 1993.

[3] G.Bennett, N.J,Kalton, *Consistency Theorem for Almost Convergence*, Trans. Amer. Math. Soc. 198,23-43(1974).

[4] Dunford, Nelson, Jacob T., Schwartz , *Linear operators. Part I: General Theory*, Pure and Applied Mathematics, vol. VII, New York–London–Sydney: Wiley-Interscience,1958

[5] Mohammed El Azhari, *On the continuous dual of the sequence space bv*, Acta Math. Univ. Comen., 2020, 89 (2), pp.295-298.

[6] W.T.Gowers, An infinite Ramsey theorem and some Banach-space dichotomies, Annals of Math. 156, pp 297-833, 2002.

[7] Ingham, A. E. *On the difference between consecutive primes.* Quarterly Journal of Mathematics. Oxford Series. 8 (1): 255–266, 1937.

[8] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull. 24 (2) (1981), 169-176.

[9] Joram Lindenstrauss *On complemented subspace of m*, Israel Journal of Mathematics, volume 5, pages153–156, 1967.

[10] G.G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. 80 pp. 167-190, 1948.

[11] James R. Munkres *Topology, A First Course* Pearson college Div, 1974

[12] Robert Whitley *Projecting m onto* c_0, The American Mathematical Monthly, Vol. 73, No. 3 pp. 285-286, 1996.

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