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Research Paper

ON SUBSPACE BALANCED CONVEX-CYCLIC OPERATORS

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Abstract. Let X be a separable Banach space and M be a subspace of X.
A bounded Linear operator T on X is subspace balanced convex-cyclic for a subspace M, if there exists a vector $x \in X$ such that the intersection of balanced convex hull of orb(T,x) with M is dense in M. We give an example of subspace balanced convex-cyclic operator that is not balanced convex-cyclic. Also we give an improvement of the Kitai-like criterion for subspace balanced convex-cyclicity and bring on with the Hahn-Banach characterization for subspace balanced convex-cyclicity.

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1. Introduction

Dynamics of linear operators are mainly with the behaviour of them.The study of subspace hypercyclic operators was started by Mador and Martinez in [6].
Convex-cyclic operators was introduced by Rezaei in [7].The study of subspace convex-cyclic operators was started in [1].Baseri, Kashkooly and Rezaei start balanced convex-cyclic operators in [3].
A bounded linear operator T on Banach space X is balanced convex-cyclic (see [3]) if there exists a vector $x \in X$ such that the balanced convex hull of orb(T,x) is dense in X. The vector x is said to be balanced convex-cyclic vector for T.
In [3] it is mentioned that between a set and its linear span there is a balanced convex hull, from this we get that every hypercyclic operator is balanced convex$-$cyclic and every balanced convex-cyclic operator is cyclic. In this paper we introduce and study the subspace balanced convex-cyclic operators.

2. Basic definitions and theorems

Let H be a real or complex separable Hilbert space.let M be a subspace of H that is closed and $B(H)$ be the algebra of all linear continuous operators on H.

Definition 2.1. The set of all balanced convex polynomials is define as $\mathcal{BCP} = \{P(z) = \sum_{j=0}^{n} a_j z^j : \sum_{j=0}^{n} |a_j| \leq 1\}$.

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Definition 2.2. Let $T \in B(H)$ and let M be a non-zero subspace of H. T is said to be subspace balanced convex-cyclic operator, if there exists $x \in H$ such that $BC - orb(T, x) \cap M$ is dense in M, where;

$$BC - orb(T, x) = \{P(T)x : P \text{ is balanced convex polynomial}\}$$
$$=\{P(T)x : P(T) = a_0 + a_1 T + a_2 T^2 ... + a_n T^n, n \in N, \sum_{j=0}^{n} |a_j| \leq 1\}$$

Such a vector x is said to be a subspace balanced convex−cyclic vector.
we will use $M-$balanced convex−cyclic instead of subspace balanced convex−cyclic.
we define $BC(T, M) := \{x \in H : BC - orb(T, x) \cap M \text{ is dense in } M\}$ as the set of all subspace balanced convex−cyclic vectors for M.

Proposition 2.3. Let $T : X \longrightarrow X$ and $S : Y \longrightarrow Y$ be bounded operators, If S⊕T is hypercyclic then so are S and T.

Proof. See [5] □

Proposition 2.4. Let X be a Banach space and $T \in B(X)$ be a balanced convex−cyclic operator, then
1. $\|T\| > 1$.
2. $sup\{\|T^n\| : n \geq 1\} = +\infty$.
3. $sup\{\|T^{*^n}\Lambda\| : n \in \mathrm{N}\} = +\infty$, for every $\Lambda \neq 0$ in X^*.
4. every component of $\sigma(T)$ must intersect the set $\mathbb{C} \diagdown \mathbb{D}$.
5. If X is an infinite−dimensional complex Banach space, then T is not compact.

Proof. See [3] □

Example 2.5. Let T be a balanced convex−cyclic operator on $\mathcal{H}$ and I be the identity operator on $\mathcal{H}$. Then $T \oplus I : \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{H}$ is subspace balanced convex−cyclic operator for subspace $M = \mathcal{H} \oplus \{0\}$ with subspace balanced convex−cyclic vector $x \oplus 0$, moreover $T \oplus I$ is not balanced convex−cyclic operator.

Proof. Since T is balanced convex−cyclic operator on $\mathcal{H}$, so there exists $x \in \mathcal{H}$ such that $BC - orb(T, x)$ is dense in $\mathcal{H}$. let $M = \mathcal{H} \oplus \{0\} \subseteq \mathcal{H}$ and $m = x \oplus 0$ thus,
$BC - orb(T \oplus I, m) = \{P(T \oplus I)m : P \text{ is a balanced convex polynomial}\} = \{P(T)x \oplus 0 : P$ is a balanced convex polynomial$\} \subseteq \mathcal{H} \oplus \{0\} = M$.
Since $BC - orb(T, x)$ is dense in $\mathcal{H}$, so $BC - orb(T \oplus I, m) \cap M$ is dense in M.Therefore $T \oplus I$ is a subspace balanced convex−cyclic operator.
Assume that $T \oplus I$ is a subspace balanced convex−cyclic on $\mathcal{H} \oplus \mathcal{H}$, then by proposition 2.3, the identity operator must be balanced convex−cyclic on $\{0\}$, which is impossible.
Because $\|I\| = 1$ and by proposition 2.4 we get a contradiction. □

We will define subspace balanced convex−transitive operators and we will show that they will be subspace balanced convex−cyclic operators.
We will use the idea from [4,5,6] changing them for balanced convex polynomial spans and generalizing them.

Definition 2.6. Let $T \in B(\mathcal{H})$ and M be a non−zero subspace of $\mathcal{H}.T$ is said to be $M-$balanced convex−transitive with respect to M if for all non−empty sets $U, V \subseteq M$ both are relatively open, there exists a balanced convex−polynomial P such that $P(T)(U) \cap V \neq \emptyset$ or $U \cap P(T)^{-1}(V) \neq \emptyset$contains a relatively open non−empty subset of M.

Proposition 2.7. Let $T \in B(\mathcal{H})$ and M ba a non−zero subspace of $\mathcal{H}$.
Then $BC(T,M) = \bigcap_{j=1}^{\infty} \bigcup_{P \in \mathcal{BCP}} P(T)^{-1}(V_j)$ where $\mathcal{BCP}$ is the collection of all balanced convex polynomials and $\{\mathrm{V_j}\}$ countable open basis for relative topology of M as a subspace of $\mathcal{H}$.

Proof. If $x \in \bigcap_{j=1}^{\infty} \bigcup_{P \in \mathcal{BCP}} P(T)^{-1}(V_j)$ then for all j=1,2,... there exists a balanced convex polynomial P such that $x \in P(T)^{-1}(V_j)$ if and only if $P(T)x \in V_j$.
But since $\{V_j\}$ is a basis for the relatively topology of M, this occurs if and only if $BC-orb(T,M) \cap M$ is dense in M, that is $x \in BC(T,M)$. □

Lemma 2.8. Let $T \in B(\mathcal{H})$ and M be a non−zero subspace of $\mathcal{H}$.Then the following are equivalent:

(1) T is M−balanced convex−transitive with respect to M.

(2) For each relatively open subsets U and V of M, there exists $P \in \mathcal{BCP}$ such that $P(T)^{-1}(\mathrm{U}) \cap V$ is relatively open subset in M.

(3) For each relatively open subsets U and V of M, there exists $P \in \mathcal{BCP}$ such that $P(T)^{-1}(\mathrm{U}) \cap V \neq \emptyset$ and $P(T)(M) \subseteq M$.

Proof. (3)$\Longrightarrow$(2) Since $P(T) : M \longrightarrow M$ is continuous and V is relatively open in M, then $P(T)^{-1}(\mathrm{U})$ is also relatively open in M. Now, if V be any open subset of M, then $P(T)^{-1}(\mathrm{U}) \cap V$ is open and $P(T)^{-1}(\mathrm{U}) \cap V \subseteq M$.

(2)$\Longrightarrow$(1) Since for each relatively open subsets U and V, $P(T)^{-1}(\mathrm{U}) \cap V$ is relatively open subset in M, so $P(T)^{-1}(\mathrm{U}) \cap V \neq \emptyset$ and $P(T)^{-1}(\mathrm{U}) \cap V$ is open in M.

(1)$\Longrightarrow$(3) By definition of M−balanced convex−transitive, there exists U and V relatively open subsets in M, such that $W = P(T)^{-1}(U) \cap V \neq \emptyset$ and W is relatively open in M, and $W \subseteq P(T)^{-1}(U)$. So $P(T)(W) \subseteq U$ and $U \subseteq M$, therefore we have $P(T)(W) \subseteq M$.
We must show that $P(T)(M) \subseteq M$.
Let $x \in M$ and $x_0 \in W$, since W is relatively open in M, so there exists $r > 0$ such that $x_0 + rx \in W$.
Since $P(T)(W) \subseteq M$, so $P(T)(x_0 + rx) = P(T)(x_0) + rP(T)x \in M$.
Thus $P(T)(x_0) \in M$ and since M is subspace so $r^{-1}(-P(T)(x_0) + P(T)(x_0) + rP(T)x) \in M$, that is $P(T)(x) \in M$.This is true for any $x \in M$, hence for $P(T)(x) \in M$, so $P(T)(M) \subseteq M$. □

Theorem 2.9. Let $T \in B(H)$ and M be a non−zero subspace of $\mathcal{H}$.If T is M−balanced convex−transitive,then T is M−balanced convex−cyclic.

Proof. By proposition 2.7 and lemma 2.8 proof is clear. □

3. Subspace balanced convex−cyclic Criterion

In this section similar to [3] we give easily applicable Kitai-like criterion for an operator to be subspace balanced convex−cyclic. Also we will relate it with invariant subspaces and we will see that the converse of theorem 2.9 in general is not true.

Proposition 3.1. Let $T \in B(\mathcal{H})$ and let M be a non−zero subspace of $\mathcal{H}$.If there exist X and Y, dense subsets of M such that for every $x \in X$ and $y \in Y$ there exists a sequence $\{P_k\}_{k=1}^{\infty}$ of balanced convex polynomials such that;

1. $P_k(T)y \longrightarrow 0$; $\quad \forall y \in Y$.
2. For each $x \in X$, there exists a sequence $\{y_k\} \subseteq$ M, such that $y_k \rightarrow 0$ and $P_k(T)y_k \rightarrow x$.
3. M is an invariant subspace for $P_k(T)$ for all $k \geqslant 1$.

Then T is $M-$balanced convex−cyclic operator.

Proof. We will use lemma 2.8 and theorem 2.9.
Let U and V be non−empty relatively open subsets of M. We will show that there exists $k \geqslant 1$ such that $P_k(T)(U) \cap V \neq \emptyset$.Since X, Y are dense in M, there exists $v \in Y \cap V$ and $u \in X \cap U$. Since U and V are relatively open, there exists $\varepsilon > 0$ such that the $M-$ball centered at v of radius ε is contained in V and the $M-$ball centered at u of radius ε is contained in U.
By hypothesis, for $u \in X$ and $v \in Y$, we can choose k large enough such that there exists $y_k \in M$ with $\|P_k(T)v\| < \frac{\varepsilon}{2}$, $\|y_k\| < \varepsilon$ and $\|P_k(T)y_k - u\| < \frac{\varepsilon}{2}$.So we have;

a) Since $v \in M$ and $y_k \in M$, so $v + y_k \in M$. Also since $\|(v + y_k)\text{-}v\| = \|y_k\| < \varepsilon$ it follows that $v + y_k$ is in $M-$ball centered at v of radius ε and hence $v + y_k \in V$.

b) Since v and y_k are in M and M is invariant under $P_k(T)$,it follows that $P_k(T)(v + y_k) \in M$. Also $\|P_k(T)(v + y_k) - u\| \leqslant \|P_k(T)(v)\| + \|P_k(T)(y_k) - u\| < \varepsilon$.

Hence $P_k(T)(v + y_k)$ is in the $M-$ball centered at u of radius ε and thus $P_k(T)(v + y_k) \in U$. So by (a) and (b), T is $M-$balanced convex−transitive and by theorem 2.9 we get that $v + y_k \in P_k(T)^{-1}(U) \cap V$, thus $P_k(T)^{-1}(U) \cap V \neq \emptyset$ which means that T is $M-$balanced convex−cyclic operator. □

4. Hahn−Banach characterization

A necessary and sufficient condition for an operator so as to have a balanced convex−cyclic vector was established in [3].
Now in this section we give a similar necessary and sufficient condition for subspace balanced convex−cyclic operators.Also we prove some properties about subspace balanced convex−cyclic operators.

Theorem 4.1. (Hahn−Banach characterization for subspace balanced convex−cyclicity) Let X be a separable infinite dimensional Banach space and M be a non−trivial closed subspace of X and $T \in B(X)$. Then for a vector $x \in X$ the following conditions are equivalent:

1. The vector x is an $M-$balanced convex−cyclic vector for T.
2. For every non−zero functional $f \in M^*$ we have $\sup\{|f(P(T)x)| :$ P is balanced convex polynomial and $P(T)x \in M\} = +\infty$.

Proof. Let $S = BC - orb(T, x) \cap M$. So S is a balanced convex set and the proof is similar to proposition 2.2 in [3] and so we omit it. □

Theorem 4.2. Let X be a separable infinite dimensional Banach space.
If T is $M-$balanced convex$-$cyclic operator, then;

(1) $\|T\| > 1$.
(2) $\sup\{\|{T^*}^n f\| : \text{n}\in\text{N}\}=+\infty$ for each non$-$zero $f \in M^*$.

Proof. (1) If $\|T\|\leqslant 1$,then for each $x \in X$, $BC - orb(T, x)\cap M$ is norm bounded and can not dense in M.So T is not $M-$balanced convex$-$cyclic which is a contradiction.

(2) By way of contradiction, suppose there exists a non$-$zero linear functional $f \in M^*$ such that $\|{T^*}^n f\|\leqslant b$ for some positive number b and all $n\geqslant 1$.Let $x \in X$ so
$|f(T^n x)| = |({T^*}^n f)x|\leqslant\|{T^*}^n f\|\|x\|\leqslant b\|x\|$. Hence for every balanced convex polynomial P,we have $|f(p(T))x|\leqslant b\|x\|$.So $BC - orb(T, x)\cap M$ can not dense in M and T is not $M-$balanced convex$-$cyclic which is a contradiction.

□

Proposition 4.3. If $T \in B(X)$ is an $M-$balanced convex$-$cyclic operator,then $\sigma_p(T^*) \cap \overline{\mathbb{D}}=\emptyset$.

Proof. Let x be a subspace balanced convex$-$cyclic vector for T and λ be an eigenvalue for T^*.So there exists a non$-$zero bounded linear functional f on X such that $T^*\circ f=\lambda f$.
Hence $f(T^n x)={T^*}^n(f(x))=\lambda^n f(x)$.So for every balanced convex polynomial P we have $f(P(T)x)=P(\lambda)f(x)$ and consequently $\sup_{\text{P}\in\,\mathcal{BCP}}|f(P(T)x)|=\sup_{\text{P}\in\,\mathcal{BCP}}|P(\lambda)|\ \|f\|=+\infty$ if and only if $|\lambda|>1$.This completes the proof. □

As a result of theorem 4.1 we have the next corollary.

Corollary 4.4. If $T \in B(X)$ is an $M-$balanced convex$-$cyclic operator,then T has dense range.

The next theorem similar to theorem 2.6 in [2] shows that if an operator $T \in B(X)$ be balanced convex$-$cyclic, then there exists a special subspace M of X such that T is an $M-$balanced convex$-$cyclic operator.

Theorem 4.5. Let $x \in X$ be a balanced convex$-$cyclic vector for an operator $T \in B(X)$,then x is an $M-$balanced convex$-$cyclic vector for a non$-$trivial closed subspace M of X.

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