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Research Paper

EXISTENCE AND UNIQUENESS SOLUTION OF THE HAMMERSTEIN TYPE FRACTIONAL EQUATIONS VIA THE FIXED POINT THEOREMS

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Abstract. The aim of this paper is to investigate the existence and uniqueness of solution for a class of nonlinear integro-differential equations known as Hammerstein type. We study fractional equations in the Banach space whose derivative is of the Caputo type. The existence of solution is studied by using the Schauder's fixed point theorem, and the uniqueness is established via a generalization of the Banach fixed point theorem. Finally, an example is given to illustrate the analytical findings.
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1. Introduction and Background

In recent years, investigation on the characteristics of solution for fractional differential and integral equations has been presented in numerous research papers and monographs. In [6], Balachandran and Trujillo studied the existence of solutions of fractional quasi-linear integro-differential equations. The study of semilinear fractional differential equations, and neutral integro-differential equations via fractional operators, were also done by Balachandran and Kiruthika [4] and Chang and Nieto [7], respectively. Aghajani et al. [3] investigated the solvability of a wide class of nonlinear fractional equations. Also, investigating the existence of the solution for fractional equations can be found in [1, 2, 5, 9, 10, 12, 13].

In this paper, we try to prove the existence and uniqueness solution of the initial value problem

$$\left({}^{C}D^{\alpha}y\right)(x) = g(x) + \int_0^x k(x,t)\, G\left({}^{C}D^{\beta}y(t)\right) dt, \qquad x \in [0,a], \tag{1.1}$$

$$y^{(i)}(0) = y_i, \quad i = 0, 1, \ldots, m-1, \tag{1.2}$$

where for $m, n \in \mathbb{N}$, $m-1 < \alpha < m$, $n-1 < \beta < n$, $\beta < \alpha$ and the fractional derivatives are considered in the Caputo sense. This type of derivative of order α for a function $y(x) \in C[0,a]$ is determined by the following relation,

$$\left({}^{C}D^{\alpha}y\right)(x) = \frac{1}{\Gamma(r-\alpha)} \int_0^x (x-\tau)^{r-\alpha-1}\, y^{(r)}(\tau) d\tau,$$

with

$$r = \lfloor \alpha \rfloor + 1 \text{ for } \alpha \notin \mathbb{N}_0; \text{ and } r = \alpha \text{ for } \alpha \in \mathbb{N}_0. \tag{1.3}$$

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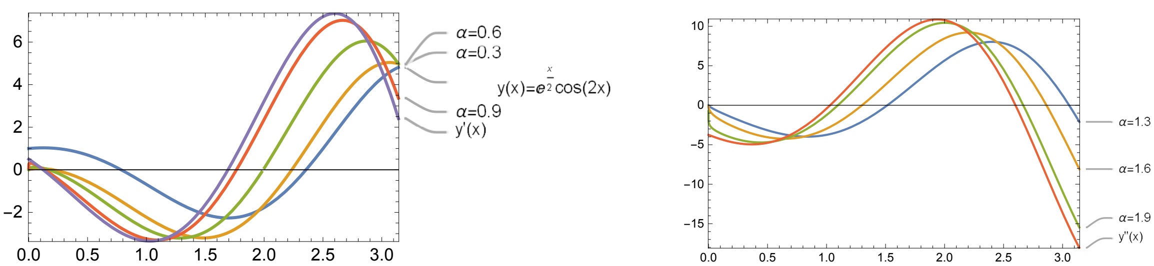
Figures 1(a) and 1(b) show the Caputo derivatives of function $y(x) = e^{\frac{x}{2}} cos(2x)$ on $[0, \pi]$ with $\alpha = 0.3, 0.6, 0.9$ and $\alpha = 1.3, 1.6, 1.9$, respectively.



(a) $0 < \alpha < 1$ (b) $1 < \alpha < 2$

FIGURE 1. 1(a) Function $y(x) = e^{\frac{x}{2}} cos(2x)$ and its Caputo derivatives for fractional order $\alpha = 0.3, 0.6, 0.9$ on the interval $[0, \pi]$. 1(b) The Caputo derivatives of $y(x) = e^{\frac{x}{2}} cos(2x)$ for fractional order $\alpha = 1.3, 1.6, 1.9$ on the interval $[0, \pi]$.

To achieve the main purpose of this work, we utilize the equivalence of equation (1.1)-(1.2) with the following integral equation [17],

$$y(x) = \sum_{i=0}^{n-1} \frac{y_i}{i!} x^i + \frac{1}{\Gamma(\beta)} \int_0^x \frac{u(s)}{(x-s)^{1-\beta}} ds, \tag{1.4}$$

with

$$u(x) = \sum_{i=n}^{m-1} \frac{y_i}{\Gamma(i-\beta+1)} x^{i-\beta} + I^{\alpha-\beta} g(x) + I^{\alpha-\beta} \int_0^x k(x,t)\ G(u(t)) dt, \tag{1.5}$$

where $n \leq m$, and $u \in C\,[0, a]$.

The use of fixed point theorems to show the existence of the solution of integral equations is desired by researchers [11, 14, 16]. In Section 3, using the Schauder's fixed point theorem, the existence of solutions will be investigated. Moreover, the uniqueness of solution will be established using a generalization of the Banach fixed point theorem. Finally, we present an example which the existence of its solution can be illustrated based on the obtained results.

2. Definitions and theorems

In this section, we give some definitions, notations and results that we need in the sequel (for more details see [8, 15]). Let $[0, a] \subset \mathbb{R}$ be a finite interval. Throughout this paper, we consider the Banach space χ with the norm $\| \, . \, \|$ which is defined as follows:

Definition 2.1. We denote by $C^n[0,a]$ a space of functions f which are n times continuously differentiable on $[0,a]$ with the norm

$$\|f\|_{C^n} = \sum_{i=0}^{n} \max_{x\in[0,a]} |f^{(i)}(x)|, \;\; n \in \mathbb{N}_0.$$

In particular, for $n = 0$, $C^0[0,a] \equiv C[0,a]$ is the space of continuous functions f on $[0,a]$ with the norm

$$\|f\|_C = \max_{x\in[0,a]} |f(x)|. \tag{2.1}$$

Definition 2.2. A set H is called uniformly bounded if there exists a constant $M > 0$ such that $\| h \|_\infty \leq M$ for every $h \in H$.

Now the Arzela-Ascoli theorem can be recalled.

Theorem 2.3. *Let H be a subset of $C[a,b]$ equipped with the Chebyshev norm. Then H is relatively compact in $C[a,b]$ if and only if, H is equicontinuous and uniformly bounded.*

We indicate a generalization of the Banach fixed point theorem given by Weissinger [18].

Theorem 2.4. *Let (U,d) be a nonempty complete metric space, and let $\omega_i \geq 0$ for any $i \in \mathbb{N}_0$ be such that the series $\sum_{i=0}^{\infty} \omega_i$ converges. Further, let $T : U \to U$ be a map that, for every $i \in \mathbb{N}$ and $u, v \in U$, the relation*

$$d\left(T^i u, T^i v\right) \leq \omega_i \; d(u,v) \;\; (i \in \mathbb{N}),$$

holds. Then the operator T has a unique fixed point $u^* \in U$. Furthermore, for any $u_0 \in U$, the sequence $\left\{T^i u_0\right\}_{i=1}^{\infty}$ converges to this fixed point u^*.

3. Main results

The aim of this section is to establish the existence and uniqueness theorems for solution of nonlinear fractional integro-differential equation (1.1). According to the equivalence of equation (1.1)-(1.2) with equation (1.4)-(1.5), it is sufficient to show that Eq. (1.5) has a solution $u \in C[0,a]$.

3.1. **Existence and uniqueness.** We will consider Eq. (1.5) under the following assumptions:

(h_1) the function $G : C[0,a] \to C[0,a]$ satisfies the Lipschitz condition, i.e. there exists a constant $M > 0$, such that for any $t \in [0,a]$ and for all $u, v \in C[0,a]$ the following inequality holds

$$|(Gu)(t) - (Gv)(t)| \leq M|u(t) - v(t)|, \tag{3.1}$$

(h_2) the function G transforms continuously the space $C[0,a]$ into itself, and there exists a nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|Gu\| \leq \psi(\|u\|)$ for any $u \in C[0,a]$. Moreover, for every function $u \in C[0,a]$ which is nonnegative on $[0,a]$, the function Gu is nonnegative on $[0,a]$,

(h_3) there exists a real number $r_0 > 0$ such that if

$$\left|\sum_{i=n}^{m-1} \frac{y_i}{\Gamma(i-\beta+1)} a^{i-\beta}\right| \leq L,$$

then

$$L+\frac{a^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\left[\|g\|_C+a\|k\|_C\ \psi(r)\right]\leq r. \tag{3.2}$$

Theorem 3.1. *(Existence) Suppose that* (h_1) - (h_3) *hold. Then, problem* (1.1) - (1.2) *has at least one solution* $y(x)\in C^{m-1}[0,a]$ *which* $({}^C D^\alpha y)(x)\in C[0,a]$.

Proof. Let $\chi=C[0,a]$ be the Banach space equipped with the max norm is defined in relation (2.1), and let $B_r=\{u\in C[0,a]:\ \|u\|\leq r\}$ be a closed bounded and convex subset of $C[0,a]$, where r is satisfied in (3.2). Using (1.5), we define a map $T:B_r\to B_r$ as

$$(Tu)(x)=\sum_{i=n}^{m-1}\frac{y_i}{\Gamma(i-\beta+1)}x^{i-\beta}+I^{\alpha-\beta}g(x)+I^{\alpha-\beta}\int_0^x k(x,t)\ G(u(t))dt.$$

We prove $T\in C(B_r,B_r)$. In other words, we show that $T:B_r\to B_r$ is a continuous and relatively compact operator in χ. First, we have

$$\begin{aligned}|(Tu)(x)|&\leq\left|\sum_{i=n}^{m-1}\frac{y_i}{\Gamma(i-\beta+1)}x^{i-\beta}\right|+|I^{\alpha-\beta}g(x)|+\left|I^{\alpha-\beta}\int_0^x k(x,t)\ G(u(t))dt\right|\\&\leq\left|\sum_{i=n}^{m-1}\frac{y_i}{\Gamma(i-\beta+1)}a^{i-\beta}\right|+\frac{a^{\alpha-\beta}\|g\|_C}{\Gamma(\alpha-\beta+1)}+\frac{a^{\alpha-\beta+1}\|k\|_C\ \psi(\|u\|)}{\Gamma(\alpha-\beta+1)}.\end{aligned}$$

Thus we have

$$|(Tu)(x)|\leq L+\frac{a^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\left[\|g\|_C+a\|k\|_C\ \psi(\|u\|)\right]. \tag{3.3}$$

From the estimates (3.2) and (3.3), we infer that there exists $r_0>0$ such that the operator T transforms the ball B_r into itself.

In view of the assumption of continuity g on $[0,a]$, k on $[0,a]\times[0,a]$, G on $C[0,a]$ and the operator $I^{\alpha-\beta}$ on $C[0,a]$, one can easily finds $Tu\in C[0,a]$ for $u\in B_r$. Let $u,v\in B_r$ such that $\|u-v\|_C\leq\epsilon$ for every $x\in[0,a]$, using Eq. (3.1), we obtain

$$\begin{aligned}|(Tu)(x)-(Tv)(x)|&\leq I^{\alpha-\beta}\int_0^x|k(x,t)||(Gu)(t)-(Gv)(t)|dt\\&\leq\|k\|_C\ M\|u-v\|_C\ I^{\alpha-\beta}\int_0^x dt\\&\leq\frac{\|k\|_C\ Ma^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+1)}\|u-v\|_C.\end{aligned}$$

Hence we have

$$\|Tu-Tv\|_C\leq\frac{\|k\|_C\ Ma^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+1)}\|u-v\|_C,$$

thus T is continuous on B_r. Now we consider the set

$$T(B_r)=\{Tu:\ u\in B_r\}.$$

From (3.2) and (3.3) for every $y\in T(B_r)$ and $x\in[0,a]$, we have

$$|y(x)|=|Tu(x)|<r,$$

which from relation (2.1), we imply $\|Tu\|_C < r$ for $u \in B_r$. Therefore from Definition 2.2, we conclude that the set $T(B_r)$ is uniformly bounded.
Now assume that

$$f(s) := \int_0^s k(s,t)(Gu)(t)dt,$$
$$\|f\| = \max_{s\in[0,a]} |f(s)|.$$

Let $x_1, x_2 \in [0,a]$, without loss of generality, we can assume that $x_1 \le x_2$. For $u \in B_r$, we have

$$\begin{aligned}
&|(Tu)(x_1) - (Tu)(x_2)| \\
&\quad \le \frac{1}{\Gamma(\alpha-\beta)} \left| \int_0^{x_1} \left[(x_1-s)^{\alpha-\beta-1} - (x_2-s)^{\alpha-\beta-1} \right] f(s)ds \right| \\
&\quad + \frac{1}{\Gamma(\alpha-\beta)} \left| \int_{x_1}^{x_2} (x_2-s)^{\alpha-\beta-1} f(s)ds \right| + \left| I^{\alpha-\beta} g(x_1) - I^{\alpha-\beta} g(x_2) \right| \\
&\quad \le \frac{2\left(\|f\| + \|g\|\right)}{\Gamma(\alpha-\beta+1)} (x_2-x_1)^{\alpha-\beta}.
\end{aligned}$$

Therefore, the set $T(B_r)$ is equicontinuous. The Arzela-Ascoli theorem implies that $T(B_r)$ is a relatively compact subset of $C[0,a]$, and hence applying the Schauder's fixed-point theorem, we conclude that T has a fixed point in $C[0,a]$ or, equivalently, Eq. (1.1) has a continuous solution over $[0,a]$. □

Theorem 3.2. *(Uniqueness) Let hypothesis (h_1) - (h_3) hold. Then Eq. (1.1) has a unique solution $y \in C^{m-1}[0,a]$ such that $({}^C D^\alpha y)(x) \in C[0,a]$.*

Proof. Let us define the iterates of operator T as the standard from:

$$T^1 = T, \; T^i = TT^{i-1} \; (i \in \mathbb{N} - \{1\}).$$

Actually, for $u, v \in C[0,a]$ we have

$$|\left(T^i u\right)(x) - \left(T^i v\right)(x)| \le \frac{\left(M\|k\|\right)^i x^{i(\alpha-\beta+1)}}{\Gamma\left(i(\alpha-\beta+1)+1\right)} \|u-v\|. \tag{3.4}$$

In fact,

$$\begin{aligned}
|\left(Tu\right)(x) - \left(Tv\right)(x)| &\le I^{\alpha-\beta} \int_0^x |k(x,t)| \left|\left(Gu\right)(t) - \left(Gv\right)(t)\right| dt \\
&\le \frac{\left(M\|k\|\right) x^{(\alpha-\beta+1)}}{\Gamma\left((\alpha-\beta+1)+1\right)} \|u-v\|.
\end{aligned}$$

Therefore (3.4) is proved for $i = 1$.
Assuming by induction that relation (3.4) is valued for i, we obtain similarly

$$\begin{aligned}
|\left(T^{i+1}u\right)(x) - \left(T^{i+1}v\right)(x)| &= |\left(TT^i u\right)(x) - \left(TT^i v\right)(x)| \\
&\le I^{\alpha-\beta} \int_0^x |k(x,t)| \left|\left(GT^i u\right)(t) - \left(GT^i v\right)(t)\right| dt \\
&\le I^{\alpha-\beta} \int_0^x M\|k\| \left|\left(T^i u\right)(t) - \left(T^i v\right)(t)\right| dt \\
&\le \frac{\left(M\|k\|\right)^{i+1} x^{(i+1)(\alpha-\beta+1)}}{\Gamma\left((i+1)(\alpha-\beta+1)+1\right)} \|u-v\|.
\end{aligned}$$

So (3.4) is proved. Let us consider the series $\sum_{i=0}^{\infty}\omega_i$, with

$$\omega_i := \frac{(M\|k\|)^i\, x^{i(\alpha-\beta+1)}}{\Gamma\left(i(\alpha-\beta+1)+1\right)}.$$

It is clear that $\omega_i \geq 0$. Since the Gamma function is an increasing function for $i \geq 2$, thus we have

$$\frac{(M\|k\|)^i\, x^{i(\alpha-\beta+1)}}{\Gamma\left(i(\alpha-\beta+1)+1\right)} \leq \frac{(M\|k\|)^i\, a^{i(\alpha-\beta+1)}}{i!} := a_i, \tag{3.5}$$

so $a_i > 0$, and by the Ratio test, we can show that the series $\sum a_i$ converges. Also, by the comparison test and (3.5), we conclude that the series $\sum_{i=0}^{\infty}\omega_i$ converges. Therefore, by Theorem 2.4, the map T has a unique fixed point which is the unique continuous solution of Eq. (1.1). □

3.2. **Application.** Consider the following fractional integro-differential equation

$$\left({}^{C}D^{1.75}y\right)(x) = x^2e^{-2x} + \int_0^x \frac{tx^2}{1+x}\left(\frac{1}{4} + \int_0^t \tau\left({}^{C}D^{0.5}y\right)(\tau)d\tau\right)dt,\ x\in[0,1],$$

$$y^{(i)}(0) = y_i,\ i = 0,1. \tag{3.6}$$

This equation is a special case of Eq. (1.1), where

$$\alpha = 1.75,\ \beta = 0.5,\ m = 2,\ n = 1,\ a = 1,$$

$$g(x) = x^2e^{-2x},\ k(x,t) = \frac{tx^2}{1+x},\ x,t\in[0,1],$$

$$(Gu)(t) = \frac{1}{4} + \int_0^t \tau\ u(\tau)d\tau,\ t\in[0,1],\ u\in C[0,1].$$

It is easy to check that the function G satisfies conditions (h_1) and (h_2) with $M = 1$ and $\psi(r) = \frac{1}{4} + r$. By assumption $|y_i| \leq 1$ $(i = 0,1)$, we have $L = \frac{2}{\sqrt{\pi}}$. Finally, in order to verify assumption (h_3), the corresponding inequality has the form

$$\frac{2}{\sqrt{\pi}} + \frac{1}{\Gamma(2.25)}\left[\frac{1}{e^2} + \frac{1}{2}(\frac{1}{4} + r)\right] \leq r.$$

We can easily verify that $r_0 = \frac{5}{2}$ is a solution of the above inequality. Now applying Theorem 3.1, we infer that Eq. (3.6) has at least one solution $y(x)\in C^1[0,1]$ such that $({}^{C}D^{1.75}y)(x)\in C[0,1]$.

Conclusion

In this paper, we investigated existence and uniqueness of solution of nonlinear fractional integro-differential equations of the Hammerstein type in the Banach space. To this end, we employed of the equivalence of the main problem with a fractional integral equation, and we used the Schauder's fixed point theorem and the Banach fixed point theorem under some suitable conditions.

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