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Research Paper

A SURVEY ON MULTIPLICITY RESULTS FOR FRACTIONAL DIFFERENCE EQUATIONS AND VARIATIONAL METHOD

MOHSEN KHALEGHI MOGHADAM*

Abstract. In this paper, we deal with the existence and multiplicity solutions, for the following fractional discrete boundary-value problem

$$\begin{cases} {}_{T+1}\nabla_k^\alpha\left({}_k\nabla_0^\alpha(u(k))\right) + {}_k\nabla_0^\alpha\left({}_{T+1}\nabla_k^\alpha(u(k))\right) = \lambda f(k,u(k)), \quad k \in [1,T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases}$$

where $0 \le \alpha \le 1$ and ${}_0\nabla_k^\alpha$ is the left nabla discrete fractional difference and ${}_k\nabla_{T+1}^\alpha$ is the right nabla discrete fractional difference and $f : [1,T]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\lambda > 0$ is a parameter. The technical approach is based on the critical point theory and some local minimum theorems for differentiable functionals. Several examples are included to illustrate the main results.

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1. Introduction and Background

The first concepts of fractional nabla differences traces back to the works of Gray and Zhang[8]. Discrete fractional calculus with the nabla operator studied in [11]. Initial value problems in discrete fractional calculus considered in [10]. In [12] authors studied two-point boundary value problems for finite fractional difference equations. This kind of problems play a fundamental role in different fields of research, for example in biological, Atici and Şengül introduced and solved Gompertz fractional difference equation for tumor growth models [13].

We refer the reader to the recent monograph on the introduction to fractional nabla calculus [26]. Another well-known monograph is [39] that is devoted to the systematic and comprehensive exposition of classical and modern results in the theory of fractional integrals and derivatives and their applications. Also [41] is a new monograph that works for differential and integral equations and systems and for many theoretical and applied problems in mathematics, mathematical physics, probability and statistics, applied computer science and numerical methods.

It is well known that variational methods is an important tool to deal with the problems for differential and difference equations with boundary value conditions.

Variational methods for dealing with fractional difference equations with boundary value conditions have appeared in [7, 9]. More, recently, in [4, 5, 25, 27, 31, 33, 34, 38] by starting

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*Corresponding author.

from the seminal papers [14, 15], the existence and multiplicity of solutions for nonlinear discrete boundary value problems have been investigated by adopting variational methods.

There seems to be increasing interest in the existence of solutions to boundary value problems for finite difference equations with fractional difference operator during the last three decades.

The other important tool in the study of nonlinear difference equations is fixed point methods; see, for instance, [23, 28, 29] and references therein. Morse theory is also other tool in the study of nonlinear fractional differential equations [36].
In recent paper [30], for the first time, the authors showed that critical point theory is an effective approach to study the existence of weak solutions of fractional boundary value problems (FBVPs) of the form

$$\begin{cases} {}_tD_T^{\alpha}\left({}_0D_t^{\alpha}(u(k))\right) = \lambda f(k,u(k)), \quad a.e.\ t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

We note that due to the difficulty of creating a framework of suitable function spaces and variable functions for FBVPs, it is often not easy to apply critical point theory to study FBVPs. Recently, in [40, 32] authors have used the variational method to investigate the existence of weak solution of fractional equations.

The aim of this paper is to establish the existence of one non-trivial solution and two non-trivial solutions and three solutions, separately, for the following discrete boundary-value problem

$$\begin{cases} {}_k\nabla_{T+1}^{\alpha}\left({}_0\nabla_k^{\alpha}(u(k))\right) + {}_0\nabla_k^{\alpha}\left({}_k\nabla_{T+1}^{\alpha}(u(k))\right) = \lambda f(k,u(k)), \quad k \in [1,T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.2}$$

where $0 \leq \alpha \leq 1$ and ${}_0\nabla_k^{\alpha}$ is the nabla discrete fractional difference and ${}_k\nabla_{T+1}^{\alpha}$ is the nabla discrete fractional difference and $\nabla u(k) = u(k) - u(k-1)$ is the backward difference operator $f : [1,T]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$f(k,x) \geq 0, \quad \forall x \leq 0, \quad k \in [1,T],$$

$\lambda > 0$ is a parameter and $T \geq 2$ is fixed positive integer and $\mathbb{N}_1 = \{1,2,3,\cdots\}$ and ${}_T\mathbb{N} = \{\cdots T-2, T-1, T\}$ and $[1,T]_{\mathbb{N}_0}$ is the discrete set $\{1,2,\cdots,T-1,T\} = \mathbb{N}_1 \bigcap {}_T\mathbb{N}$. By a non-trivial solution of (1.2), we mean a function

$$u \in \{w : [0,T+1]_{\mathbb{N}_0} \to \mathbb{R} : w(0) = w(T+1) = 0, w = (w(1), w(2), ..., w(T))^{\dagger}\},$$

that satisfies the equation in (1.2) on $[1,T]_{\mathbb{N}_0}$.
In [25] the Riemann-Liouville fractional integrals of order β were applied to investigate the existence of at least three solutions of the fractional boundary value problem in the continuous case.
The rest of this paper is arranged as follows. In the following we point out a special case of one of the our main results that proves later. In section 2, we provide our main tools and some definitions and integration by parts for fractional difference theorem and in Section 3, we establish variational framework and provide the matric form of (1.2) and auxiliary inequalities and fundamental functional and lemmas. In Section 4 and Section 5, we investigate the existence and multiple solutions, respectively that contains several theorems and examples

and diagrams to illustrate the results.

In this paper, based on a local minimum theorem (Theorem 2.1) due to Bonanno, Candito and D'Aguì[17], we ensure an exact interval of the parameter λ, in which the problem (1.2) admits at least a non-trivial solution.

As an example, here, we point out the following special case of one of the our main results.

Theorem 1.1. *Let fixed $\alpha \in (0,1)$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then for any*

$$\lambda \in \left]0, \frac{\lambda_{\min}}{2T} \sup_{c>0} \frac{c^2}{\max_{|\xi|\le c} \int_0^\xi f(s)ds}\right[,$$

the problem

$$\begin{cases} {}_k\nabla^\alpha_{T+1}\left({}_0\nabla^\alpha_k(u(k))\right) + {}_0\nabla^\alpha_k\left({}_k\nabla^\alpha_{T+1}(u(k))\right) = \lambda f(u(k)), \quad k \in [1,T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases}$$

(1) has at least one non-trivial solution.
(2) has at least two solutions provided that $\liminf_{\xi\to\infty} \frac{f(\xi)}{\xi} = \infty$.

2. Preliminaries

In this paper, our main tools are the following local minimum theorems.
(H) Let $\Phi,\ \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functional with Φ coercive and

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Clearly, if $\Phi(\overline{x}) = 0$, then $\overline{x} = 0$.

Theorem 2.1. *(Bonanno, Candito and D'Aguì* [17, Theorem 3.3]*) Assume that (H) holds and let $r > 0$.*
Then, for each $\lambda \in \Lambda :=]0, \frac{r}{\sup_{\Phi^{-1}([0,r])}\Psi}[$, the function $I_\lambda = \Phi - \lambda\Psi$ admits at least a local minimum $\overline{u} \in X$ such that $\Phi(\overline{u}) < r$, $I_\lambda(\overline{u}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}([0,r])$ and $I'_\lambda(\overline{u}) = 0$.

Another local minimum theorem ensures the existence of a non-zero local minimum in the following.

Theorem 2.2. *(Bonanno, Candito and D'Aguì* [17, Theorem 3.4]*) Assume that (H) holds. In addition, suppose that there exist $r \in \mathbb{R}$ and $w \in X$, with $0 < \Phi(w) < r$, such that*

$$\frac{\sup_{\Phi^{-1}([0,r])}\Psi}{r} < \frac{\Psi(w)}{\Phi(w)}.$$

Then, for each $\lambda \in \Lambda_w :=]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi^{-1}([0,r])}\Psi}[$, the function $I_\lambda = \Phi - \lambda\Psi$ admits at least a local minimum $\overline{u} \in X$ such that $\overline{u} \ne 0$, $\Phi(\overline{u}) < r$, $I_\lambda(\overline{u}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}([0,r])$ and $I'_\lambda(\overline{u}) = 0$.

The third main tool is the following two critical points theorem.

Theorem 2.3. *(Bonanno* [16, Theorem 3.2]*) Let X be a real finite dimensional Banach space and let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is* *bounded from below and* $\Phi(0) = \Psi(0) = 0$. *Fix* $r > 0$ *such that* $\frac{\sup_{\Phi^{-1}(]-\infty,r[)} \Psi}{r} < +\infty$ *and, assume that for each*

$$\lambda \in \Lambda := \left]0, \frac{r}{\sup_{\Phi^{-1}(]-\infty,r[)} \Psi}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ *satisfies the (PS)-condition and it is unbounded from below.*

Then, for each $\lambda \in \Lambda$, *the function* I_λ *admits at least two distinct critical points.*

It is worth noticing that in the previous result, Theorem 2.3, one of the two critical points may be zero. But in the next result, Theorem 2.4, both of the two critical points can not be zero.

Theorem 2.4. *(Bonanno and D'Aguì* [19, Theorem 2.1]*) Let* X *be a real finite dimensional Banach space and let* $\Phi, \Psi : X \longrightarrow \mathbb{R}$ *be two continuously Gâteaux differentiable functionals such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there are* $r \in \mathbb{R}$ *and* $\tilde{u} \in X$, *with* $0 < \Phi(\tilde{u}) < r$, *such that*

$$\frac{\sup_{\Phi^{-1}(]-\infty,r])} \Psi}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2.1}$$

and, for each

$$\lambda \in \Lambda_{\tilde{u}} := \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{\Phi^{-1}(]-\infty,r])} \Psi}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ *satisfies the (PS)-condition and it is unbounded from below.*

Then, for each $\lambda \in \Lambda_{\tilde{u}}$, *the function* I_λ *admits at least two non-zero critical points* $u_{\lambda,1}$, $u_{\lambda,2}$ *such that* $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.

The forth main tool is the following three critical points theorem.

Theorem 2.5. *(Bonanno, Candito and D'Aguì* [17, Theorem 4.1]*) Assume that (H) holds and there exist* $r \in \mathbb{R}$ *and* $w \in X$, *with* $0 < r < \Phi(w)$, *such that*

(a_1) $\frac{\sup_{\Phi^{-1}([0,r])} \Psi}{r} < \frac{\Psi(w)}{\Phi(w)}$,

(a_2) *for each* $\lambda \in \Lambda := \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi^{-1}([0,r])} \Psi}\right[$, *the function* $I_\lambda = \Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda$, *the function* I_λ *admits at least three distinct critical points.*

Remark 2.6. Theorems 2.5 is the finite dimensional versions of [21, Theorem 3.6].

We refer to the paper [18] in which Theorem 2.1 and Theorem 2.2 have been successfully employed to the existence of positive results for a nonlinear parameter-depending algebraic system. We make reference to the paper [20] in which Theorem 2.1 has been successfully employed to the existence of at least one non-trivial solution for two-point boundary value problems. Also we refer to the paper [37] in which Theorems 2.3 and 2.4 has been successfully employed to the existence of at least two positive solutions to Kirchhoff-type fourth-order impulsive elastic beam equations. Also we speak of- the paper [22] in which Theorem 2.4 has been successfully employed to the existence of at least two positive solutions for a nonlinear parameter-depending algebraic system. We refer to the paper [35] in which Theorem 2.5 has been successfully employed to the existence of at least three solutions for a discrete anisotropic boundary value problem.
The following definitions will be helpful to our discuss.

Definition 2.7. [2] **(i)** Let m be a natural number, then the m rising factorial of t is written as

$$t^{\overline{m}} = \prod_{k=0}^{m-1}(t+k), \qquad t^{\overline{0}} = 1. \tag{2.2}$$

(ii) For any real number, the α rising function is increasing on $\mathbb{N}_0$ and

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \qquad \text{such that } t \in \mathbb{R}\backslash\{\cdots,-2,-1,0\}, 0^{\overline{\alpha}} = 0. \tag{2.3}$$

Definition 2.8. let f be defined on $\mathbb{N}_{a-1}\bigcap {}_{b+1}\mathbb{N}$, $a<b$, $\alpha \in (0,1)$, then the nabla discrete new (left Gerasimov-Caputo) fractional difference is defined by

$$\left({}_k^C\nabla_{a-1}^{\alpha} f\right)(k) = \frac{1}{\Gamma(1-\alpha)}\sum_{s=a}^{k}\nabla_s f(s)(k-\rho(s))^{\overline{-\alpha}}, \quad k \in \mathbb{N}_a \tag{2.4}$$

and the right Gerasimov-Caputo one by

$$\left({}_{b+1}^C\nabla_k^{\alpha} f\right)(k) = \frac{1}{\Gamma(1-\alpha)}\sum_{s=k}^{b}(-\Delta_s f)(s)(s-\rho(k))^{\overline{-\alpha}}, \quad k \in {}_b\mathbb{N}, \tag{2.5}$$

and in the left Riemann-Liouville sense by

$$\left({}_k^R\nabla_{a-1}^{\alpha} f\right)(k) = \frac{1}{\Gamma(1-\alpha)}\nabla_k\sum_{s=a}^{k} f(s)(k-\rho(s))^{\overline{-\alpha}}, \quad k \in \mathbb{N}_a, \tag{2.6}$$

$$= \frac{1}{\Gamma(-\alpha)}\sum_{s=a}^{k} f(s)(k-\rho(s))^{\overline{-\alpha-1}}, \quad k \in \mathbb{N}_a, \tag{2.7}$$

and the right Riemann-Liouville one by

$$\left({}_{b+1}^R\nabla_k^{\alpha} f\right)(k) = \frac{1}{\Gamma(1-\alpha)}(-\Delta_k)\sum_{s=k}^{b} f(s)(s-\rho(k))^{\overline{-\alpha}}, \quad k \in {}_b\mathbb{N}, \tag{2.8}$$

$$= \frac{1}{\Gamma(-\alpha)}\sum_{s=k}^{b} f(s)(s-\rho(k))^{\overline{-\alpha-1}}, \quad k \in {}_b\mathbb{N}, \tag{2.9}$$

where $\rho(k) = k-1$ be the backward jump operator.

For example, Let $f(k) = 1$ be defined on $\mathbb{N}_{a-1}\bigcap {}_{b+1}\mathbb{N}$, therefore from (2.4) and (2.5), we have [1]

$${}_{b+1}^C\nabla_k^{\alpha} 1 = {}_k^C\nabla_{a-1}^{\alpha} 1 = 0, \qquad k \in \mathbb{N}_a \bigcap {}_b\mathbb{N}. \tag{2.10}$$

The relation between the nabla left and right Gerasimov-Caputo and Riemann-Liouville fractional differences are as follow:

$$\left({}_k^C\nabla_{a-1}^{\alpha} f\right)(k) = \left({}_k^R\nabla_{a-1}^{\alpha} f\right)(k) - \frac{(k-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}f(a-1), \tag{2.11}$$

$$\left({}_{b+1}^C\nabla_k^{\alpha} f\right)(k) = \left({}_{b+1}^R\nabla_k^{\alpha} f\right)(k) - \frac{(b+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}f(b+1). \tag{2.12}$$

Thus by (2.10), (2.11) and (2.12), we have for any $k \in \mathbb{N}_a \bigcap {}_b\mathbb{N}$,

$$ {}^{R}_{b+1}\nabla^{\alpha}_{k} 1 = \frac{(b+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}, \quad {}^{R}_{k}\nabla^{\alpha}_{a-1} 1 = \frac{(k-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}. \tag{2.13} $$

Regarding the domains of the fractional type differences we observe:

(i) The nabla left fractional difference ${}_{a-1}\nabla^{\alpha}_{k}$ maps functions defined on ${}_{a-1}\mathbb{N}$ to functions defined on ${}_a\mathbb{N}$.

(ii) The nabla right fractional difference ${}_{k}\nabla^{\alpha}_{b+1}$ maps functions defined on ${}_{b+1}\mathbb{N}$ to functions defined on ${}_b\mathbb{N}$.

As in [6] one can show that, for $\alpha \to 0$, one has ${}_a\nabla^{\alpha}_{k}(f(k)) \to f(t)$ and for $\alpha \to 1$, one has ${}_a\nabla^{\alpha}_{k}(f(k)) \to \nabla f(t)$
we note that the nabla Riemann-Liouville and Gerasimov-Caputo fractional differences, for $0 < \alpha < 1$, coincide when f vanishes at the end points that is $f(a-1) = 0 = f(b+1)$ [1]. Indeed, when $0 < \alpha < 1$, those conclude from (2.11) and (2.12). So, for convenience, from now on we will use the symbol ∇^{α} instead of ${}^{R}\nabla^{\alpha}$ or ${}^{C}\nabla^{\alpha}$.
Now we present summation by parts formula in new discrete fractional calculus.

Theorem 2.9. *([3, Theorem 4.4] Integration by parts for fractional difference) For functions f and g defined on $\mathbb{N}_a \bigcap {}_b\mathbb{N}$, $a \equiv b$ (mod 1), and $0 < \alpha < 1$, one has*

$$ \sum_{k=a}^{b} f(k) \left({}_{k}\nabla^{\alpha}_{a-1} g\right)(k) = \sum_{k=a}^{b} g(k) \left({}_{b+1}\nabla^{\alpha}_{k} f\right)(k). \tag{2.14} $$

Similarly,

$$ \sum_{k=a}^{b} f(k) \left({}_{b+1}\nabla^{\alpha}_{k} g\right)(k) = \sum_{k=a}^{b} g(k) \left({}_{k}\nabla^{\alpha}_{a-1} f\right)(k). \tag{2.15} $$

3. Preliminary results

Now, we establish variational framework. Define the finite $T-$dimensional Hilbert space

$$ W := \{u : [0, T+1]_{\mathbb{N}_0} \to \mathbb{R} : u(0) = u(T+1) = 0, u = (u(1), u(2), ..., u(T))^{\dagger}\}, $$

which $u^{\dagger}$ denotes the transpose of u and W is equipped with the usual inner product and the norm

$$ \langle u, v \rangle = \sum_{k=1}^{T} u(k)v(k), \qquad \|u\|_2 := \left(\sum_{k=1}^{T} |u(k)|^2 \right)^{\frac{1}{2}}. $$

It is known that the following norm

$$ \|u\| = \left\{ \sum_{k=1}^{T} |\left({}_{k}\nabla^{\alpha}_{0} u\right)(k)|^2 + |\left({}_{T+1}\nabla^{\alpha}_{k} u\right)(k)|^2 \right\}^{\frac{1}{2}} $$

is an equivalent norm in W.

Next, observe by Definition 2.8 that, for $k \in [1,T]_{\mathbb{N}_0}$

$$\left({}_k\nabla_0^\alpha u\right)(k) = \nabla_k \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{k} u(s)(k-\rho(s))^{\overline{-\alpha}},$$

$$\left({}_{T+1}\nabla_k^\alpha u\right)(k) \quad = \quad (-\Delta_k) \frac{1}{\Gamma(1-\alpha)} \sum_{s=k}^{T} u(s)(s-\rho(k))^{\overline{-\alpha}}$$

we let

$$z(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{k} u(s)(k-\rho(s))^{\overline{-\alpha}}, \quad w(k) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=k}^{T} u(s)(s-\rho(k))^{\overline{-\alpha}}$$

$\widetilde{z}(k) = (\nabla_k z)(k) = z(k) - z(k-1), \quad \widetilde{w}(k) = (-\Delta_k w)(k) = w(k) - w(k+1),$
thus from (3.8), one can conclude that

$$\begin{aligned} \|u\|^2 &= \sum_{k=1}^{T} |\left({}_k\nabla_0^\alpha u\right)(k)|^2 + |\left({}_{T+1}\nabla_k^\alpha u\right)(k)|^2 \\ &= \sum_{k=1}^{T} |(\nabla_k z)(k)|^2 + |(-\Delta_k z)(k)|^2 \\ &= \sum_{k=1}^{T} |\widetilde{z}(k)|^2 + |\widetilde{w}(k)|^2 \\ &= \|\widetilde{z}\|_2^2 + \|\widetilde{w}\|_2^2, \end{aligned} \tag{3.1}$$

$$z := (z(1), z(2), ..., z(T))^\dagger, \quad w := (w(1), w(2), ..., w(T))^\dagger$$
$$\widetilde{z} := (\widetilde{z}(1), \widetilde{z}(2), ..., \widetilde{z}(T))^\dagger, \quad \widetilde{w} := (\widetilde{w}(1), \widetilde{w}(2), ..., \widetilde{w}(T))^\dagger$$

then
$z(0) = 0,$
$z(1) = u(1),$
$z(2) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{2} u(s)(2-\rho(s))^{\overline{-\alpha}} = (1-\alpha)u(1) + u(2),$
$z(3) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{3} u(s)(3-\rho(s))^{\overline{-\alpha}} = \frac{1}{2!}(2-\alpha)(1-\alpha)u(1) + (1-\alpha)u(2) + u(3),$
$\vdots$
$z(T) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{T} u(s)(T-\rho(s))^{\overline{-\alpha}} = \frac{1}{(T-1)!}(T-\alpha-1)(T-\alpha-2)\cdots(1-\alpha)u(1) + \frac{1}{(T-2)!}(T-\alpha-2)(T-\alpha-3)\cdots(1-\alpha)u(2) + \cdots + (1-\alpha)u(T-1) + u(T),$

and

$w(1) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{T} u(s)(s)^{\overline{-\alpha}} = u(1) + (1-\alpha)u(2) + \frac{1}{2!}(2-\alpha)(1-\alpha)u(3) \cdots + \frac{1}{(T-1)!}(T-\alpha-1)(T-\alpha-2)\cdots(1-\alpha)u(T),$
$w(2) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=2}^{T} u(s)(s-1)^{\overline{-\alpha}} = u(2) + (1-\alpha)u(3) + \frac{1}{2!}(2-\alpha)(1-\alpha)u(4) + \frac{1}{(T-2)!}(T-\alpha-2)(T-\alpha-3)\cdots(1-\alpha)u(T),$

$w(3) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=3}^{T} u(s)(s-2)^{\overline{-\alpha}} = u(3) + (1-\alpha)u(4) + \frac{1}{2!}(2-\alpha)(1-\alpha)u(5) + \frac{1}{(T-2)!}(T-\alpha-2)(T-\alpha-3)\cdots(1-\alpha)u(T)$,

$\vdots$

$w(T-1) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=T-1}^{T} u(s)(s-T+2)^{\overline{-\alpha}} = u(T-1) + (1-\alpha)u(T)$,

$w(T) = u(T)$,

hence, $z = Bu$, $\widetilde{z} = Dz$ and $w = B^{\dagger}u$, $\widetilde{w} = D^{\dagger}w$ where $\dagger$ denotes the transpose and

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ (1-\alpha) & 1 & 0 & \cdots & 0 \\ \frac{1}{2!}(2-\alpha)(1-\alpha) & (1-\alpha) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(T-\alpha-1)(T-\alpha-2)\cdots(1-\alpha)}{(T-1)!} & \frac{(T-\alpha-2)(T-\alpha-3)\cdots(1-\alpha)}{(T-2)!} & \cdots & \cdots & 1 \end{bmatrix}_{T\times T},$$

$$D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & -1 & 1 \end{bmatrix}_{T\times T}.$$

It is clear that $BD = DB$ and $B^{\dagger}D^{\dagger} = D^{\dagger}B^{\dagger}$, let

$$A = (DB)^{\dagger}DB, \qquad \widetilde{A} = DB(DB)^{\dagger}.$$

Hence, for all $u \in W$

$$u^{\dagger}Au = u^{\dagger}(DB)^{\dagger}DBu = u^{\dagger}B^{\dagger}D^{\dagger}DBu = z^{\dagger}D^{\dagger}Dz = \widetilde{z}^{\dagger}\widetilde{z} = \|\widetilde{z}\|_2^2, \tag{3.2}$$

$$u^{\dagger}\widetilde{A}u = u^{\dagger}DB(DB)^{\dagger}u = u^{\dagger}BDD^{\dagger}B^{\dagger}u = w^{\dagger}DD^{\dagger}w = \widetilde{w}^{\dagger}\widetilde{w} = \|\widetilde{w}\|_2^2, \tag{3.3}$$

Let $\mathbb{A} = A + \widetilde{A}$, thus

$$u^{\dagger}\mathbb{A}u = u^{\dagger}Au + u^{\dagger}\widetilde{A}u = \|\widetilde{z}\|_2^2 + \|\widetilde{w}\|_2^2,$$

therefore from (3.1) and (3.2) and (3.3), we have

$$\|u\|^2 = u^{\dagger}\mathbb{A}u, \tag{3.4}$$

Let $\lambda_{\min}$ and $\lambda_{\max}$ denote respectively the minimum and the maximum eigenvalues of $\mathbb{A}$, for any $u \in W$, we have

$$\lambda_{\min}\|u\|_2^2 < u^{\dagger}\mathbb{A}u < \lambda_{\max}\|u\|_2^2, \tag{3.5}$$

and then from (3.4),

$$\sqrt{\lambda_{\min}}\|u\|_2 < \|u\| < \sqrt{\lambda_{\max}}\|u\|_2. \tag{3.6}$$

Therefor from (3.6), $\|u\| \to +\infty$ if and only if $\|u\|_2 \to +\infty$.

thus the following statements hold

1) The matrix $\mathbb{A}$ is real symmetric matrixes.
2) The quadratic form of the matrix $\mathbb{A}$ is positive.
3) The matrixes $\mathbb{A}$ is positive definite matrixes.
4) All the eigenvalues of $\mathbb{A}$ is positive.
5) The eigenvalues of the matrix A are as the same as the matrix $\widetilde{A}$.

6) The eigenvectors of A and $\widetilde{A}$ are not the same.

Taking the definition $\mathbb{A}$ into account

$$(3.7)\qquad \left({}_{T+1}\nabla_k^{\alpha}\left({}_k\nabla_0^{\alpha}\right) + {}_k\nabla_0^{\alpha}\left({}_{T+1}\nabla_k^{\alpha}\right)\right)\begin{bmatrix} u(1) \\ u(2) \\ u(3) \\ \vdots \\ u(T) \end{bmatrix} = \mathbb{A}\begin{bmatrix} u(1) \\ u(2) \\ u(3) \\ \vdots \\ u(T) \end{bmatrix}.$$

Let $\Phi : W \to \mathbb{R}$ be the functional

$$(3.8)\qquad \Phi(u) := \frac{1}{2}\sum_{k=1}^{T} |\left({}_k\nabla_0^{\alpha}u\right)(k)|^2 + |\left({}_{T+1}\nabla_k^{\alpha}u\right)(k)|^2 = \frac{1}{2}u^{\dagger}\mathbb{A}u.$$

An easy computation ensures that Φ turns out to be of class C^1 on W and Gateaux differentiable with

$$\Phi'(u)(v) = \sum_{k=1}^{T}\left({}_k\nabla_0^{\alpha}(u(k))\right)\left({}_k\nabla_0^{\alpha}v(k)\right) + \left({}_{T+1}\nabla_k^{\alpha}(u(k))\right)\left({}_{T+1}\nabla_k^{\alpha}v(k)\right) = u^{\dagger}\mathbb{A}v,$$

for all $u, v \in W$. To study the problem (1.2), for every $\lambda > 0$, we consider the functional $I_\lambda : W \to \mathbb{R}$ defined by

$$(3.9)\qquad I_\lambda(u) := \Phi(u) - \lambda\Psi(u), \qquad \Psi := \sum_{k=1}^{T} F(k,u).$$

where $F(k,u) = \int_0^u f(k,t)dt$.

Lemma 3.1. *The function u be a critical point of I_λ in W, iff u be a solution of the problem* (1.2).

Proof. First, let $\overline{u}$ be a critical point of I_λ in W. Then by previous argument for all $v \in W$, $I'_\lambda(\overline{u})(v) = 0$ and $\overline{u}(0) = \overline{u}(T+1) = v(0) = v(T+1) = 0$. We applying the summation by parts formulas (2.14) and (2.15) in Theorem 2.9. Thus, by selecting $f(k) = \left({}_{a-1}\nabla_k^{\alpha}(\overline{u}(k))\right)$ and $g(k) = v(k)$ defined on $\mathbb{N}_1 \bigcap {}_T\mathbb{N}$ in (2.14) and selecting $f(k) = \left({}_k\nabla_{T+1}^{\alpha}(\overline{u}(k))\right)$ and $g(k) = v(k)$

defined on $\mathbb{N}_1 \bigcap {}_T\mathbb{N}$ in (2.15), one can conclude that

$$\begin{aligned}
0 &= I'_\lambda(\overline{u})(v)\\
&= \sum_{k=1}^{T} \left({}_0\nabla_k^\alpha(\overline{u}(k))\right)\left({}_0\nabla_k^\alpha v(k)\right) + \left({}_k\nabla_{T+1}^\alpha(\overline{u}(k))\right)\left({}_k\nabla_{T+1}^\alpha v(k)\right)\\
&- \lambda \sum_{k=a}^{b} \left[f(k,\overline{u}(k))\right] v(k)\\
&= \sum_{k=1}^{T} v(k)\left({}_k\nabla_{T+1}^\alpha\left({}_0\nabla_k^\alpha(\overline{u}(k))\right)\right) + \sum_{k=1}^{T} v(k)\left({}_0\nabla_k^\alpha\left({}_k\nabla_{T+1}^\alpha(\overline{u}(k))\right)\right)\\
&- \lambda \sum_{k=1}^{T} \left[f(k,\overline{u}(k))\right] v(k)\\
&= \sum_{k=1}^{T} v(k)\left\{\left({}_k\nabla_{T+1}^\alpha\left({}_0\nabla_k^\alpha(\overline{u}(k))\right)\right) + \left({}_0\nabla_k^\alpha\left({}_k\nabla_{T+1}^\alpha(\overline{u}(k))\right)\right)\right\}\\
&- \lambda \sum_{k=1}^{T} \left[f(k,\overline{u}(k))\right] v(k).
\end{aligned}$$

Bearing in mind $v \in W$ is arbitrary, one can conclude that

$$\left({}_k\nabla_{T+1}^\alpha\left({}_0\nabla_k^\alpha(\overline{u}(k))\right)\right) + \left({}_0\nabla_k^\alpha\left({}_k\nabla_{T+1}^\alpha(\overline{u}(k))\right)\right) - \lambda f(k,\overline{u}(k)) = 0,$$

for every $k \in [1,T]_{\mathbb{N}_0}$. Therefore, $\overline{u}$ is a solution of (1.2). Since $\overline{u}$ be arbitrary, we conclude that every critical point of the functional I_λ in W, is a solution of the problem (1.2). On the other hand, if $\overline{u}$ be a solution of (1.2), arguing backward, the proof is completed. □

Now we provide some lemmas used throughout the paper, which hold on the space W. In the sequel, we will use the following inequality

Lemma 3.2. *For every* $0 < \alpha < 1$ *and* $u \in W$, *we have*

$$\|u\|_\infty := \max_{k\in[1,T]} |u(k)| \le \|u\|_2, \tag{3.10}$$

Proof. Let $k \in \{1, 2, \cdots, T-1, T\}$ be arbitrary. It is clear that

$$|u(k)|^2 \le \|u\|_2^2.$$

Thus $|u(k)| \le \|u\|$ for any $k \in \{1, 2, \cdots, T-1, T\}$. So, the inequality (3.10) conclude. □

Lemma 3.3. *the functional* Φ *is coercive, i.e.* $\Phi(u) \to +\infty$ *as* $\|u\| \to +\infty$.

Proof. From (3.1) and (3.2) and (3.3), we have

$$\Phi(u) = \frac{1}{2}\|u\|^2. \tag{3.11}$$

Hence $\Phi(u) \to +\infty$ as $\|u\| \to +\infty$. □

4. Existence of a solution

Based on a local minimum theorem (Theorem 2.1), we ensure an exact interval of the parameter λ, in which the problem (1.2) admits at least a non-trivial solution.
Now, we present our main results. The first result gives the existence of one solution for each λ close to zero. Here we point out an immediate consequence of Theorem 2.1 as follows.

Theorem 4.1. *Let fixed $\alpha \in (0,1)$ and c as a fixed positive constant. Then for any*

$$\lambda \in \left]0, \frac{\left(\frac{\lambda_{\min}}{2}\right)c^2}{\sum_{k=1}^{T}\max_{|\xi|\le c} F(k,u(k))}\right[,$$

the problem (1.2) has at least one non-trivial solution $u_0 \in W$ such that $\|u_0\|_\infty < c$.

Proof. Our aim is to apply Theorem 2.1 to our problem. Thus, take $X = W$, and put Φ, Ψ and I_λ as in (3.8) and (3.9). Clearly, $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$ and by Lemma 3.3, one can conclude that (H) holds. By the similar arguing in [17], put $r = \left(\frac{\lambda_{\min}}{2}\right)c^2$, and for all $u \in W$ such that $0 \le \Phi(u) \le r$, taking (3.2) into account, one has $\max_{k\in[1,T]}|u(k)| \le \|u\|_2 \le \sqrt{\frac{2\Phi(u)}{\lambda_{\min}}} \le c$. Hence, for all $u \in W$ such that $0 \le \Phi(u) \le r$, one has $\max_{k\in[1,T]}|u(k)| \le c$. Therefore,

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}[0,r]}\Psi(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}[0,r]}\sum_{k=1}^{T}F(k,u(k))}{r}\\ &\le \frac{\sum_{k=1}^{T}\max_{|\xi|\le c}F(k,\xi)}{\left(\frac{\lambda_{\min}}{2}\right)c^2}.\end{aligned} \tag{4.1}$$

Therefore, owing to Theorem 2.1, for each

$$\lambda \in \left]0, \frac{\left(\frac{\lambda_{\min}}{2}\right)c^2}{\sum_{k=1}^{T}\max_{|\xi|\le c} F(k,\xi)}\right[\subset \Lambda,$$

the functional I_λ admits one critical point $\overline{u} \in W$ such that $\|\overline{u}\|_\infty < c$. Hence, the proof is complete. □

We now present an example to illustrate the result of Theorem 4.1.

Example 4.2. The problem

$$\begin{cases} {}_k\nabla_4^{0.5}\left({}_0\nabla_k^{0.5}(u(k))\right) + {}_0\nabla_k^{0.5}\left({}_k\nabla_4^{0.5}(u(k))\right) = \frac{3}{4}\lambda u(k)^2(\ln\frac{k+1}{k}), & k \in [1,3],\\ u(0) = u(4) = 0, \end{cases}$$

for every $\lambda \in]0, \frac{0.91}{5\ln 4}[$ has at least one solution $\overline{u}$ such that $\|\overline{u}\|_\infty < 10$. Indeed, $T = 3$ and $\alpha = 0.5$, so $\lambda_{\min} = \lambda_1 \simeq 0.91 < \lambda_2 \simeq 2.515 < 3.605 \simeq \lambda_{\max}$ where

$$B = \begin{bmatrix} 1 & 0 & 0\\ 0.5 & 1 & 0\\ \frac{3}{8} & \frac{1}{2} & 1\end{bmatrix},\ D = \begin{bmatrix} 1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix},\ \mathbb{A} = \begin{bmatrix} 2.265625 & -0.9375 & -0.25\\ -0.9375 & 2.5 & -0.9375\\ -0.25 & -0.9375 & 2.265625\end{bmatrix},$$

and by taking $c = 10$, then,

$$\frac{\left(\frac{\lambda_{\min}}{2}\right) c^2}{\sum_{k=1}^{T} \max_{|\xi|\leq c} F(k,\xi)} = \frac{0.91}{5\ln 4}.$$

Theorem 4.3. *Let fixed $\alpha \in (0,1)$ and $T \geq 2$ and $f : [1,T]\times\mathbb{R} \to \mathbb{R}$ be a continuous function. Then for any*

$$\lambda \in \left]0, \frac{\lambda_{\min}}{2} \sup_{c>0} \frac{c^2}{\sum_{k=1}^{T} \max_{|\xi|\leq c} F(k,u(k))}\right[,$$

the problem (1.2) has at least one solution.

Proof. Fix λ as in the conclusion and c such that $\lambda < \frac{\lambda_{\min}}{2} \frac{c^2}{\sum_{k=1}^{T} \max_{|\xi|\leq c} F(k,u(k))}$. Arguing as in the proof of Theorem 4.1 and putting $r = \left(\frac{\lambda_{\min}}{2}\right) c^2$, we obtain

$$\frac{\sup_{u\in\Phi^{-1}[0,r]} \Psi(u)}{r} \leq \frac{\sum_{k=1}^{T} \max_{|\xi|\leq c} F(k,\xi)}{\left(\frac{\lambda_{\min}}{2}\right) c^2} < +\infty.$$

Therefore, owing to Theorem 4.1, for each

$$\lambda \in \left]0, \frac{\left(\frac{\lambda_{\min}}{2}\right) c^2}{\sum_{k=1}^{T} \max_{|\xi|\leq c} F(k,\xi)}\right[,$$

the conclusion is achieved. □

Corollary 4.4. *The proof of Theorem 1.1(1) is a conclusion that follows from Theorem 4.3*

The second result ensures the existence of one non-trivial solution such that λ can not close to zero. Here we point out an immediate consequence of Theorem 2.2 as follows.

Theorem 4.5. *Let fixed $\alpha \in (0,1)$ and assume that there exist two positive constants c, d, with*

$$\frac{d^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} \left((k)^{\overline{-\alpha}}\right)^2 < \left(\frac{\lambda_{\min}}{2}\right) c^2, \tag{4.2}$$

such that

$$\frac{\sum_{k=1}^{T} \max_{|\xi|\leq c} F(k,\xi)}{\left(\frac{\lambda_{\min}}{2}\right) c^2} < \frac{\sum_{k=1}^{T} F(k,d)}{\frac{d^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} \left((k)^{\overline{-\alpha}}\right)^2}. \tag{4.3}$$

Then for any

$$\lambda \in \left] \frac{\frac{d^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} \left((k)^{\overline{-\alpha}}\right)^2}{\sum_{k=1}^{T} F(k,d)}, \frac{\left(\frac{\lambda_{\min}}{2}\right) c^2}{\sum_{k=1}^{T} \max_{|\xi|\leq c} F(k,u(k))}\right[,$$

the problem (1.2) has at least one non-trivial solution $\overline{u} \in W$ such that $\|\overline{u}\|_\infty < c$.

Proof. Our aim it to apply Theorem 2.2 to our problem. By the similar arguing in [17] and arguing as in the proof of Theorem 4.1 and putting $r = \left(\frac{\lambda_{\min}}{2}\right) c^2$, we obtain (4.2). Let

$$w(k) = \begin{cases} d & k \in [1,T]_{\mathbb{N}_0}, \\ 0 & k = 0, T+1. \end{cases} \tag{4.4}$$

Clearly $w \in W$. Since w vanishes at the end points that is $w(0) = 0 = w(T+1)$, thus its nabla Riemann-Liouville and Gerasimov-Caputo fractional differences coincide, hence for any $k \in \mathbb{N}_1 \bigcap {}_T\mathbb{N}$

$$\left({}_{T+1}\nabla_k^{\alpha} w\right)(k) = \left({}_{T+1}^{R}\nabla_k^{\alpha} w\right)(k) = \left({}_{T+1}^{C}\nabla_k^{\alpha} w\right)(k) = \frac{d(T+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)},$$

$$\left({}_{k}\nabla_0^{\alpha} w\right)(k) = \left({}_{k}^{R}\nabla_0^{\alpha} w\right)(k) = \left({}_{k}^{C}\nabla_0^{\alpha} w\right)(k) = \frac{d(k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}.$$

So, we have

$$\begin{aligned}
\Phi(w) &= \frac{1}{2}\sum_{k=1}^{T} |\left({}_{k}\nabla_0^{\alpha} w\right)(k)|^2 + |\left({}_{T+1}\nabla_k^{\alpha} w\right)(k)|^2 \\
&= \frac{1}{2}\sum_{k=1}^{T} |\frac{d(k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}|^2 + |\frac{d(T+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}|^2 \\
&= \frac{d^2}{2\left(\Gamma(1-\alpha)\right)^2}\sum_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 + |(T+1-k)^{\overline{-\alpha}}|^2 \\
&= \frac{d^2}{\left(\Gamma(1-\alpha)\right)^2}\sum_{k=1}^{T} |(k)^{\overline{-\alpha}}|^2 \\
&= \frac{d^2}{\left(\Gamma(1-\alpha)\right)^2}\sum_{k=1}^{T} \left((k)^{\overline{-\alpha}}\right)^2 > 0.
\end{aligned}$$

Therefore

$$\frac{\Psi(w)}{\Phi(w)} = \frac{\sum_{k=1}^{T} F(k, w(k))}{\frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2} = \frac{\sum_{k=1}^{T} F(k, d)}{\frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2}. \tag{4.5}$$

Hence, from (4.2), (4.1) and (4.5) and assumption (4.4) one has $0 < \Phi(w) < r$ and

$$\frac{\sup_{u\in\Phi^{-1}[0,r]}\Psi(u)}{r} < \frac{\Psi(w)}{\Phi(w)}.$$

Therefore, owing to Theorem 2.2, for each

$$\lambda \in \left]\frac{\frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2}{\sum_{k=1}^{T} F(k,d)}, \frac{\left(\frac{\lambda_{\min}}{2}\right)c^2}{\sum_{k=1}^{T}\max_{|\xi|\le c} F(k, u(k))}\right[\subset \Lambda_w,$$

the desired result concludes and the proof is complete. $\square$

We now present an example to illustrate the result of Theorem 4.5.

Example 4.6. The problem

$$\begin{cases} {}_{5}\nabla_{k}^{0.75}\left({}_{k}\nabla_{0}^{0.75}(u(k))\right)+{}_{k}\nabla_{0}^{0.75}\left({}_{5}\nabla_{k}^{0.75}(u(k))\right)=8\lambda\frac{u^3(k)sinh(u^4(k))}{cosh(u^4(k))^3}(\ln\frac{k+1}{k}), \quad k\in[1,4],\\ u(0)=u(5)=0,\end{cases}$$

for every $\lambda\in]1.18,15.84[$ has at least one non-trivial solution $\overline{u}$ such that $\|\overline{u}\|_\infty<10$. Indeed, $T=4$ and $\alpha=0.75$, and

$$B=\begin{bmatrix}1&0&0&0\\ \frac{1}{4}&1&0&0\\ \frac{5}{32}&\frac{1}{4}&1&0\\ \frac{15}{128}&\frac{5}{32}&\frac{1}{4}&1\end{bmatrix},$$

$$D=\begin{bmatrix}1&0&0&0\\ -1&1&0&0\\ 0&-1&1&0\\ 0&0&-1&1\end{bmatrix}, \mathbb{A}=\begin{bmatrix}42153/16384&-5841/4096&-81/512&-5/64\\ -5841/4096&3209/1024&-87/64&-81/512\\ -81/512&-87/64&3209/1024&-5841/4096\\ -5/64&-81/512&-5841/4096&42153/16384\end{bmatrix},$$

$$\lambda_{\max}=\lambda_1=117049/32768+15/32768\sqrt{11719633}\simeq 5.1392,$$

$$\lambda_2=117049/32768-15/32768\sqrt{11719633}\simeq 2.0050,$$

$$\lambda_3=69945/32768+1/32768\sqrt{2834119345}\simeq 3.7593$$

$$\lambda_{\min}=\lambda_4=69945/32768-1/32768\sqrt{2834119345}\simeq 0.5099,$$

and by taking $c=10$, $d=1$ then,

$$\frac{d^2}{(\Gamma(0.25))^2}\sum_{k=1}^{4}\left((k)^{\overline{-0.75}}\right)^2=1.100646972,$$

$$\left(\frac{\lambda_{\min}}{2}\right)c^2=\left(\frac{69945/32768-1/32768\sqrt{2834119345}}{2}\right)c^2=25.497,$$

thus the condition (4.2) holds. Put $f(k,x)=g(x)(\ln\frac{k+1}{k})=8\frac{x^3sinh(x^4)}{(cosh(x^4))^3}(\ln\frac{k+1}{k})$, (see the graph of $g(x)$ in the Figure 1), then $F(k,x)=\left(1-1/(cosh(x^4))^2\right)(\ln\frac{k+1}{k})$, so

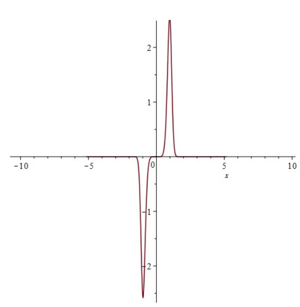

FIGURE 1. The graph of the function g

$$\begin{aligned}\sum_{k=1}^{4}\max_{|\xi|\le c}F(k,\xi) &= \left(1-1/(cosh(10))^2\right)\left(\sum_{k=1}^{4}\ln\frac{k+1}{k}\right)\\ &= \left(1-1/(cosh(10))^2\right)(\ln 5)=1.6094\end{aligned}$$

$$\sum_{k=1}^{4}F(k,d)=\left(1-1/(cosh(1))^2\right)(\ln 5)=0.93352$$

$$0.063\simeq\frac{\sum_{k=1}^{4}\max_{|\xi|\le c}F(k,\xi)}{\left(\frac{\lambda_{\min}}{2}\right)c^2}<\frac{\sum_{k=1}^{4}F(k,d)}{\frac{d^2}{(\Gamma(0.25))^2}\sum_{k=1}^{4}\left((k)^{\overline{-0.75}}\right)^2}\simeq 0.848.$$

5. Multiple solutions

By using multiple locale critical points theorem, one can show that problem (1.2) has multiple solutions, precisely it has at least two or three solutions.
Before providing the mentioned result we recall that a continuously differentiable functional I defined on a real Banach space X satisfies the Palais-Smale condition, the (PS) - condition for short, if every sequence $\{u_n\}$ such that $\{I(u_n)\}$ is bounded and $I^{'}(u_n)\longrightarrow 0$ in X^* as $n\longrightarrow\infty$ has a convergent subsequence.
In the following, we prove that the functional I_λ satisfies the Palais–Smale condition.
Put

$$L_\infty:=\min_{k\in[1,T]}\left(\liminf_{\xi\to\infty}\frac{F(k,\xi)}{|\xi|^2}\right),\quad and\quad \lambda^*:=\frac{\lambda_{max}}{2L_\infty}.$$

Lemma 5.1. *Let $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. If $L_\infty>0$ then the functional I_λ satisfies the Palais–Smale condition and it is unbounded from below for all $\lambda\in]\lambda^*,+\infty[$.*

Proof. Let us fix $\lambda>\lambda^*$. Assume that a sequence $\{u_n\}$ is such that $\{I_\lambda(u_n)\}$ is bounded and $I'_\lambda(u_n)\to 0$ as $n\to\infty$. Since W is finite dimensional, it is sufficient to show that $\{u_n\}$ is bounded. Put

$$u_n^+(k):=\max\{0,u_n(k)\}\text{ and }u_n^-(k):=\max\{0,-u_n(k)\}$$

for all $n\in\mathbb{N}$ and for all $k\in[0,T]$. By straightforward computation we can check that for all $n\in\mathbb{N}$ and $k,k'\in[1,T]$

$$u_n^+(k)u_n^-(k)=0\ \text{ and }\ u_n^+(k)u_n^-(k')\ge 0,\quad k\ne k'. \tag{5.1}$$

Due to be negative the elements of the matrix $\mathbb{A}$ except the original diameter, $(u_n^+)^\dagger\,\mathbb{A}u^-\le 0$ and $(u_n^-)^\dagger\,\mathbb{A}u^+\le 0$. So,

$$\begin{aligned}\Phi'(u_n)(u_n^-) &= \sum_{k=1}^{T}\left({}_k\nabla_0^\alpha(u_n(k))\right)\left({}_k\nabla_0^\alpha u_n^-(k)\right)+\left({}_{T+1}\nabla_k^\alpha(u_n(k))\right)\left({}_{T+1}\nabla_k^\alpha u_n^-(k)\right)\\ &= \Phi'\left(u_n^+-u_n^-\right)(u_n^-)=\Phi'\left(u_n^+\right)(u_n^-)-\Phi'\left(u_n^-\right)(u_n^-)\\ &= \left(u_n^+\right)^\dagger\mathbb{A}u_n^--\left(u_n^-\right)^\dagger\mathbb{A}u_n^-\\ &= \left(u_n^+\right)^\dagger\mathbb{A}u_n^--\|u_n^-\|^2\le-\|u_n^-\|^2\le 0,\end{aligned}$$

and

$$\Phi'(u_n^-)(u_n^-)=\left(u_n^-\right)^\dagger\mathbb{A}u_n^-=\|u_n^-\|^2\ge 0.$$

Moreover, by the assumption of f and by the definition of u_n^- we deduce that

$$\Psi'(u_n)(u_n^-) = \sum_{k=1}^{T} f(k, u_n(k))u_n^-(k) \geq 0,$$

and in a consequence

$$0 \leq \|u_n^-\|^2 \leq -\Phi'(u_n)(u_n^-) \leq -\Phi'(u_n)(u_n^-) + \lambda\Psi'(u_n)(u_n^-) = -I'_\lambda(u_n)(u_n^-)$$

for all $n \in \mathbb{N}$.

Since $I'_\lambda(u_n) \to 0$ as $n \to \infty$, hence $\|u_n^-\| \to 0$ as $n \to \infty$. Thus, there exists $N > 0$ such that $|(u_n^-)^\dagger \mathbb{A} u_n^-| < 1$ for any $n \geq N$ and according to the constant $\mathbb{A}$, then $0 \leq u_n^-(k) < L$ for any $k \in [1, T]_{\mathbb{N}_0}$ and for any $n \in \mathbb{N}$, where $L = \max\{1, \max_{k\in[1,T]_{\mathbb{N}_0}}\{u_1^-(k), u_2^-(k), u_3^-(k), ..., u_{N-1}^-(k)\}\}$. This means that $\{u_n^-\}$ is bounded, and $u_n(k) > -L$ for any $k \in [1, T]_{\mathbb{N}_0}$ and for any $n \in \mathbb{N}$.

Now, by the similar arguing in [24] and arguing by a contradiction, we will show that $\{u_n\}$ is bounded. Suppose that $\{u_n\}$ is unbounded. We may assume that $\|u_n\| \to \infty$. From $\liminf_{\xi\to\infty} \frac{F(k,\xi)}{|\xi|^2} \geq L_\infty$ for every $k \in [1, T]$ there exists $\delta_k > 0$ such that

$$F(k, \xi) > L_\infty |\xi|^2 \text{ for all } \xi > \delta_k.$$

Moreover, for all $\xi \in [-L, \delta_k]$ we have

$$\begin{aligned} F(k, \xi) &\geq \min_{\xi\in[-L,\delta_k]} F(k, \xi) \geq \min_{\xi\in[-L,\delta_k]} F(k, \xi) + L_\infty \left(|\xi|^2 - (\max\{\delta_k, L\})^2\right) \\ &\geq L_\infty |\xi|^2 - \max\{L_\infty(\max\{\delta_k, L\})^2 - \min_{\xi\in[-L,\delta_k]} F(k, \xi), 0\} \\ &= L_\infty |\xi|^2 - Q(k), \end{aligned}$$

where $Q(k) = \max\{L_\infty(\max\{\delta_k, L\})^2 - \min_{\xi\in[-L,\delta_k]} F(k, \xi), 0\}$ possesses only non-negative values for every $k \in [1, T]$. Eventually,

$$F(k, \xi) \geq L_\infty |\xi|^2 - Q(k), \quad \forall\xi \in (-L, +\infty), \quad \forall k \in [1, T].$$

Due to $u_n > -L$ for all $n \in \mathbb{N}$, we conclude that

$$F(k, u_n(k)) \geq L_\infty |u_n(k)|^2 - Q(k), \quad \forall n \in \mathbb{N}, \quad \forall k \in [1, T]. \tag{5.2}$$

By (5.2) and (3.5) we get

$$\Psi(u_n(k)) = \sum_{k=1}^{T} F(k, u_n(k)) \geq L_\infty\|u_n\|_2^2 - \overline{Q} \geq L_\infty \frac{\|u_n\|^2}{\lambda_{max}} - \overline{Q}, \tag{5.3}$$

where

$$\overline{Q} = \sum_{k=1}^{T} Q(k).$$

By (5.3) and (3.11) we infer that

$$\begin{aligned} I_\lambda(u_n) &= \Phi(u_n) - \lambda\Psi(u_n) \\ &\le \frac{\|u_n\|^2}{2} - \lambda L_\infty \frac{\|u_n\|^2}{\lambda_{max}} + \lambda\overline{Q} \\ &= \left(\frac{1}{2} - \lambda\frac{L_\infty}{\lambda_{max}}\right)\|u_n\|^2 + \lambda\overline{Q} \\ &= \frac{L_\infty}{\lambda_{max}}\left(\frac{\lambda_{max}}{2L_\infty} - \lambda\right)\|u_n\|^2 + \lambda\overline{Q} \\ &= \frac{L_\infty}{\lambda_{max}}(\lambda^* - \lambda)\|u_n\|^2 + \lambda\overline{Q} \end{aligned}$$

Since $\|u_n\| \to \infty$ and $\lambda^* - \lambda < 0$, so $I_\lambda(u_n) \to -\infty$ and this is an absurd. Hence I_λ satisfies the Palais–Smale condition for all $\lambda \in]\lambda^*, \infty[$.

It remains to establish that I_λ is unbounded from below. Let a sequence $\{u_n\}$ be such that $\{u_n^-\}$ is bounded and $\{u_n^+\}$ is unbounded and then $\|u_n\| \to \infty$. Arguing as before one has $I_\lambda(u_n) \to -\infty$ for all $\lambda \in]\lambda^*, +\infty[$ and the proof is complete. □

Now, we provide the main result of this section. Main results ensures the existence of two solution with requiring condition at infinity. Here we point out a consequence of Theorem 2.3 as follows.

Theorem 5.2. *Let fixed $\alpha \in (0,1)$ and $T \ge 2$ and $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and*

$$\liminf_{\xi\to\infty} \frac{F(k,\xi)}{|\xi|^2} = \infty.$$

Then for any

$$\lambda \in \Lambda = \left]0, \frac{\lambda_{\min}}{2} \sup_{c>0} \frac{c^2}{\sum_{k=1}^{T} \max_{|\xi|\le c} F(k,u(k))}\right[,$$

the problem (1.2) has at least two solutions.

Proof. Fix λ as in the conclusion and c such that $\lambda < \frac{\lambda_{\min}}{2}\frac{c^2}{\sum_{k=1}^{T}\max_{|\xi|\le c} F(k,u(k))}$. Owing to Lemma 5.8 and $L_\infty = +\infty$, the functional I_λ satisfies the Palais–Smale condition and it is unbounded from below. Arguing as in the proof of Theorem 4.1 and putting $r = \left(\frac{\lambda_{\min}}{2}\right)c^2$, we obtain

$$\frac{\sup_{u\in\Phi^{-1}]-\infty,r[}\Psi(u)}{r} \le \frac{\sum_{k=1}^{T}\max_{|\xi|\le c} F(k,\xi)}{\left(\frac{\lambda_{\min}}{2}\right)c^2} < +\infty.$$

Therefore, owing to Theorem 2.3, for each

$$\lambda \in \left]0, \frac{\left(\frac{\lambda_{\min}}{2}\right)c^2}{\sum_{k=1}^{T}\max_{|\xi|\le c} F(k,\xi)}\right[\subset \Lambda,$$

the conclusion is achieved. □

Corollary 5.3. *The proof of Theorem 1.1(2) is a conclusion that follows from Theorem 5.2.*

Example 5.4. For any
$$\lambda \in \left]0, \frac{\lambda_{\min}}{8T}\right[,$$
The problem
$$\begin{cases} {}_{T+1}\nabla_k^{0.75}\left({}_k\nabla_0^{0.75}(u(k))\right) + {}_k\nabla_0^{0.75}\left({}_{T+1}\nabla_k^{0.75}(u(k))\right) = \lambda[2u(k)(e^{u(k)})^2 \\ \qquad\qquad\qquad +2u(k)^2(e^{u(k)})^2 + 1], \quad k \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases}$$
has at least two solutions. In fact, that is enough to apply Theorem 5.2 to the function
$$f(s) = \begin{cases} 2s(e^s)^2 + 2s^2(e^s)^2 + 1, & s \geq 0\ , \\ 1, & s \leq 0. \end{cases}$$
taking into account that $\sup_{c>0} \frac{c^2}{\max_{|\xi|\leq c} F(\xi)} \geq \frac{1}{4}$, where $F(s) = s^2(e^s)^2 + s$, for $s > 0$ and $L_\infty = +\infty$.

Also we point out a consequence of Theorem 2.4 as follows.

Theorem 5.5. *Let fixed $\alpha \in (0,1)$ and $T \geq 2$ and $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $L_\infty > 0$ and assume that there are two positive constants c, d, with*
$$\frac{d^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} \left((k)^{\overline{-\alpha}}\right)^2 < \left(\frac{\lambda_{\min}}{2}\right) c^2, \tag{5.4}$$
such that
$$\frac{\sum_{k=1}^{T} \max_{|\xi|\leq c} F(k,\xi)}{\left(\frac{\lambda_{\min}}{2}\right) c^2} < \min\left\{\frac{\sum_{k=1}^{T} F(k,d)}{\frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2}, \frac{2L_\infty}{\lambda_{max}}\right\}. \tag{5.5}$$
Then for any
$$\lambda \in \left]\max\left\{\frac{\frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2}{\sum_{k=1}^{T} F(k,d)}, \frac{\lambda_{max}}{2L_\infty}\right\}, \frac{\left(\frac{\lambda_{\min}}{2}\right)c^2}{\sum_{k=1}^{T}\max_{|\xi|\leq c} F(k,u(k))}\right],$$
the problem (1.2) has at least two non-trivial solutions $u_1, u_2 \in W$ such that $\|u_1\|_\infty < c$.

Proof. Our aim is to apply Theorem 2.4 to our problem. Thus, take $X = W$, and put Φ, Ψ and I_λ as in (3.8) and (3.9). Clearly, $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$ and (H) holds. Arguing as in the proof of Theorem 4.1 and putting $r = \left(\frac{\lambda_{\min}}{2}\right) c^2$, we obtain (5.4) and
$$\frac{\sup_{u\in\Phi^{-1}]-\infty,r]} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})},$$
where $\bar{u}$ be as in (4.4). So the condition (2.1) holds. By (5.5) and Lemma 5.8 for any
$$\lambda \in \left]\max\left\{\frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\frac{\left((k)^{\overline{-\alpha}}\right)^2}{F(k,d)}, \lambda^*\right\}, \frac{\left(\frac{\lambda_{\min}}{2}\right)c^2}{\sum_{k=1}^{T}\max_{|\xi|\leq c} F(k,u(k))}\right] \subseteq [\lambda^*, +\infty)$$
the functional I_λ satisfies the Palais–Smale condition and it is unbounded from below. Therefore, owing to Theorem 2.4, the proof is complete. $\square$

We now present an example to illustrate the result of Theorem 5.5.

Example 5.6. Suppose that

$$f(x) = \frac{d}{dx}\left(x^2 \cosh\left(\frac{10e^{\frac{-1}{10\sinh^2(x)}}}{\sinh(x)}\right)\right), \quad U_0(u(x)) = \begin{cases} 0, & u(x) < 0 \\ 1, & u(x) \geq 0, \end{cases}$$

(see the graph of $f(x)$ in the Figure 2). The problem

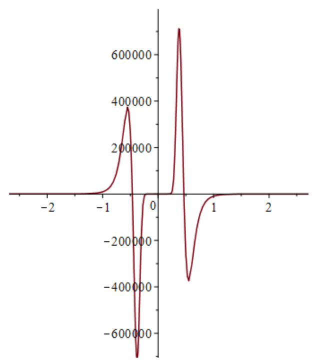

FIGURE 2. The graph of the function $f(x)$

$$\begin{cases} {}_5\nabla_k^{0.75}\left({}_k\nabla_0^{0.75}(u(k))\right) + {}_k\nabla_0^{0.75}\left({}_5\nabla_k^{0.75}(u(k))\right) = \lambda U_0(u(k)) f(u(k))(10^{-7k}), \\ k \in [1,4], \quad u(0) = u(5) = 0, \end{cases}$$

for every $\lambda \in]205.57, 1862.58[$ has at least two non-trivial solutions. Indeed, $T = 4$, $\alpha = 0.75$, B, D, $\mathbb{A}$, $\lambda_{\max}$ and $\lambda_{\min}$ are the same as in the Example 4.6. Now by taking $c = 1$, $d = 0.434$ then,

$$\frac{d^2}{(\Gamma(0.25))^2}\sum_{k=1}^{4}\left((k)^{\overline{-0.75}}\right)^2 = 0.2073134614,$$

$$\left(\frac{\lambda_{\min}}{2}\right)c^2 = \left(\frac{69945/32768 - 1/32768\sqrt{2834119345}}{2}\right)c^2 = 25.497,$$

thus the condition (5.4) holds. Let $F(k,\xi) = \int_0^\xi U_0(t)f(t)10^{-7k}dt$, then for $\xi > 0$, $F(k,\xi) = \int_0^\xi U_0(t)f(t)10^{-7k}dt = 10^{-7k}\xi^2\cosh\left(\frac{10e^{\frac{-1}{10\sinh^2(\xi)}}}{\sinh(\xi)}\right)$ and for $\xi \leq 0$, $F(k,\xi) = \int_0^\xi U_0(t)f(t)10^{-7k}dt = 0$, (see the graph of $F(x)$ in the Figure 3).so $L_\infty = 1$ and

$$\sum_{k=1}^{4}\max_{|\xi|\leq c} F(k,\xi) \quad = \quad \sum_{k=1}^{4} 10^{-7k} \times \begin{cases} F(0.46492), & c \geq 0.46492 \\ F(c), & c < 0.46492. \end{cases}$$

where $F(0.46492) = 78107.79252$ and

$$\sum_{k=1}^{4} F(k,d) = F(d)\sum_{k=1}^{4} 10^{-7k} = 0.007311944163,$$

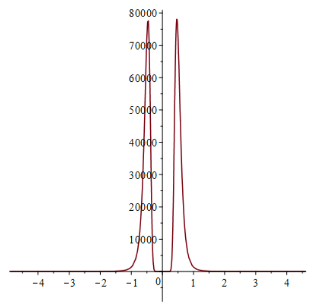

FIGURE 3. The graph of the function $F(x)$, where $F' = f$

and

$$\frac{\sum_{k=1}^{4} \max_{|\xi|\le c} F(k,\xi)}{\left(\frac{\lambda_{\min}}{2}\right) c^2} = 0.0005368869455,$$

$$\frac{\sum_{k=1}^{4} F(k,d)}{\frac{d^2}{(\Gamma(0.25))^2}\sum_{k=1}^{4}\left((k)^{\overline{-0.75}}\right)^2} = 0.03526999219,$$

$$\frac{2L_\infty}{\lambda_{max}} = 0.3891688210,$$

thus the condition (5.5) holds. Then for any

$$\begin{aligned}\lambda &\in \left]\max\left\{\frac{1}{0.03526999219}, \frac{1}{0.3891688210}\right\}, \frac{1}{0.0005368869455}\right] \\ &= \left]205.5663139, 1862.589524\right],\end{aligned}$$

the problem has at least two non-trivial solutions $u_{\lambda,1}, u_{\lambda,2} \in W$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

Example 5.7. The problem

$$\begin{cases} {}_5\nabla_k^{0.75}\left({}_k\nabla_0^{0.75}(u(k))\right) + {}_k\nabla_0^{0.75}\left({}_5\nabla_k^{0.75}(u(k))\right) = \\ \lambda U_0(u(k))\left[\frac{u^3(k) sinh(u^4(k))}{cosh(u^4(k))^3} + (\frac{u(k)}{2})^{10}\right], \quad k \in [1,4], \\ u(0) = u(5) = 0, \end{cases}$$

for every $\lambda \in]0.412, 0.45[$ has at least two non-trivial solutions. Indeed, $T = 4$, $\alpha = 0.75$, B, D, $\mathbb{A}$, $\lambda_{\max}$ and $\lambda_{\min}$ are as the same as in the Example 4.6. Now by taking $c = 3$, $d = 1.1$ then,

$$\frac{d^2}{(\Gamma(0.25))^2}\sum_{k=1}^{4}\left((k)^{\overline{-0.75}}\right)^2 = 1.331782836,$$

$$\left(\frac{\lambda_{\min}}{2}\right) c^2 = \left(\frac{69945/32768 - 1/32768\sqrt{2834119345}}{2}\right) c^2 = 2.2946,$$

thus the condition (5.4) holds. Put $f(k,x) = \frac{x^3\sinh(x^4)}{\cosh^3(x^4)} + (\frac{x}{3})^{10}$, (see the graph of $f(k,x)$ in the Figure 4), then $F(k,x) = \left(1 - 1/(\cosh^2(x^4)) + \frac{1}{11}\frac{x^{11}}{3^{10}}\right)$, so $L^\infty = +\infty$,

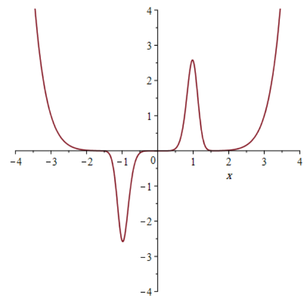

FIGURE 4. The graph of the function $f(x,k)$

$$\begin{aligned}\sum_{k=1}^{4}\max_{|\xi|\le c}F(k,\xi) &= \left(1-1/(cosh(3))^2+\frac{1}{11}\frac{3^{11}}{3^{10}}\right)\\ &= 5.0909\end{aligned}$$

$$\sum_{k=1}^{4}F(k,d)=\left(1-1/(cosh(1.1))^2+\frac{1}{11}\frac{1.1^{11}}{3^{10}}\right)(\ln 5)=3.228838252$$

$$2.218643772\simeq\frac{\sum_{k=1}^{4}\max_{|\xi|\le c}F(k,\xi)}{\left(\frac{\lambda_{\min}}{2}\right)c^2}<\frac{\sum_{k=1}^{4}F(k,d)}{\frac{d^2}{(\Gamma(0.25))^2}\sum_{k=1}^{4}\left((k)^{\overline{-0.75}}\right)^2}\simeq 2.424448014.$$

Now, we provide the existence of at least three solutions for $\lambda>0$ being in a certain interval.
In the following, we prove that the functional I_λ satisfies the coercive provided that the required assumption holds.

Put

$$L^\infty:=\max_{k\in[1,T]}\left(\limsup_{\xi\to\infty}\frac{F(k,\xi)}{|\xi|^2}\right).$$

Lemma 5.8. *The functional I_λ satisfies the coercive condition for any $\lambda\in]0,\frac{\lambda_{\min}}{2L^\infty}[$ provided that $0\le L^\infty<\infty$.*

Proof. The proof is similar to the proof of [17, Lemma 5.1(i)]. □

Corollary 5.9. *In Theorem 4.5, if one adds the condition $L^\infty=0$, then by Theorem 2.5, we give a result of at least three solutions for any*

$$\lambda\in\left(\frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\frac{\left((k)^{\overline{-\alpha}}\right)^2}{F(k,d)},\frac{\left(\frac{\lambda_{\min}}{2}\right)c^2}{\sum_{k=1}^{T}\max_{|\xi|\le c}F(k,u(k))}\right).$$

In the next example, the results should be more precise than Example 4.6, by Corollary 5.9.

Example 5.10. In the Example 4.6, it is clear that $L^\infty=0$, so by Corollary 5.9 one can obtain more solutions that is, the problem for every $\lambda\in]1.18,15.84[$ has at least three solutions

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Author details
Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, Sari, Iran
Email: mohsen.khaleghi@rocketmail.com and m.khaleghi@sanru.ac.ir

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(Mohsen Khaleghi Moghadam) Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, Sari, Iran.
Email address: m.khaleghi@sanru.ac.ir