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Research Paper

SOME PROPERTIES OF φ-CONVEX FUNCTIONS

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Abstract. In this paper, some basic results under various conditions for φ-convex functions are investigated. We prove that, under special hypotheses, every φ-convex function f is continuous on (a, b). Moreover, we introduce the notion of (φ, n)-convex functions.

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1. Introduction

The concept of convexity and its generalizations are important in applied mathematics, mathematical economics and optimization theory.

Recently, several extensions have been considered for the classical convex functions, such as pseudo-convex functions [3], strongly convex functions [4, 6], h-convex functions [8], strongly h-convex functions [1, 5], E-convex functions [9], E-quasi-convex functions [7].

Let I be an interval in $\mathbb{R}$. Then the mapping $f : I \longrightarrow \mathbb{R}$ is called convex if for all $x, y \in I$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y). \tag{1.1}$$

Moreover, f is called affine if the equality in (1.1) hold.

Motivated by (1.1), the following generalization of convex functions was introduced in [2].

Definition 1.1. Let $\varphi : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be a bifunction. A function $f : I \longrightarrow \mathbb{R}$ is called

(a) φ-convex, if for all $x, y \in I$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq f(y) + \lambda\varphi(f(x), f(y)),$$

(b) φ-affine, if

$$f(\lambda x + (1-\lambda)y) = f(y) + \lambda\varphi(f(x), f(y)), \quad \lambda, x, y \in \mathbb{R}.$$

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It is obvious that if $\varphi(x,y) = x - y$, then the above concepts coincide with the classical definitions of convex and affine functions.

By [2, Example 2.2], for a convex function f we may find another function φ different from $\varphi(x,y) = x - y$, such that f is φ-convex. Moreover, there is a φ-convex function f which is not convex.

In [2], the authors investigated the Jensen and Hermite-Hadamard type inequalities concerning for φ-convex functions.

In this paper, we generalize some basic results of convex functions to the φ-convex functions. As a consequence we show that, under certain condition, every φ-convex function f is continuous, and for each φ-affine map f from $\mathbb{R}$ to $\mathbb{R}$, the map $x \longrightarrow \varphi(f(x), f(0))$ is linear. Finally, we introduce the notion of (φ, n)-convex functions.

2. φ-convex function

We commence with the following characterization of φ-convex function.

Theorem 2.1. *Let $f : (a,b) \longrightarrow \mathbb{R}$ be a function. Then f is φ-convex if and only if for all $a < s < u < t < b$,*

$$\frac{f(u) - f(s)}{u - s} \leq \frac{\varphi(f(t), f(s))}{t - s}. \tag{2.1}$$

Proof. Assume that f is φ-convex and take $\alpha = \frac{u-s}{t-s}$. Then $0 < \alpha < 1$ and $u = \alpha t + (1-\alpha)s$. Therefore we get

$$f(u) = f(\alpha t + (1-\alpha)s) \leq f(s) + \alpha\varphi(f(t), f(s)).$$

Hence

$$f(u) - f(s) \leq \frac{u-s}{t-s}\varphi(f(t), f(s)),$$

which impolies that

$$\frac{f(u) - f(s)}{u - s} \leq \frac{\varphi(f(t), f(s))}{t - s}.$$

For the converse let the inequality (2.1) holds for all $a < s < u < t < b$. Let $x, y \in (a,b)$ and $x < y$. Then for each $x < z < y$ there exist $0 < \alpha < 1$ such that $z = \alpha y + (1-\alpha)x$. Since $a < x < z < y < b$, from (2.1) it follows that

$$\frac{f(\alpha y + (1-\alpha)x) - f(x)}{\alpha y + (1-\alpha)x - x} \leq \frac{\varphi(f(y), f(x))}{y - x}, \tag{2.2}$$

and hence

$$f(\alpha y + (1-\alpha)x) \leq f(x) + \alpha\varphi(f(y), f(x)),$$

for all $x, y \in (a,b)$. Therefore f is φ-convex. $\square$

Corollary 2.2. *The function $f : (a,b) \longrightarrow \mathbb{R}$ is convex if and only if*

$$\frac{f(u) - f(s)}{u - s} \leq \frac{f(t) - f(s)}{t - s},$$

for all $a < s < u < t < b$.

Lemma 2.3. *Let* $f : (a, b) \longrightarrow \mathbb{R}$ *be a* φ*-convex function. If* φ *is bounded, then so is* f.

Proof. Let $x, y \in (a, b)$ be fixed and $x < y$. Then for each $x < z < y$ there exist $0 < \alpha < 1$ such that $z = \alpha x + (1 - \alpha)y$. Put

$$M = max\{f(x), f(y) + \alpha\varphi(f(x), f(y))\}.$$

Then

$$f(z) = f(\alpha x + (1 - \alpha)y) \le f(y) + \alpha\varphi(f(x), f(y)) \le M. \tag{2.3}$$

If $r = z - \frac{x+y}{2}$, then

$$z = r + \frac{x + y}{2}, \quad w = \frac{x + y}{2} - r$$

lies in (x, y), and $\frac{x+y}{2} = \frac{1}{2}w + \frac{1}{2}z$. As f is φ-convex, we have

$$f(\frac{x + y}{2}) = f(\frac{1}{2}w + \frac{1}{2}z) \le f(z) + \frac{1}{2}\varphi(f(w), f(z)) \le f(z) + \frac{1}{2}K.$$

for all $x, y \in (a, b)$, where K is the upper bound of φ. Take $m = f(\frac{x+y}{2}) - \frac{1}{2}K$. Thus, for all $z \in (x, y)$,

$$m \le f(z) \le M.$$

Therefore f is bounded on (x, y) and so it is bounded on (a, b). □

It is known that every convex function $f : (a, b) \longrightarrow \mathbb{R}$ is continuous. Now we generalize it for φ-convex functions.

Theorem 2.4. *Let* $f : (a, b) \longrightarrow \mathbb{R}$ *be a* φ*-convex function. If* φ *is bounded, then* f *is continuous on* $[c, d] \subset (a, b)$.

Proof. Suppose $x, y \in [c, d]$ and $x < y$. Let p, q be fixed number such that

$$a < q < c \le x < y \le d < p < b.$$

Since f is φ-convex, by Theorem 2.1 we have

$$\frac{f(y) - f(x)}{y - x} \le \frac{\varphi(f(p), f(x))}{p - x} \le \frac{K}{p - d}.$$

On the other hand, from Lemma 2.3, $m \le f(z) \le M$, for each $z \in (a, b)$. Thus, $m - M \le f(y) - f(x)$ and hence

$$\frac{m - M}{p - q} \le \frac{f(y) - f(x)}{y - x}.$$

Now let $\alpha = |\frac{m-M}{p-q}|$, $\beta = |\frac{K}{p-d}|$ and take

$$N = max\{\alpha, \beta\}.$$

Then

$$|f(y) - f(x)| \le N|y - x|.$$

Consequently, f is continuous on $[c, d]$. □

The following example shows that every φ-convex function f on closed interval $[a,b]$ is not necessary continuous, even φ is bounded.

Example 2.5. Define $\varphi : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ by $\varphi(x,y) = x - y$ and $f : [0,1] \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & 0 \leq x < 1 \\ 2 & x = 1 \end{cases}$$

Then $|\varphi(x,y)| \leq 1$ and hence φ is bounded. The mapping f is a φ-convex, but it is not continuous.

Theorem 2.6. *Let $f : (a,b) \longrightarrow \mathbb{R}$ be a φ-convex function. If for all $x, y \in \mathbb{R}$,*

$$\varphi(x,y) \leq x - y,$$

then f is continuous.

Proof. It follows from Theorem 2.1 and assumption that

$$f(u) - f(s) \leq \frac{u-s}{v-s}\varphi(f(v), f(s)) \leq \frac{u-s}{v-s}(f(v) - f(s)),$$

for all $a < s < u < v < t < b$. Let $\alpha = \frac{v-s}{u-s}$ and $\beta = \frac{v-u}{u-s}$. Then

$$\alpha f(u) - \beta f(s) \leq f(v). \tag{2.4}$$

Since $\alpha = 1 + \beta$, so from (2.4), we get

$$f(u) + \beta(f(u) - f(s)) \leq f(v). \tag{2.5}$$

On the other hand, by Theorem 2.1,

$$f(v) \leq f(u) + \frac{v-u}{t-u}\varphi(f(t), f(u)). \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\begin{aligned} f(u) + \beta(f(u) - f(s)) &\leq f(v) \\ &\leq f(u) + \frac{v-u}{t-u}\varphi(f(t), f(u)) \\ &\leq f(u) + \frac{v-u}{t-u}(f(t) - f(u)). \end{aligned}$$

Let $\{v_n\}$ be a sequence such that $v_n \geq u$ and $v_n \longrightarrow u$. Then by the above inequality, we have

$$f(u) \leq \lim_n f(v_n) \leq f(u).$$

Consequently, $f(x) \longrightarrow f(u)$, where $x \longrightarrow u^+$. Similarly, if $x \longrightarrow u^-$, then $f(x) \longrightarrow f(u)$. Thus, f is continuous in u. This finishes the proof. $\square$

Corollary 2.7. *Suppose that $f : (a,b) \longrightarrow \mathbb{R}$ is a φ-convex function. If for all $x, y \in \mathbb{R}$,*

$$\varphi(x,y) \leq x - y,$$

then f satisfies the Lipschitz condition locally on (a,b).

Theorem 2.8. *Every φ-affine mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ is affine.*

Proof. Let f be φ-affine and $\mu \in \mathbb{R}$. Then

$$f(\mu) = f(\mu + (1-\mu)0) = f(0) + \mu\varphi(f(1), f(0)). \tag{2.7}$$

Thus, for all $\lambda, x, y \in \mathbb{R}$ by (2.7), we obtain

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= f(0) + \big(\lambda x + (1-\lambda)y\big)\varphi(f(1), f(0)) \\ &= \lambda f(0) + (1-\lambda)f(0) + \big(\lambda x + (1-\lambda)y\big)\varphi(f(1), f(0)) \\ &= \lambda[f(0) + x\varphi(f(1), f(0))] + (1-\lambda)[f(0) + y\varphi(f(1), f(0))] \\ &= \lambda f(x) + (1-\lambda)f(y). \end{aligned}$$

Therefore

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y),$$

and hence f is affine. □

Theorem 2.9. *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is φ-affine and φ is linear in first variable, then the map $T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $T(x) = \varphi(f(x), f(0))$ is linear.*

Proof. Suppose that f is φ-affine, then

$$f(\lambda x + (1-\lambda)y) = f(y) + \lambda\varphi(f(x), f(y)),$$

for all $\lambda, x, y \in \mathbb{R}$. Setting $x = y = 0$, we get $\varphi(f(0), f(0)) = 0$. On the other hand, by Theorem 2.8, f is affine. Now for all $\lambda, x \in \mathbb{R}$, we have

$$\begin{aligned} T(\lambda x) &= \varphi(f(\lambda x), f(0)) \\ &= \varphi\big(f(\lambda x + (1-\lambda)0), f(0)\big) \\ &= \varphi\big(\lambda f(x) + (1-\lambda)f(0), f(0)\big) \\ &= \lambda\varphi(f(x), f(0)) + (1-\lambda)\varphi(f(0), f(0) \\ &= \lambda\varphi(f(x), f(0)) \\ &= \lambda T(x). \end{aligned}$$

Therefore for all $\lambda, x \in \mathbb{R}$,

$$T(\lambda x) = \lambda T(x). \tag{2.8}$$

From (2.8), we have

$$\begin{aligned} \frac{1}{2}T(x+y) &= T(\frac{x+y}{2}) \\ &= \varphi(f(\frac{x+y}{2}), f(0)) \\ &= \varphi\big(\frac{1}{2}f(x) + \frac{1}{2}f(y), f(0)\big) \\ &= \frac{1}{2}\varphi(f(x), f(0)) + \frac{1}{2}\varphi(f(y), f(0)) \\ &= \frac{1}{2}T(x) + \frac{1}{2}T(y). \end{aligned}$$

Thus, $T(x+y) = T(x) + T(y)$ and hence T is additive. Consequently, T is linear. □

We mention that in Theorem 2.9, if the assumption φ-affine is replace by φ-convex, then the conclusion is not true, in general. For example let $f(x) = x^2$ and $\varphi(x,y) = x(1+2y)$, for all $x, y \geq 0$. Then φ is linear in first variable, f is φ-convex but it is not φ-affine. However, $T(x) = \varphi(f(x), f(0))$ is not linear.

Corollary 2.10. *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is affine, then $T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $T(x) = f(x) - f(0)$ is linear.*

Definition 2.11. Let $n \in \mathbb{N}$. We say that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is (φ, n)-convex, if

$$f(\lambda x^n + (1-\lambda)y^n) \leq f(y^n) + \lambda\varphi(f(x^n), f(y^n)),$$

for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$.

Clearly, $(\varphi, 1)$-convex function is φ-convex and every φ-convex function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is (φ, n)-convex, but the converse is fails, in general. The following example illustrate this fact.

Example 2.12. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x$ and consider $\varphi(x,y) = x + y$. Then for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$ we have

$$\begin{aligned} f(\lambda x^2 + (1-\lambda)y^2) &= \lambda x^2 + (1-\lambda)y^2 \\ &= y^2 + \lambda(x^2 - y^2) \\ &\leq y^2 + \lambda(x^2 + y^2) \\ &= f(y^2) + \lambda\varphi(f(x^2), f(y^2)). \end{aligned}$$

Thus, f is $(\varphi, 2)$-convex function, but it is not φ-convex.

Proposition 2.13. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a (φ, n)-convex, and let A be a convex subset of $\{x^n : x \in \mathbb{R}\}$. Then the restriction of f to A is φ-convex.*

Proof. Let $a, b \in A$, then there exists $x, y \in \mathbb{R}$ such that $a = x^n$ and $b = y^n$. Suppose that g is the restriction of f to A. Then

$$\begin{aligned} g(\lambda a + (1-\lambda)b) &= f(\lambda a + (1-\lambda)b) \\ &= f(\lambda x^n + (1-\lambda)y^n) \\ &\leq f(y^n) + \lambda\varphi(f(x^n), f(y^n)) \\ &= f(b) + \lambda\varphi(f(a), f(b)) \\ &= g(b) + \lambda\varphi(g(a), g(b)). \end{aligned}$$

Therefore g is φ-convex on A. □

Corollary 2.14. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a (φ, n)-convex. Then the restriction of f to $[0,1]$ is φ-convex.*

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