Mathematical Analysis & Convex Optimization
Vol. 2 (2021), No. 1, 63–69
https:\\ maco.lu.ac.ir
DOI: 10.52547/maco.2.1.6
Research Paper

AN EXTENSION OF THE INTERPOLATION THEOREM

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Abstract. In this paper we prove the Riesz-Thorian interpolation theorem for weighted Orlicz and weighted Morrey Spaces.

MSC(2010):46E30.
Keywords: Interpolation Theorem, Weighted Orlicz space, Young function, Weighted Morrey space.

1. Introduction and Preliminaries

Orlicz and Morrey spaces are two important generalizations of the usual Lebesgue spaces which so many research papers are based on them in the last decade; see the below two subsections for definition and some references of the weighted ones. Recently, the Riesz-Thorin interpolation theorem was proved in setting of Lebesgue-Morrey spaces in [15]; see [1] as a monograph. In this work, by a similar method, we give an extension of this theorem in setting of (weighted) Orlicz and Morrey spaces.

1.1. **Weighted Morrey Spaces.** For each $a \in \mathbb{R}^n$ and $t > 0$, the set $\{a+y \,:\, a \in \mathbb{R}^n, y \in [0,t]^n\}$ is called a cube in $\mathbb{R}^n$. Let $p \in [1,\infty)$ and $\lambda \in [0,1]$. Then, the *Morrey norm* is defined by

$$\|f\|_{\mathcal{M}^{p,\lambda}} := \sup\left\{ |Q|^{\frac{-\lambda}{p}} \|f\|_{L^p(Q)} \,:\, Q \text{ is a cube in } \mathbb{R}^n \right\},$$

for all measurable function $f : \mathbb{R}^n \to \mathbb{C}$. Then, the set of all complex-valued measurable functions f on $\mathbb{R}^n$ with $\|f\|_{\mathcal{M}^{p,\lambda}} < \infty$ is denoted by $\mathcal{M}^{p,\lambda}$ and called a *Morrey space*. Morrey Spaces are generalization of Lebesgue spaces. In fact, for each $p \geq 1$ we have $\mathcal{M}^{p,1} = L^p(\mathbb{R}^n)$. These spaces were initiated by C.B. Morrey in [3] while he was investigating elliptic differential equations, and then refined by Peetre [6]; see [9, 2, 10] as some recent works on this field.

Let $w : \mathbb{R}^n \to (0,\infty)$ be a measurable function. For each measurable function $f : \mathbb{R}^n \to \mathbb{C}$ we denote

$$\|f\|_{(p,\lambda,w)} := \|wf\|_{\mathcal{M}^{p,\lambda}}.$$

Date: Received: December 6, 2020, Accepted: May 22, 2021.
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The set of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ with $\|f\|_{(p,\lambda,w)} < \infty$ is denoted by $\mathcal{M}_w^{p,\lambda}$ and is called the *weighted Morrey space*. Simply, we put

$$\|f\|_{Q,p,w} := \|fw\|_{L^p(Q)},$$

where Q is a cube in $\mathbb{R}^n$.

1.2. **Weighted Orlicz Spaces.** The books [7, 8] are two main monographs for Orlicz spaces. For giving the definition of an Orlicz space, one needs to recall Young functions. A convex even function $\Phi : \mathbb{R} \to [0, \infty)$ is called a *Young function* if $\Phi(0) = \lim_{x\to 0} \Phi(x) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$. We say that a Young function Φ satisfies Δ_2-*condition* (and write $\Phi \in \Delta_2$) if for some constants $c > 0$ and $x_0 \geq 0$,

$$\Phi(2x) \leq c\,\Phi(x), \quad (x \geq x_0).$$

A continuous Young function $\Phi : \mathbb{R} \to [0, \infty)$ is called a *nice Young function* (or simply *N-function*) if $\lim_{x\to 0} \Phi(x)/x = 0$, $\lim_{x\to\infty} \Phi(x)/x = \infty$, and $\Phi(x) = 0$ implies that $x = 0$.

The *complementary* of a Young function Φ is defined by

$$\Psi(x) := \sup\{y|x| - \Phi(y) : y \geq 0\}, \quad (x \in \mathbb{R}).$$

In this case, (Φ, Ψ) is called a *complementary pair*.

In sequel $(\mathcal{X}, \mathcal{A}, \mu)$ would be a measure space, and we assume that the non-negative measure μ has *the finite subset property* i.e. for each $E \in \mathcal{A}$ with $\mu(E) > 0$, there exists a set $F \in \mathcal{A}$ such that $F \subseteq E$ and $0 < \mu(F) < \infty$ (see [7, page 46]). For each measurable function $f : \mathcal{X} \to \mathbb{C}$ we denote

$$\|f\|_\Phi := \inf\left\{\lambda > 0 : \int_{\mathcal{X}} \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1\right\}.$$

Then, the set of all measurable functions $f : \mathcal{X} \to \mathbb{C}$ with $\|f\|_\Phi < \infty$ is denoted by $L^\Phi(\mathcal{X})$ and is called an *Orlicz space*. Since, by our assumption, μ has the finite subset property, $L^\Phi(\mathcal{X})$ is a complete normed space [7]. For each $1 < p < \infty$, the function Φ_p defined by $\Phi_p(x) := |x|^p$ for all $x \in \mathbb{R}$, is a Young function and the Orlicz space $L^{\Phi_p}(\mathcal{X})$ is same as the usual Lebesgue space $L^p(\mathcal{X})$. Orlicz spaces, as extensions of Lebesgue spaces, have been studied in several recent decades; see for example [4, 5, 11, 12, 13, 14] as some recent works regarding Orlicz spaces in the context of locally compact groups and hypergroups.

Any measurable function $w : \mathcal{X} \to (0, \infty)$ is called a *weight* on $\mathcal{X}$, and we write $w^{-1} := \frac{1}{w}$. The space of all measurable functions f on $\mathcal{X}$ such that $wf \in L^\Phi(\mathcal{X})$ is called the *weighted Orlicz space* and is denoted by $L_w^\Phi(\mathcal{X})$. For each $f \in L_w^\Phi(\mathcal{X})$ we put $\|f\|_{\Phi,w} := \|wf\|_\Phi$. Then, $(L_w^\Phi(\mathcal{X}), \|\cdot\|_{\Phi,w})$ is also a Banach space. If $\Phi \in \Delta_2$, then the dual of the Banach space $L_w^\Phi(\mathcal{X})$ equals $L_{w^{-1}}^\Psi(\mathcal{X})$ (see [4]) via the duality formula

$$\langle f, g\rangle = \int_{\mathcal{X}} f(x)g(x)\,d\mu(x).$$

2. Main Results

In this section, we give Riesz-Thorian interpolation theorem for weighted Orlicz and weighted Morrey spaces. First we recall the following concept from [7, Chapter VI].

Definition 2.1. Let (Φ_0, Φ_1) be a pair of Young functions and fix a number $0 < \theta < 1$. Then, the corresponding intermediate function Φ_θ is defined by

$$\Phi_\theta^{-1} := (\Phi_0^{-1})^{1-\theta} \, (\Phi_1^{-1})^{\theta}. \tag{2.1}$$

Now, we recall the following lemma from [7, Proposition 4, Chapter VI] which plays a key role in the proof of the main result of this paper.

Lemma 2.2 (Three-Line Theorem)**.** *Let F be a bounded and continuous finction on $\{z \in \mathbb{C} : 0 \le \mathrm{Re} z \le 1\}$ and analytic on $\{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}$. Let $M_0, M_1 > 0$ be constant numbers such that*

$$|F(it)| \le M_0, \quad |F(1+it)| \le M_1, \quad (-\infty < t < \infty).$$

Then, for each $0 < \theta < 1$ we have

$$|F(\theta + it)| \le M_0^{1-\theta} M_1^{\theta}, \quad (-\infty < t < \infty).$$

In the next theorem, for each $p > 0$, we assume that $1/p + 1/p' = 1$.

Theorem 2.3. *Assume that $(\Phi_i, \Psi_i) \quad (i = 0, 1)$ are complimentary pairs of N-functions such that $\Phi_i \in \Delta_2$ for $i = 0, 1$. Let $1 \le p_i < \infty,\ 0 \le \lambda_i \le 1$ $(i = 0, 1)$, and $0 < \theta < 1$ be a fixed number. Let v_0 and v_1 be weight functions on $\mathcal{X}$, and w_0 and w_1 be weight functions on $\mathbb{R}^n$. Let the mappings*

$$k : \mathbb{R}^n \times \mathbb{C} \to \mathbb{C} \quad \textit{and} \quad k' : \mathcal{X} \times \mathbb{C} \to \mathbb{C}$$

satisfy the following properties:

(1) *for each $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $|k(y, it)| \le w_0(y)$ and $|k(y, 1 + it)| \le w_1(y)$.*
(2) *for each $x \in \mathcal{X}$ and $t \in \mathbb{R}$, $|k'(x, it)| \, v_0(x) \le 1$ and $|k'(x, 1 + it)| \, v_1(x) \le 1$.*
(3) *for each $x \in \mathcal{X}$ and $y \in \mathbb{R}^n$, the mappings $k(x, \cdot)$ and $k'(y, \cdot)$ are analytic. Also, for each $z \in \mathbb{C}$, the mappings $k(\cdot, z)$ and $k'(\cdot, z)$ are measurable.*

Let

$$w_\theta(y) := k(y, \theta)^{-p'_\theta} \quad \textit{and} \quad v_\theta(x) := \frac{1}{k'(x, \theta)} \qquad (x \in \mathcal{X},\ y \in \mathbb{R}^n).$$

Assume that for each $f \in L^{\Phi_0}_{v_0}(\mathcal{X})$ and $g \in L^{\Phi_1}_{v_1}(\mathcal{X})$,

$$\|T(f)\|_{(p_0, \lambda_0, w_0)} \le M_0 \, \|f\|_{\Phi_0, v_0}. \tag{2.2}$$

and

$$\|T(g)\|_{(p_1, \lambda_1, w_1)} \le M_1 \, \|g\|_{\Phi_1, v_1}. \tag{2.3}$$

Then,

$$\|Tf\|_{(p_\theta, \lambda_\theta, w_\theta)} \le M_0^{1-\theta} M_1^{\theta} \, \|f\|_{\Phi_\theta, v_\theta},$$

for all $f \in L^{\Phi_\theta}_{v_\theta}(\mathcal{X})$*, where* Φ_θ *is the intermediate function corresponding to* Φ_0 *and* Φ_1*, and*

$$p_\theta := \left((1-\theta)p_0^{-1} + \theta p_1^{-1}\right)^{-1}, \qquad \lambda_\theta := (1-\theta)\lambda_0 p_\theta p_0^{-1} + \theta\lambda_1 p_\theta p_1^{-1}. \tag{2.4}$$

Proof. For each complex number z put $\operatorname{sgn}(z) = \frac{z}{|z|}$ if $z \neq 0$, and $\operatorname{sgn}(z) = 0$ of $z = 0$. Let f be a simple function on $\mathcal{X}$ with $\|f\|_{\Phi_\theta, v_\theta} = 1$. Define

$$A(x,z) := \operatorname{sgn}(f(x)) \cdot [\Phi_0^{-1}(\Phi_\theta(|f(x)|v_\theta(x)))]^{1-z} \cdot [\Phi_1^{-1}(\Phi_\theta(|f(x)|v_\theta(x)))]^{z} \cdot k'(x,z)$$

for all $x \in \mathcal{X}$ and $z \in \mathbb{C}$. Fix a cube Q in $\mathbb{R}^n$. Let g be a simple function on $\mathbb{R}^n$ with $\|g\|_{Q,p'_\theta,w_\theta^{-1}} = 1$. Define

$$B(y,z) := \operatorname{sgn}(g(y))\, |g(y)|^{\frac{p'_\theta}{p'_z}}\, (w_\theta(y))^{\frac{1}{p'_z}}\, k(y,z)$$

for all $y \in \mathbb{R}^n$ and $z \in \mathbb{C}$, where

$$p_z := \left((1-z)p_0^{-1} + z p_1^{-1}\right)^{-1}, \qquad \lambda_z := (1-z)\lambda_0 p_z p_0^{-1} + z\lambda_1 p_z p_1^{-1}.$$

Then, for all $t \in \mathbb{R}$,

$$\begin{aligned}\int_Q \left(|B(y,it)|\, w_0^{-1}(y)\right)^{p'_0} dy &= \int_Q \left(\left((|g(y)| w_\theta^{-1}(y))^{p'_\theta} \right)^{\frac{1}{p'_0}} \cdot |k(y,it)|\, w_0^{-1}(y) \right)^{p'_0} dy \\ &\leq \int_Q (|g(y)| w_\theta^{-1}(y))^{p'_\theta}\, dy \\ &= \|g\|^{p'_\theta}_{Q,p'_\theta,w_\theta^{-1}} \leq 1.\end{aligned}$$

This implies that

$$\|B(\cdot, it)\|_{Q,p'_0,w_0^{-1}} \leq 1. \tag{2.5}$$

Similarly, for all $t \in \mathbb{R}$ we have

$$\begin{aligned}\int_Q \left(|B(y,1+it)|\, w_1^{-1}(y)\right)^{p'_1} dy &= \int_Q \left(\left((|g(y)| w_\theta^{-1}(y))^{p'_\theta} \right)^{\frac{1}{p'_1}} \cdot |k(y,it)|\, w_1^{-1}(y) \right)^{p'_1} dy \\ &\leq \int_Q (|g(y)| w_\theta^{-1}(y))^{p'_\theta}\, dy \\ &= \|g\|^{p'_\theta}_{Q,p'_\theta,w_\theta^{-1}} \leq 1,\end{aligned}$$

and so,

$$\|B(\cdot, 1+it)\|_{Q,p'_1,w_1^{-1}} \leq 1. \tag{2.6}$$

Also, for all $t \in \mathbb{R}$,

$$\begin{aligned}\int_{\mathcal{X}} \Phi_0\left(|A(x,it)|\, v_0(x)\right)\, d\mu(x) &= \int_{\mathcal{X}} \Phi_0\left(\Phi_0^{-1}(\Phi_\theta(|f(x)|v_\theta(x)))\,\cdot k'(x,it)\, v_0(x)\right)\, d\mu(y)\\ &\le \int_{\mathcal{X}} \Phi_0\left(\Phi_0^{-1}(\Phi_\theta(|f(x)|v_\theta(x)))\right)\, d\mu(y)\\ &\le \int_{\mathcal{X}} \Phi_\theta(|f(x)|v_\theta(x))\, d\mu(y)\\ &\le 1.\end{aligned}$$

This implies that

$$\|A(\cdot,it)\|_{\Phi_0,v_0} \le 1. \tag{2.7}$$

Similarly, by the hypothesis one can see that

$$\|A(\cdot,1+it)\|_{\Phi_1,v_1} \le 1 \tag{2.8}$$

for all $t \in \mathbb{R}$, since

$$\begin{aligned}\int_{\mathcal{X}} \Phi_1\left(|A(x,1+it)|\, v_1(x)\right)\, d\mu(x) &= \int_{\mathcal{X}} \Phi_1\Big(\Phi_1^{-1}(\Phi_\theta(|f(x)|v_\theta(x)))\\ &\qquad\qquad \cdot k'(x,1+it)\, v_1(x)\Big)\, d\mu(y)\\ &\le \int_{\mathcal{X}} \Phi_1\left(\Phi_1^{-1}(\Phi_\theta(|f(x)|v_\theta(x)))\right)\, d\mu(y)\\ &\le \int_{\mathcal{X}} \Phi_\theta(|f(x)|v_\theta(x))\, d\mu(y)\\ &\le 1.\end{aligned}$$

Now, define

$$F_Q(z) := |Q|^{\frac{-\lambda_z}{p_z}} \int_Q T(A(\cdot,z))(y)\, B(y,z)\, dy, \qquad (z \in \mathbb{C}).$$

Then, for each $t \in \mathbb{R}$ we have

$$\begin{aligned}|F_Q(it)| &\le |Q|^{\frac{-\lambda_0}{p_0}} \int_Q |T(A(\cdot,it))(y)|\ |B(y,it)|\, dy\\ &\le |Q|^{\frac{-\lambda_0}{p_0}} \|T(A(\cdot,it))\|_{Q,p_0,w_0}\ \|B(\cdot,it)\|_{Q,p_0',w_0^{-1}}\\ &\le |Q|^{\frac{-\lambda_0}{p_0}} \|T(A(\cdot,it))\|_{Q,p_0,w_0}\\ &\le \|T(A(\cdot,it))\|_{(p_0,\lambda_0,w_0)}\\ &\le M_0\, \|A(\cdot,it)\|_{\Phi_0,v_0} \le M_0,\end{aligned}$$

thanks to the relations (2.5), (2.7) and (2.2). Similarly, by the relations (2.8), (2.6) and (2.3), for each $t \in \mathbb{R}$ we have

$$\begin{aligned}|F_Q(1+it)| &\leq |Q|^{\frac{-\lambda_1}{p_1}} \int_Q |T(A(\cdot,1+it))(y)|\ |B(y,1+it)|\,dy \\ &\leq |Q|^{\frac{-\lambda_1}{p_1}} \|T(A(\cdot,1+it))\|_{Q,p_1,w_1}\ \|B(\cdot,1+it)\|_{Q,p_1',w_1^{-1}} \\ &\leq |Q|^{\frac{-\lambda_1}{p_1}} \|T(A(\cdot,1+it))\|_{Q,p_1,w_1} \\ &\leq \|T(A(\cdot,1+it))\|_{(p_1,\lambda_1,w_1)} \\ &\leq M_1\,\|A(\cdot,1+it)\|_{\Phi_1,v_1} \leq M_1.\end{aligned}$$

So, by Three-Line Theorem we have

$$|Q|^{\frac{-\lambda_\theta}{p_\theta}} \left|\int_Q T(f)(y)\,g(y)\,dy\rangle\right| = |F_Q(\theta)| \leq M_0^{1-\theta}\,M_1^{\theta}, \tag{2.9}$$

since $f = A(\cdot,\theta)$ and $g = B(\cdot,\theta)$. Finally,

$$\begin{aligned}\|T(f)\|_{(p_\theta,\lambda_\theta,w_\theta)} &= \sup\left\{|Q|^{\frac{-\lambda_\theta}{p_\theta}}\ \|T(f)\,w_\theta\|_{L^{p_\theta}(Q)}\ :\ Q \text{ is a cube in } \mathbb{R}^n\right\} \\ &= \sup\Big\{|Q|^{\frac{-\lambda_\theta}{p_\theta}} \left|\int_Q T(f)(y)\,h(y)\,dy\rangle\right|\ :\ Q \text{ is a cube in } \mathbb{R}^n, \\ &\qquad\qquad h \text{ is simple and } \|h\|_{Q,p_\theta',w_\theta^{-1}} = 1\Big\} \\ &\leq M_0^{1-\theta}\,M_1^{\theta}.\end{aligned}$$

This completes the proof because the set of all simple functions is dense in $\mathcal{M}^{p_\theta,\lambda_\theta}_{w_\theta^{-1}}$. □

Example 2.4. Let (Φ_i,Ψ_i) $(i = 0,1)$ be complimentary pairs of N-functions. Suppose that $0 < \theta < 1$ is a fixed number, $1 \leq p_i < \infty$, $0 \leq \lambda_i \leq 1$ $(i = 0,1)$, v_0 and v_1 are weight functions on $\mathcal{X}$, and w_0 and w_1 are weight functions on $\mathbb{R}^n$. Then, the functions k and k' defined by:

$$k(y,z) := w_0(y)^{1-z}\,w_1(y)^z$$

and

$$k'(x,z) := v_0(x)^{z-1}v_1(x)^{-z},$$

where $x \in \mathcal{X}$, $y \in \mathbb{R}^n$ and $z \in \mathbb{C}$, satisfy the hypothesis of Theorem 2.3.

References

[1] J. Bergh and J. Löfstr"om, *Interpolation Spaces; An Introduction*, Springer-Verlag, Berlin, 1976.
[2] H. Gunawan, D.I. Hakim and M. Idris, Proper inclusions of Morrey spaces, https://arxiv.org/abs/1702.07053v2 to appear in Math. Glasnik.
[3] C.B. Morrey, On the solutions of quasi linear elliptic partial differential equations, Trans. Amer. Math. Soc., 43 (1938), 126–166.
[4] A. Osançlıol and S. Öztop, *Weighted Orlicz algebras on locally compact groups*, J. Aust. Math. Soc., **99** (2015), 399-414.

[5] S. Öztop and S.M. Tabatabaie, *Weighted Orlicz algebras on hypergroups*, Filomat, **34**(9) (2020), 2991-3002.
[6] J. Peetre, On the theory of $\mathcal{L}_{p,\lambda}$, J. Func. Anal., 4 (1969) 71–87.
[7] M.M. Rao and Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1991.
[8] M.M. Rao and Z.D. Ren, *Applications of Orlicz Spaces*, Marcel Dekker, New York, 2002.
[9] Y. Sawano, A non-dense subspace in $\mathcal{M}_q^p$ with $1 < q < p < \infty$, Trans. A. Razmadze Math. Inst. 171 (2017) no. 3, 379–380.
[10] Y. Sawano, S. Sugano and H. Tanaka, Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces. Trans. Amer. Math. Soc., 363 (12) (2011) 6481–6503.
[11] Y. Sawano and S.M. Tabatabaie, *Inclusions in generalized Orlicz spaces*, Bull. Iran. Math. Soc. (2020). https://doi.org/10.1007/s41980-020-00437-y
[12] S. M. Tabatabaie and A. R. Bagheri Salec, *Convolution of two weighted Orlicz spaces on hypergroups*, Revista Colombiana de Matemáticas Revista, **54** (2020) 117-128.
[13] S. M. Tabatabaie, A. R. Bagheri Salec and M. Zare Sanjari, *A note on Orlicz algebras*, Oper. Matrices, **14**(1) (2020) 139–144.
[14] S. M. Tabatabaie, A. R. Bagheri Salec and M. Zare Sanjari, *Remarks on weighted Orlicz spaces on locally compact groups*, Math. Ineq. Appl. **23**(3) (2020) 1015–1025.
[15] S. M. Umarkhadzhiev, *Riesz-Thorin-Stein-Weiss interpolation theorem in a Lebesgue-Morrey setting*, Operator Theory: Advances and Applications, **229** (2013) 387-392.

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