Mathematical Analysis & Convex Optimization

Vol. 1 (2020), No. 1, 1–5
https:\\ maco.lu.ac.ir
DOI: 10.29252/maco.1.1.1

Research Paper

ON THE EXISTENCE OF SOLUTIONS OF A GENERALIZED MONOTONE EQUILIBRIUM PROBLEM

HADI KHATIBZADEH* AND MAHNAZ REZAEI

ABSTRACT. Blum and Oettli in their seminal paper studied the existence of equilibrium points for monotone bifunctions. In this work, we extend their main result by replacing monotone bifunction with a more general bifunction and prove the existence of an equilibrium point.

MSC(2010): 47H05; 47H04; 49J40; 49J53.
Keywords: Equilibrium problem, monotone bifunction, generalized monotonicity.

1. Introduction and Background

Throughout the paper, we assume that X is a real Banach space with norm $\|\cdot\|$ and K is a closed convex subset of X. By a *bifunction* we mean any function $f : K \times K \to \mathbb{R}$ such that $f(x, x) = 0, \ \ \forall\, x \in K$.

Definition 1.1. Let $f : K \times K \to \mathbb{R}$ be a bifunction. Consider the *equilibrium problem* (EP) of finding $\overline{x} \in K$ such that

$$f(\overline{x}, y) \geq 0, \quad \forall y \in K.$$

$\overline{x}$ is called an *equilibrium point* for f and K. The set of all equilibrium points for f and K is denoted by $EP(f, K)$.

Definition 1.2. Given a nonempty subset K of a Banach space X, the bifunction $f : K \times K \to \mathbb{R}$ is said to be

- *monotone* iff
$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in K.$$
- *pseudo-monotone* if $f(x, y) \geq 0$ with $x, y \in K$, then $f(y, x) \leq 0$.
- *quasi-monotone* if $f(x, y) > 0$ with $x, y \in K$, then $f(y, x) \leq 0$.
- *θ-monotone* if there is a function $\theta : K \times K \to \mathbb{R}$ such that
$$f(x, y) + f(y, x) \leq \theta(x, y)\|x - y\|, \quad \forall x, y \in K$$

Date: Received: October 2, 2019, Accepted: March 13, 2020.
*Corresponding author.

Example. Let $f : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ with $f(x,y) = x^2 - xy$. Obviously it is not monotone but it is θ-monotone with $\theta(x,y) = |x-y|$.

Existence of an equilibrium point for a monotone bifunction first studied by Blum and Oettli in [6]. An equilibrium point for a monotone bifunction can be a fixed point for a nonexpansive mapping, a solution of a variational inequality for a maximal monotone operator and a minimum point of a convex function. It has also some other interpratations in nonlinear problems. Therefore equilibrium problems unify several problems in nonlinear analysis and optimization (see [6]). Equilibrium problems for monotone and some variants of generalized monotone bifunctions has been studied by several authors (see for example [3, 4, 5, 7, 8, 9, 10, 11, 12, 13]). But the researchers have paid more attention to some generalized monotonicity of pseudo- and quasi-monotone type so far. Recently some general monotonicity conditions of different types for operators and bifunctions studied by authors (see [1, 2, 10, 11]). One of this conditions is θ-monotonicity that was defined in above. In this paper, we extend the existence theorem of Blum and Oettli [6] form monotone bifunctions to θ-monotone bifunctions.

2. Main Results

In this section we prove a basic existence result for the equilibrium problem in the case where $f(x,y) = g(x,y) + h(x,y)$. All assumptions on g and h are the same as assumed by Blum and Oettli [6] except monotonicity of g that we replace it by θ-monotonicity. Before the main theorem we recall a definition.

Definition 2.1. Let K and C be convex sets with $C \subset K$. Then $\text{core}_K C$ is defined through $a \in \text{core}_K C$ if $a \in C$, and $C \cap (a,y) \neq \varnothing$, for all $y \in K \diagdown C$, where $(a,y) = \{ta + (1-t)\,y;\ 0 < t < 1\}$.

Theorem 2.2. *Let the following assumptions hold*

- $g : K \times K \to \mathbb{R}$ *has the following properties:*
  - $g(x,x) = 0, \quad \forall x \in K$;
  - *For all* $x, y \in K$ *the function* $t \in [0,1] \mapsto g(ty+(1-t)x, y)$ *is upper-semicontinuous at* $t = 0$*; (g is called uper-hemicontinuous respect to the first argument);*
  - g *is convex and lower semicontinuous in the second argument;*
  - $g(x,y) + g(y,x) \leq \theta(x,y)\|x-y\|, \quad \forall x,y \in K,\ (\theta - monotonicity)$;

  where
- $\theta : K \times K \to \mathbb{R}$ *satisfies the following conditions:*
  - $\theta(x,x) = 0,\ \forall\ x \in K$;
  - θ *is upper semicontinuous respect to the second argument;*
- $h : K \times K \to \mathbb{R}$ *has the following properties:*
  - $h(x,x) = 0, \quad \forall x \in K$;
  - h *is upper semicontinuous in the first argument;*
  - h *is convex in the second argument.*
- *(Coercivity condition) There exists* $C \subset K$ *nonempty, compact and convex such that for every* $x \in C \diagdown \text{core}_K C$ *there exists* $a \in \text{core}_K C$ *such that* $g(x,a) + h(x,a) \leq 0$.

Then there exists $\overline{x} \in C$ *such that* $0 \leq g(\overline{x},y) + h(\overline{x},y), \quad \forall y \in K$.

The proof goes over the following three lemmas, for which the hypotheses remain the same as for Theorem 2.2.

Lemma 2.3. *There exists* $\overline{x} \in C$ *such that*

$$g(y, \overline{x}) \leq \theta(y, \overline{x})\|y - \overline{x}\| + h(\overline{x}, y), \ \forall\ y \in C.$$

Proof. For each $y \in C$ define

$$S(y) = \{x \in C : \ g(y, x) \leq \theta(y, x)\|y - x\| + h(x, y)\}, \quad \forall y \in C$$

By the assumptions on g and θ, $S(y)$ is closed and since C is compact then $S(y)$ is compact for every $y \in C$. It is enough we show $\{S(y) : \ \ y \in C\}$ has finite intersection property. Let $\{y_i : \ \ i \in I\}$ be a finite and arbitrary subset of C and $\zeta \in \text{co}\{y_i : \ \ i \in I\}$ (convex hull of $\{y_i : \ \ i \in I\}$) be arbitrary. Therefore there are nonnegative scalers μ_i such that $\sum_{i\in I} \mu_i = 1$ and $\zeta = \sum_{i\in I} \mu_i y_i$. Now suppose that ζ is not in $S(y_i), \ \ \forall i \in I$. i.e.

$$g(y_i, \zeta) > \theta(y_i, \zeta)\|y_i - \zeta\| + h(\zeta, y_i). \tag{2.1}$$

Multiplying both sides by $\sum_{i\in I} \mu_i$, then by the conditions on h, we get

$$\begin{aligned}
\sum_{i\in I} \mu_i g(y_i, \zeta) &> \sum_{i\in I} \mu_i(\theta(y_i, \zeta)\|y_i - \zeta\| + h(\zeta, y_i)) \\
&\geq \sum_{i\in I} \mu_i\theta(y_i, \zeta)\|y_i - \zeta\| + \sum_{i\in I} \mu_i h(\zeta, y_i) \\
&\geq \sum_{i\in I} \mu_i\theta(y_i, \zeta)\|y_i - \zeta\| + h(\zeta, \zeta) \\
&\geq \sum_{i\in I} \mu_i\theta(y_i, \zeta)\|y_i - \zeta\| \qquad (2.2)
\end{aligned}$$

Adding both sides of the recent inequality by $\sum_{i\in I} \mu_i g(\zeta, y_i)$, and using the θ-monotonicity of g, we get

$$\begin{aligned}
\sum_{i\in I} \mu_i\theta(y_i, \zeta)\|y_i - \zeta\| &\geq \sum_{i\in I} \mu_i(g(y_i, \zeta) + g(\zeta, y_i)) \\
&> \sum_{i\in I} \mu_i\theta(y_i, \zeta)\|y_i - \zeta\| + \sum_{i\in I} \mu_i g(\zeta, y_i) \\
&\geq \sum_{i\in I} \mu_i\theta(y_i, \zeta)\|y_i - \zeta\| + g(\zeta, \zeta) \\
&= \sum_{i\in I} \mu_i\theta(y_i, \zeta)\|y_i - \zeta\|,
\end{aligned}$$

which is a contradiction. Then there is $i \in I$ such that

$$g(y_i, \zeta) \leq \theta(y_i, \zeta)\|y_i - \zeta\| + h(\zeta, y_i).$$

Therefore for some $i \in I$, $\zeta \in S(y_i)$. Since ζ is an arbitrary element of $\text{co}\{y_i : \ \ i \in I\}$, we conclude that

$$\text{co}\{y_i : \ i \in I\} \subset \cup_{i\in I} S(y_i).$$

Then by the KKM theorem, we get $\cap_{y\in C} S(y) \neq \varnothing$. □

Lemma 2.4. *The following statements are equivalent*
(*a*) $\exists\ \overline{x} \in C, \ g(y, \overline{x}) \leq \theta(\overline{x}, y)\|\overline{x} - y\| + h(\overline{x}, y), \ \forall\ y \in C;$
(*b*) $\exists\ \overline{x} \in C, \ 0 \leq g(\overline{x}, y) + h(\overline{x}, y), \ \forall y \in C.$

Proof. $(b) \Rightarrow (a)$: From θ-monotonicity of g, we have

$$g(\bar{x}, y) + g(y, \bar{x}) + h(\bar{x}, y) \leq \theta(\bar{x}, y)\|\bar{x} - y\| + h(\bar{x}, y)$$

From (b), we get

$$g(y, \bar{x}) \leq \theta(\bar{x}, y)\|\bar{x} - y\| + h(\bar{x}, y)$$

$(a) \Rightarrow (b)$: Let $y \in C$ be arbitrary, and take $x_t = ty + (1-t)\bar{x}$ and $0 < t \leq 1$. Since C is convex, then $x_t \in C$. Take $y = x_t$ in (a), then

$$g(x_t, \bar{x}) \leq \theta(\bar{x}, x_t)\|x_t - \bar{x}\| + h(\bar{x}, x_t)$$

and

$$\begin{aligned}
0 &= g(x_t, x_t) \\
&\leq tg(x_t, y) + (1-t)g(x_t, \bar{x}) \\
&\leq tg(x_t, y) + (1-t)(\theta(\bar{x}, x_t)\|\bar{x} - x_t\| + h(\bar{x}, x_t)) \\
&\leq tg(x_t, y) + (1-t)\big(\theta(\bar{x}, x_t)\|\bar{x} - x_t\| + th(\bar{x}, y) + (1-t)h(\bar{x}, \bar{x})\big) \\
&= t(g(x_t, y) + (1-t)h(\bar{x}, y)) + (1-t)\theta(\bar{x}, x_t)\|x_t - \bar{x}\| \\
&= t(g(x_t, y) + (1-t)h(\bar{x}, y)) + (1-t)t\theta(\bar{x}, x_t)\|y - \bar{x}\|.
\end{aligned}$$

Dividing both sides by t and letting $t \to 0$, by semicontinuous of g and θ, we get the result. □

Lemma 2.5. [6] *Assume that $\Psi : K \to \mathbb{R}$ is convex, $x_0 \in core_K C$, $\Psi(x_0) \leq 0$, and $\Psi(y) \geq 0$, $\forall\ y \in C$. Then $\Psi(y) \geq 0$, $\forall y \in K$.*

Now we present the proof of Theorem 2.2.

Proof. From Lemma 2.3, we obtain $\overline{x} \in C$ with

$$g(y, \overline{x}) \leq \theta(y, \overline{x})\|y - \overline{x}\| + h(\overline{x}, y),\ \forall\ y \in C$$

From Lemma 2.4 follows that

$$0 \leq g(\overline{x}, y) + h(\overline{x}, y),\ \forall y \in C$$

Set $\Psi(\cdot) := g(\overline{x}, \cdot) + h(\overline{x}, .)$, then $\Psi(\cdot)$ is convex and $\Psi(y) \geq 0$, $\forall\ y \in C$. If $\overline{x} \in C$, then set $x_0 := \overline{x}$. If $\overline{x} \in C \setminus \mathrm{core}_K C$, then set $x_0 := a$, where a is as in coercivity assumption for $x = \overline{x}$. In both cases $x_0 \in \mathrm{core}_K C$, *and* $\Psi(x_0) \leq 0$. Hence it follows from Lemma 2.5 that $\Psi(y) \geq 0$, $\forall y \in K$, *i.e.*, $g(\overline{x}, y) + h(\overline{x}, y) \geq 0$, $\forall y \in K$. □

Corollary 2.6. *Let the following assumptions hold*

- *$g : K \times K \to \mathbb{R}$ has the following properties:*
  - *$g(x, x) = 0$, $\forall\ x \in K$;*
  - *For all $x, y \in K$ the function $t \in [0, 1] \mapsto g(ty+(1-t)x, y)$ is upper-semicontinuous at $t = 0$ (hemicontinuity);*
  - *g is convex and lower semicontinuous in the second argument;*
  - *$g(x, y) + g(y, x) \leq \theta(x, y)\|x - y\|$, $\forall\ x, y \in K(\theta - monotonicity)$;*

  where
- *$\theta : K \times K \to \mathbb{R}^+$ has the following properties:*
  - *$\theta(x, x) = 0$, $\forall\ x \in K$;*
  - *θ is upper semicontinuous respect to the second argument.*

- *There exists $C \subset K$ nonempty, compact and convex such that for every $x \in C \diagdown \mathrm{core}_K C$ there exists $a \in core_K C$ such that* $g(x,a) \leq 0$.

Then there exists $\overline{x} \in C$ such that $0 \leq g(\overline{x}, y), \ \forall y \in K$.

Acknowledgments

The authors are grateful to the referee for his(her) valuable comments.

References

[1] M. H. Alizadeh, N. Hadjisavvas, and M. Roohi, Local boundedness properties for generalized monotone operators, *J. Convex Anal.*, **19**:49–61, 2012.
[2] M. H. Alizadeh M. Roohi, Some results on pre-monotone operators, *Bull Iranian Math Soc.*, **43**:2085–2097, 2017.
[3] M. Bianchi and R. Pini, Coercivity conditions for equilibrium problems, *J. Optim. Theory Appl.*, **124**:79–92, 2005.
[4] M. Bianchi and S. Schaible, Generalized monotone bifunctions and equilibrium problems, *J. Optim. Theory Appl.*, **90**:31–43, 1996.
[5] M. Bianchi and S. Schaible, Equilibrium problems under generalized convexity and generalized monotonicity, *J. Global Optim.*, **30**:121–134, 2004.
[6] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student*, **63**:2951–2955, 1994.
[7] O. Chadli, Z. Chbani, H. Riahi, Equilibrium problems with generalized monotone bifunctions and applications to variational inequalities, *J. Optim. theory Appl.*, **105**:299–323, 2000.
[8] O. Chadli, A. Lahmdani and J.C. Yao, Existence results for equilibrium problems with applications to evolutions equations, *Appl. Anal.*, **94**:1709–1735, 2015.
[9] J. Cotrina and Y. Garcia, Equilibrium Problems: Existence results and applications, *Set-valued Var. Anal.*, **26**:159–177, 2018.
[10] A. Iusem, G. Kassay and W. Sosa, On certain conditions for the existence of solutions of equilibrium problems, *Math. Program. Ser B.*, **116**:259–273, 2009.
[11] A. Iusem, G. Kassay and W. Sosa, An existence result for equilibrium problems with some surjectivity consequences. *J. Convex Anal.*, **16**:80–826, 2009.
[12] G. Kassay and M. Mihaela, Existence results for vector equilibrium problems given by a sum of two functions, *J. Global Optim.*, **63**:195–211, 2015.
[13] I. V. Konnov and D. A. Dyabilkin, Nonmonotone equilibrium problems: coercivity conditions and weak regularization, *J. Glob. Optim.*, **49**:575–587, 2011.

(Hadi Khatibzadeh) Department of Mathematics, University of Zanjan, Zanjan, Iran.
Email address: hkhatibzadeh@znu.ac.ir

(Mahnaz Rezaei) Department of Mathematics, University of Zanjan, Zanjan, Iran.
Email address: mahnaz.rezaei615@gmail.com