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Research Paper

ON COMMON FIXED POINT OF NON SELF MAPPINGS ENJOYS THE T-APPROXIMATE STRICT FIXED POINT PROPERTY

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ABSTRACT. In this work, the common $T-$ approximate strict fixed point property for multi-valued mappings $F, G : X \to P_{cl,bd}(X)$ is introduced to prove necessary and sufficient condition for existence of a common strict fixed point of multi-valued mappings involved therein. Our results extend and unify comparable results in the existing literature. We also provide some examples and applications to support our results.

MSC(2010): 47H10, 54H25.
Keywords: Multi-valued mapping, common approximate $T-$ strict fixed point property, Common strict fixed point, Hausdorff metric.

1. Introduction and Background

Suppose $P(X)$ denotes the class of all subsets of a metric space X. Define

$$P_p(X) = \Big\{ A \subseteq X : A \neq \emptyset \text{ } has \text{ } a \text{ } property \text{ } p \Big\}.$$

Thus $P_{bd}(X)$, $P_{cl}(X)$, $P_{cp}(X)$ and $P_{cl,bd}(X)$ denote the classes of bounded, closed, compact and closed bounded subsets of X, respectively.

Let $F, G : X \longrightarrow P(X)$ be multi-valued mappings. A point $x \in X$ is said to be :

(i): a fixed point of F if $x \in Fx$.

(ii): an strict fixed point of F if $F(x) = \{x\}$.

(iii): a common strict fixed point for pair (F, G) if $Fx = Gx = \{x\}$.

Denote $Fix(F)$, $End(F)$, $Fix(F, G)$, $End(F, G)$, the set of all fixed points of F, set of all endpoints of F, set of all common fixed points of (F, G) and set of all common endpoints of (F, G), respectively. Obviously, $End(F) \subseteq Fix(F)$ and $End(F, G) \subseteq Fix(F, G)$.

For $A, B \in P_{cl,bd}(X)$, the Hausdorff distance $H(A, B)$ between A and B induced by a metric d on X is given by

$$H(A, B) := \max \Big\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \Big\}, \tag{1.1}$$

where $d(x, A) = \inf\{d(x, a) : a \in A\}$ is the distance of the point x from the set A.

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A mapping $F : X \to P_{cl,bd}(X)$ is said to be a contraction if for some $0 \leq \alpha < 1$, the cndition

$$H(Fx, Fy) \leq \alpha d(x, y), \tag{1.2}$$

holds for all $x, y \in X$.

Banach contraction principle [2] was extended to multi-valued mappings by Nadler [23] in the following theorem.

Theorem 1.1. *Let (X, d) be a complete metric space and $F : X \to P_{cl,bd}(X)$ be a contraction mapping Then there exists a point $x \in X$ such that $x \in Fx$.*

The study of fixed points for multi-valued contractions and non-expansive mappings using the Hausdorff metric was initiated by Markin [19]. Later, an interesting and rich fixed point theory for such maps was developed (see [12, 20, 21, 26, 27, 34, 35, 36, 38, 39, 41, 18, 31]). The theory of multi valued maps has application in control theory, convex optimization, differential equations and economics. Recently many authors have studied the existence and uniqueness of endpoints of multi-valued mappings (see, for example [32, 22, 37] and the references therein). In this paper, we obtain a necessary and sufficient condition for the existence of a common strict fixed point for a pair of multi-valued mappings. As an application, we obtain some common fixed point result for a hybrid pair of mappings. Our results extend the results in [15] and [37].

2. Main Results

The following definition play a crucial role throughout this work.

Definition 2.1. Let $F, G : X \to P_{cl,bd}(X)$ be multi-valued mappings, $T : X \to X$ be a single valued mapping and K be a nonempty closed subset of X . A pair $(F, G$) is said to have the common $T-$ approximate strict fixed point property on K (see [16, Definition 2.1]), if there exists a sequence $\{x_n\} \subset K$ such that

$$\lim_{n\to\infty} H(\{Tx_n\}, TFx_n) = 0, \ \lim_{n\to\infty} H(\{Tx_n\}, TGx_n) = 0. \tag{2.1}$$

Definition 2.2. Let $f, g : X \to X$ be single valued mappings, $T : X \to X$ and K a nonempty closed subset of X . A pair $(f, g$) is said to have the common $T-$ approximate fixed point property on K, if there exists a sequence $\{x_n\} \subset K$ such that

$$\lim_{n\to\infty} d(Tx_n, Tfx_n) = 0, \ \lim_{n\to\infty} H(Tx_n, Tgx_n) = 0. \tag{2.2}$$

Definition 2.3. Let $F, G : X \to P_{cl,bd}(X)$ be multi-valued mappings, and $T : X \to X$. A pair (F, G) is said to be $T-$Hardy-Rogers on X if there exist $a_i \geq 0$, $i = 1, 2, ..., 5$ with $\sum_{i=1}^{5} a_i < 1$ such that for x,y in X, we have

$$\begin{aligned} H(TFx, TGy) \quad \leq \quad & a_1 d(Tx, Ty) + a_2 D(Tx, TFx) \\ & + a_3 D(Ty, TGy) + a_4 D(Tx, TGy) + a_5 D(Ty, TFx). \end{aligned} \tag{2.3}$$

Definition 2.4. [17] A mapping $T : X \to X$ is said to be a closed graph mapping, if for any sequence $\{x_n\}$ with $\lim_{n\to\infty} Tx_n = a$, there exists $b \in X$ such that $Tb = a$. For example, an identity function on X is a closed graph mapping.

Theorem 2.5. *Let X be a complete metric space, $T : X \to X$ an injective and closed graph mapping, K a nonempty, closed subset of X with $T(K) \subset K$ and $F, G : K \to P_{cl,bd}(K)$ be a $T-$Hardy-Rogers pair on K. Then (F, G) has a unique common strict fixed* *point on K, if and only if (F,G) has common $T-$approximate strict fixed point property on K.*

Proof. It is straightforward to check that if F and G have common strict fixed point, then the pair (F,G) satisfies the common $T-$approximate strict fixed point property on K. Conversely, suppose that the pair (F,G) has the common $T-$approximate strict fixed point property on K. By assumption, there exists a sequence $\{x_n\} \subset K$ such that $\lim_n H(\{Tx_n\}, TFx_n) = 0$ and $\lim_n H(\{Tx_n\}, TGx_n) = 0$. For all $m, n \in \mathbb{N}$ we have
(2.4)

$$
\begin{aligned}
d(Tx_n, Tx_m) &\le H(\{Tx_n\}, TFx_n) + H(TFx_n, TGx_m) + H(\{Tx_m\}, TGx_m) \\
&\le H(\{Tx_n\}, TFx_n) + H(\{Tx_m\}, TGx_m) + a_1 d(Tx_n, Tx_m) \\
&\quad + a_2 D(Tx_n, TFx_n) + a_3 D(Tx_m, TGx_m) + a_4 D(Tx_n, TGx_m) \\
&\quad + a_5 D(Tx_m, TFx_n) \\
&\le H(\{Tx_n\}, TFx_n) + H(\{Tx_m\}, TGx_m) + a_1 d(Tx_n, Tx_m) \\
&\quad + a_2 H(\{Tx_n\}, TFx_n) + a_3 H(\{Tx_m\}, TGx_m) \\
&\quad + a_4 [d(Tx_n, Tx_m) + D(Tx_m, TGx_m)] \\
&\quad + a_5 [d(Tx_n, Tx_m) + D(Tx_n, TFx_n)] \\
&\le H(\{Tx_n\}, TFx_n) + H(\{Tx_m\}, TGx_m) + a_1 d(Tx_n, Tx_m) \\
&\quad + a_2 H(\{Tx_n\}, TFx_n) + a_3 H(\{Tx_m\}, TGx_m) \\
&\quad + a_4 [d(Tx_n, Tx_m) + H(\{Tx_m\}, TGx_m)] \\
&\quad + a_5 [d(Tx_n, Tx_m) + H(\{Tx_n\}, TFx_n)] \\
&= (1 + a_2 + a_5) H(\{Tx_n\}, TFx_n) \\
&\quad + (1 + a_3 + a_4) H(\{Tx_m\}, TGx_m) + (a_1 + a_4 + a_5) d(Tx_n, Tx_m).
\end{aligned}
$$

Which implies that
(2.5)

$$
d(Tx_n, Tx_m) \le \left(\frac{1+a_3+a_4}{1-a_1-a_4-a_5} \right) H(\{Tx_m\}, TGx_m) + \left(\frac{1+a_2+a_5}{1-a_1-a_4-a_5} \right) H(\{Tx_n\}, TFx_n).
$$

By taking the limit, from both side of the above inequality as $m, n \to \infty$

$$
\lim_{m,n\to\infty} d(Tx_n, Tx_m) = 0.
$$

Thus $\{Tx_n\}$ is a Cauchy sequence and so converges to $y \in K$. As T is closed graph mapping, there exists $x \in K$ such that $Tx = y$. Suppose that $D(Tx, TFx) > 0$ then
(2.6)

$$
\begin{aligned}
H(\{Tx_n\}, TFx) - H(\{Tx_n\}, TGx_n) &\le H(TFx, TGx_n) \\
&\le a_1 d(Tx, Tx_n) + a_2 D(Tx, TFx) \\
&\quad + a_3 D(Tx, TFx) + a_4 D(Tx_n, TGx_n) \\
&\quad + a_5 D(Tx_n, TFx_n) \\
&\le a_1 d(Tx, Tx_n) + a_2 D(Tx, TFx) \\
&\quad + a_3 D(Tx_n, TGx_n) + a_4 [d(Tx, Tx_n) \\
&\quad + D(Tx_n, TGx_n)] + a_5 [d(Tx_n, Tx) \\
&\quad + D(Tx, TFx)] \\
&\le a_1 d(Tx, Tx_n) + a_2 D(Tx, TFx) \\
&\quad + a_3 D(Tx_n, TGx_n) \\
&\quad + a_4 [d(Tx, Tx_n) + H(\{Tx_n\}, TGx_n)] \\
&\quad + a_5 [d(Tx_n, Tx) + D(Tx, TFx)].
\end{aligned}
$$

Considering the limit of the above inequality, we get

$$
H(\{Tx\}, TFx) \le (a_2 + a_5) D(Tx, TFx) < D(Tx, TFx), \tag{2.7}
$$

that is a contradiction. Therefore, $Tx \in TFx$. As T is injective, so $x \in Fx$. Following similar arguments for G, it is concluded that $x \in Gx$. Rewriting (2.6) and (2.7) for F and G again one can conclude that $H(\{Tx\}, TFx) = 0$ and $H(\{Tx\}, TGx) = 0$, i.e.,

$Gx = Fx = \{x\}$. Moreover, if $z \in Fix(G,F)\backslash\{x\}$ then

$$\begin{aligned} d(Tz,Tx) &= D(Tz,TFx) \\ &\leq H(TGz,TFx) \\ &\leq a_1 d(Tz,Tx) + a_2 D(Tz,TFz) \\ &+ a_3 D(Tx,TGx) + a_4 D(Tx,TGz) + a_5 D(Tz,TFx) \\ &= a_1 d(Tz,Tx) + a_4 D(Tx,TGz) + a_5 d(Tz,Tx) \end{aligned} \tag{2.8}$$

Therefore,

$$d(Tz,Tx) \leq \frac{a_4}{1-a_1-a_5} D(Tx,TGz) < D(Tx,TGz)$$

and this is a contradiction. Thus, $Fix(F,G) = End(F,G)$. $\square$

Corollary 2.6. *Let X be a complete metric space, K be a closed subset of X, $T : X \to X$ be an injective, closed graph map with $T(K) \subset K$. If $f, g : K \to X$ satisfy*

$$\begin{aligned} d(Tfx,Tgy) \leq\ & a_1 d(Tx,Ty) + a_2 d(Tx,Tfx) \\ & + a_3 d(Ty,Tgy) + a_4 d(Tx,Tgy) \\ & + a_5 d(Ty,Tfx) \end{aligned} \tag{2.9}$$

Then f and g have common approximate fixed point on K, if and only if f and g have the unique common fixed point on K.

Proof. Take $Fx = \{fx\}$, $Gx = \{gx\}$ and apply Theorem 2.5. $\square$

Corollary 2.7. *Let X be a complete metric space and K be a closed subset of X, and $T : X \to X$ be an injective, closed graph map with $T(K) \subset K$. If $f, g : K \to X$ with $gf(K) \subset K$, $f(K) \subset K$ satisfy the following:*

(a):

$$\begin{aligned} d(Tfx,Tgy) \leq\ & a_1 d(Tx,Ty) + a_2 d(Tx,Tfx) \\ & + a_3 d(Ty,Tgy) + a_4 d(Tx,Tgy) \\ & + a_5 d(Ty,Tfx) \end{aligned} \tag{2.10}$$

(b): $d(x,gx) \leq d(x,fx)$ whenever $x, y \in K$.

Then f, g have the common approximate fixed point on K in a sense of Definition 2.2.

Proof. Let $x_0 \in K$ and let $x_1 = fx_0 \in K$. Also, $x_2 = gx_1 = gfx_0 \in K$. So we can choose a sequence $\{x_n\} \subset K$ such that $\{x_{2n}\} \subset K$. We have $x_{2n+2} = gx_{2n+1}$ and $x_{2n+1} = fx_{2n}$, for all $n \in \mathbb{N}$. By replacing x by x_{2n} and y by x_{2n+1} in (2.10) we have

$$\begin{aligned} d(Tx_{2n+1},Tx_{2n+2}) &= d(Tfx_{2n},Tgx_{2n+1}) \\ &\leq a_1 d(Tx_{2n},Tx_{2n+1}) + a_2 d(Tx_{2n},Tfx_{2n}) \\ &\quad + a_3 d(Tx_{2n+1},Tgx_{2n+1}) + a_4 d(Tx_{2n},Tgx_{2n+1}) \\ &\quad + a_5 d(Tx_{2n+1},Tfx_{2n}) \\ &= a_1 d(Tx_{2n},Tx_{2n+1}) + a_2 d(Tx_{2n},Tx_{2n+1}) \\ &\quad + a_3 d(Tx_{2n+1},Tx_{2n+2}) + a_4 d(Tx_{2n},Tx_{2n+2}) \\ &\quad + a_5 d(Tx_{2n+1},Tx_{2n+1}) \\ &\leq (a_1+a_2+a_4) d(Tx_{2n},Tx_{2n+1}) + (a_3+a_4) d(Tx_{2n+1},Tx_{2n+2}). \end{aligned}$$

that is,

$$(1-a_3-a_4) d(Tx_{2n+1},Tx_{2n+2}) \leq (a_1+a_2+a_4) d(Tx_{2n},Tx_{2n+1}) \tag{2.11}$$

Again by replacing $x = x_{2n+1}$ and $y = x_{2n}$ in (2.10) we have

$$\begin{aligned} d(Tx_{2n+2}, Tx_{2n+1}) &= d(Tfx_{2n+1}, Tgx_{2n}) \\ &\le a_1 d(Tx_{2n+1}, Tx_{2n}) + a_2 d(Tx_{2n+1}, Tfx_{2n+1}) \\ &\quad + a_3 d(Tx_{2n}, Tgx_{2n}) + a_4 d(Tx_{2n+1}, Tgx_{2n}) \\ &\quad + a_5 d(Tx_{2n}, Tfx_{2n+1}) \\ &= a_1 d(Tx_{2n}, Tx_{2n+1}) + a_2 d(Tx_{2n+1}, Tx_{2n+2}) \\ &\quad + a_3 d(Tx_{2n}, Tx_{2n+1}) + a_4 d(Tx_{2n+1}, Tx_{2n+1}) \\ &\quad + a_5 d(Tx_{2n}, Tx_{2n+2}) \\ &\le (a_1 + a_3 + a_5) d(Tx_{2n}, Tx_{2n+1}) + (a_2 + a_5) d(Tx_{2n+1}, Tx_{2n+2}). \end{aligned}$$

that is

$$(1 - a_2 - a_5) d(Tx_{2n+1}, Tx_{2n+2}) \le (a_1 + a_4 + a_5) d(Tx_{2n}, Tx_{2n+1}) \tag{2.12}$$

By adding (2.11) and (2.12), we have

$$(2 - a_2 - a_3 - a_4 - a_5) d(Tx_{2n+1}, Tx_{2n+2}) \le (2a_1 + a_2 + a_3 + a_4 + a_5) d(Tx_{2n}, Tx_{2n+1})$$

that is

$$d(Tx_{2n+1}, Tx_{2n+2}) \le \delta d(Tx_{2n}, Tx_{2n+1}) \tag{2.13}$$

where $\delta = \frac{2a_1+a_2+a_3+a_4+a_5}{2-a_2-a_3-a_4-a_5}$. Obviously $\delta < 1$. Similarly, one can show that

$$d(Tx_{2n+2}, Tx_{2n+3}) \le \delta d(Tx_{2n+1}, Tx_{2n+2}) \tag{2.14}$$

If we consider $a_n = d(Tx_{2n}, Tfx_{2n})$ then (2.13) and (2.14) show that

$$a_{n+1} \le \delta a_n \tag{2.15}$$

and also bounded below by zero and so is convergent to $r \ge 0$. If $r > 0$ then on taking limit on both sides of (2.15) we have $\delta \ge 1$ and this is a contradiction. Thus $r = 0$. By (B) we have $\{d(Tx_{2n}, Tgx_{2n})\}$ is convergent to zero too. It means that, f, g have the common $T-$ approximate fixed point property on K. □

Corollary 2.8. *Let X be a complete metric space and K be a closed subset of X. Suppose that $T : X \to X$ be an injective, closed graph maps which $T(K) \subset K$, $f, g : K \to X$ be two single-valued mappings such that $gf(K) \subset K$, $f(K) \subset K$ and f, g satisfy the following:*

(A): *For each $x, y \in K$*

$$\begin{aligned} d(Tfx, Tgy) &\le a_1 d(Tx, Ty) + a_2 d(Tx, Tfx) \\ &\quad + a_3 d(Ty, Tgy) + a_4 d(Tx, Tgy) \\ &\quad + a_5 d(Ty, Tfx) \end{aligned} \tag{2.16}$$

(B): $d(x, gx) \le d(x, fx)$.

Then f, g have the unique common fixed point on K.

Proof. Combining Corollary 2.7 and 2.6, the result follows. □

Corollary 2.9. *Let X be a complete metric space, K a closed subset of X. If $f, g : K \to X$ satisfy*

$$d(fx, gy) \le a_1 d(x, y) + a_2 d(x, fx) + a_3 d(y, gy) + a_4 d(x, gy) + a_5 d(y, fx),$$

where $a_i \ge 0$, $i = 1, 2, ..., 5$ with $\sum_{i=1}^{5} a_i < 1$. Then f and g have $I-$common approximate fixed point on K if and only if f and g have the unique fixed point on K, where I is an identity map.

Proof. Taking $T = I$(identity map)in Corollary 2.6 one can conclude desired result. □

3. Some Examples

In this section we give some examples to support our main results.

Example 3.1. Let $X = \{0, 1, 2\}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} 1, & \text{if} \quad x \neq 2 \\ 0 & \text{if} \quad x = 2 \end{cases}$$

and $F, G : X \to P_{cl,\ bd}(X)$ be defined by

$$Fx = \begin{cases} \{0, 1\}, & \text{if} \quad x = 0 \\ \{1\}, & \text{if} \quad x = 1 \\ \{0\} & \text{if} \quad x = 2 \end{cases} \quad \text{and} \quad Gx = \begin{cases} \{0\}, & \text{if} \quad x = 0 \\ \{1\}, & \text{if} \quad x = 1 \\ \{0, 1\} & \text{if} \quad x = 2 \end{cases}$$

Since

$$TF0 = TF1 = TF2 = TG0 = TG1 = TG2 = 1.$$

This implies

$$H(TFx, TGy) = 0$$

for all $x, y \in X$. Hence

$$\begin{aligned} H(TFx, TGy) \quad \leq \quad & a_1 d(Tx, Ty) + a_2 D(Tx, TFx) + a_3 D(Ty, TGy) \\ & + a_4 D(Tx, TGy) + a_5 D(Ty, TFx) \end{aligned}$$

is satisfied for all $x, y \in X$. So (F, G) of multi-valued mappings is $T-\ Hardy - Rogers$ pair on X. Moreover if $K = \{0, 1\}$, then for the sequences $\{x_n\} \subseteq K$ where $x_n = 0$ and $x_n = 1$ such that

$$\lim_{n\to\infty} H(\{Tx_n\}, \{TFx_n\}) = 0 \text{ and } \lim_{n\to\infty} H(\{Tx_n\}, \{TGx_n\}) = 0.$$

Hence (F, G) have the common $T-$ approximate strict fixed point property on the boundary of K.

Now we give an example to illustrate Theorem 2.5.

Example 3.2. Let $X = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Obviously X is a complete metric space. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{3}, & \text{if} \quad x = \frac{1}{2} \\ \frac{1}{2}, & \text{if} \quad x = \frac{1}{3} \\ \frac{1}{5}, & \text{if} \quad x = \frac{1}{4} \\ \frac{1}{4} & \text{if} \quad x = \frac{1}{5} \end{cases}$$

Clearly T is injective and a closed graph mapping. Let $K = \{\frac{1}{2}, \frac{1}{3}\} = K$ be a closed subset of X and $F, G : K \to P_{cl,\ bd}(X)$ be defined by

$$Fx = \begin{cases} \frac{1}{2}, & \text{if} \quad x = \frac{1}{2} \\ \{\frac{1}{4}, \frac{1}{5}\} & \text{if} \quad x = \frac{1}{3} \end{cases} \quad \text{and} \quad Gx = \begin{cases} \frac{1}{2}, & \text{if} \quad x = \frac{1}{2} \\ \{\frac{1}{2}, \frac{1}{4}\} & \text{if} \quad x = \frac{1}{3} \end{cases}$$

Let $x = \frac{1}{2}$, and $y = \frac{1}{3}$ then

$$H(TF\tfrac{1}{2}, TG\tfrac{1}{3}) = H(T\tfrac{1}{2}, T\{\tfrac{1}{2}, \tfrac{1}{4}\}) = H(\{\tfrac{1}{3}\}, \{\tfrac{1}{3}, \tfrac{1}{5}\}) = \tfrac{2}{15}.$$

On the other hand $d(T\frac{1}{2}, T\frac{1}{3}) = d(\frac{1}{3}, \frac{1}{2}) = \frac{1}{6}$. Let $a_1 = \frac{6}{7}$, and $a_2 = a_3 = a_4 = a_5 = 0$ then clearly $\frac{2}{15} < \frac{6}{7}\frac{1}{6} = \frac{1}{7}$, hence

$$\begin{aligned} H(TF\tfrac{1}{2}, TG\tfrac{1}{3}) \quad \leq \quad & a_1 d(T\tfrac{1}{2}, T\tfrac{1}{3}) + a_2 D(T\tfrac{1}{2}, TF\tfrac{1}{2}) + a_3 D(T\tfrac{1}{3}, TG\tfrac{1}{3}) \\ & + a_4 D(T\tfrac{1}{2}, TG\tfrac{1}{3}) + a_5 D(T\tfrac{1}{3}, TF\tfrac{1}{2}). \end{aligned}$$

Now let $x = \frac{1}{3}$, and $y = \frac{1}{2}$ then

$$H(TF\tfrac{1}{3}, TG\tfrac{1}{2}) = H(T\{\tfrac{1}{4}, \tfrac{1}{5}\}, T\tfrac{1}{2}\}) = H(\{\tfrac{1}{4}, \tfrac{1}{5}\}, \{\tfrac{1}{3}\}) = \tfrac{2}{15}.$$

On the other hand $d(T\frac{1}{3}, T\frac{1}{2}) = d(\frac{1}{2}, \frac{1}{3}) = \frac{1}{6}$. Let $a_1 = \frac{6}{7}$, and $a_2 = a_3 = a_4 = a_5 = 0$ then clearly $\frac{2}{15} < \frac{6}{7}\frac{1}{6} = \frac{1}{7}$, hence

$$\begin{aligned} H(TF\tfrac{1}{3}, TG\tfrac{1}{2}) \quad \leq \quad & a_1 d(T\tfrac{1}{3}, T\tfrac{1}{2}) + a_2 D(T\tfrac{1}{3}, TF\tfrac{1}{3}) + a_3 D(T\tfrac{1}{2}, TG\tfrac{1}{2}) \\ & + a_4 D(T\tfrac{1}{3}, TG\tfrac{1}{2}) + a_5 D(T\tfrac{1}{2}, TF\tfrac{1}{3}). \end{aligned}$$

So (F, G) is a $T-\ Hardy-Rogers$ pair on K. Hence (F, G) have the common $T-$ approximate strict fixed point property on the boundary of K. All the conditions of Theorem 2.5 are satisfied. Moreover $\{\frac{1}{2}\} = G\frac{1}{2} = F\frac{1}{2}$.

Example 3.3. Let $X = [0, 1]$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Obviously X is a complete metric space. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{3}{4}x, & if \quad x \neq 1 \\ 1 & if \quad x = 1 \end{cases}$$

T is injective and closed graph mapping. Let $K = \{0, 1\} = K$ be a closed subset of X and $F, G : K \to P_{cl,\ bd}(X)$ be defined by

$$Fx = \begin{cases} 0, & if \quad x = 0 \\ [0, \frac{1}{2}] & if \quad x = 1 \end{cases} \quad and \quad Gx = \begin{cases} 0, & if \quad x = 0 \\ [0, \frac{1}{3}] & if \quad x = 1 \end{cases}$$

Let $x = 0$, and $y = 1$ then

$$H(TF0, TG1) = H(T0, T[0, \frac{1}{3}]) = H(\{0\}, [0, \frac{1}{4}]) = \tfrac{1}{4}.$$

On the other hand $d(T0, T1) = d(0, 1) = 1$. Let $a_1 = \frac{1}{2}$, and $a_2 = a_3 = a_4 = a_5 = 0$ then clearly $\frac{1}{4} < \frac{1}{2}$, hence

$$\begin{aligned} H(TF0, TG1) \quad \leq \quad & a_1 d(T0, T1) + a_2 D(T0, TF0) + a_3 D(T1, TG1) \\ & + a_4 D(T0, TG1) + a_5 D(T1, TF0). \end{aligned}$$

Now let $x = 1$, and $y = 0$ then

$$H(TF1, TG0) = H(T[0, \frac{1}{2}], T0\}) = H([0, \frac{3}{8}], \{0\}) = \tfrac{3}{8}.$$

On the other hand $d(T1, T0) = d(1, 0) = 1$. Let $a_1 = \frac{2}{3}$, and $a_2 = a_3 = a_4 = a_5 = 0$ then clearly $\frac{3}{8} < \frac{2}{3}$, hence

$$\begin{aligned} H(TF1, TG0) \quad \leq \quad & a_1 d(T1, T0) + a_2 D(T1, TF1) + a_3 D(T0, TG0) \\ & + a_4 D(T1, TG0) + a_5 D(T0, TF1). \end{aligned}$$

So (F, G) is a $T-\ Hardy-Rogers$ pair on K. Hence (F, G) have the common $T-$approximate strict fixed point property on the boundary of K . All the conditions of Theorem 2.5 are satisfied. Moreover $\frac{1}{2}$ is the unique common strict fixed point of the pair (F, G).

4. Applications

In the following we give three applications to support our results:

(**I**) : Solve an integral equations system.

Theorem 4.1. *Let $X = C[0,1]$ and let*

$$K = \{x \in C[0,1] : x(-t) = x(t) \ for \ all \ t \in [0,1]\}.$$

Consider the following problem:

Problem (A):

$$\begin{cases} x(t) = c_0 + \int_0^t f_1(s,x(s))p_1(s)\ ds - \int_0^t f_2(s,x(s))p_2(s)\ ds & , \quad x(0)=0, \\ y(t) = c_0 + \int_0^t g_1(s,y(s))p_1(s)\ ds - \int_0^t g_2(s,y(s))p_2(s)\ ds. & , \quad y(0)=0 \end{cases}$$

where for each $s \in [0,1]$, $c_0 > 0$ and $x, y \in K$

(1). $f_2(s,x(s)) \geq f_1(s,x(s)) \geq g_i(s,x(s))$,
(2). $f_i(-s,-x(s)) = -f_i(s,x(s))$,
(3). $g_i(-s,-y(s)) = -g_i(s,y(s))$,
(4). $|f_1(s,x(s)) - g_1(s,y(s))| \leq ||x-y||_\infty$,
(5). $|f_2(s,x(s)) - g_2(s,y(s))| \leq |x(s) - c_0|$.

Suppose that

$$p_1(t) = ln(\tfrac{1+\sqrt{17}}{4})Cosh(ln(\tfrac{1+\sqrt{17}}{4})t),$$
$$p_2(t) = ln(\tfrac{1+\sqrt{26}}{5})Cosh(ln(\tfrac{1+\sqrt{26}}{5})t).$$

Then, Problem (A) has a unique solution in K.

Proof. Define

$$F(x(t)) = c_0 + \int_0^t f_1(s,x(s))p_1(s)\ ds - \int_0^t f_2(s,x(s))p_2(s)\ ds, \quad x(0)=0,$$
$$\&$$
$$G(y(t)) = c_0 + \int_0^t g_1(s,y(s))p_1(s)\ ds - \int_0^t g_2(s,y(s))p_2(s)\ ds, \quad y(0)=0$$

where $x, y \in K$ and $T(x) = x$. One can easily verified that $F(K) \subseteq K$ and $G(F(K)) \subseteq K$. Also,

$$\int_0^1 p_1(t)\ dt = \tfrac{1}{4},$$
$$\int_0^1 p_2(t)\ dt = \tfrac{1}{5}.$$

Moreover,

$$\begin{array}{rcl} |TF(x(t)) - TG(y(t))| & \leq & \int_0^t |f_1(s,x(s)) - g_1(s,y(s))||p_1(s)|\ ds \\ & + & \int_0^t |f_2(s,x(s)) - g_2(s,y(s))||p_2(s)|\ ds \\ & \leq & \frac{1}{4}||x-y||_\infty + \frac{1}{5}||x - F(x)||_\infty \\ & \leq & a_1||Tx - Ty||_\infty + a_2||Tx - TFx||_\infty \\ & + & a_3||Ty - TGy||_\infty + a_4||Tx - TGy||_\infty \\ & + & a_5||Ty - TFx||_\infty \end{array}$$

Taking $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{5}$ and $a_3 = a_4 = a_5 = 0$ in Corollary 2.8 one can find a unique common fixed point in K which is the unique solution for Problem (A). □

(II) : Existence and uniqueness of common solution of system of equations arising in dynamic programming.

In this section, we assume that U and V are Banach spaces, $W \subseteq U$ and $D \subseteq V$. Suppose that

$$\begin{array}{rcl} \tau & : & W \times D \longrightarrow W \\ g & : & W \times D \longrightarrow \mathbb{R} \\ G, F & : & W \times D \times \mathbb{R} \longrightarrow \mathbb{R}. \end{array}$$

Considering W and D as the state and decision spaces respectively, the problem of dynamic programming reduces to the problem of solving the functional equations:

$$p(x) = \sup_{y\in D}\{g(x,y)+G(x,y,p(\tau(x,y)))\}, \text{ for } x\in W \tag{4.1}$$

$$q(x) = \sup_{y\in D}\{g(x,y)+F(x,y,q(\tau(x,y)))\}, \text{ for } x\in W. \tag{4.2}$$

For more on multistage process involving such functional equations, we refer to ([13, 3, 5, 4, 9, 25]). Now we study the existence and uniqueness of the common and bounded solution of the functional equations (4.1)-(4.2) arising in dynamic programming in the setup of metric spaces.

Let $B(W)$ denotes the closed subspace of $C(W)$, the set of all bounded real valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x\in W}|h(x)|$. Then $(B(W), \|\cdot\|)$ is a Banach space endowed with the metric d defined as $d(h,k) = \sup\limits_{x\in W}|hx-kx|$. Now consider

$$d(h,k) = \sup_{x\in W}|hx-kx|, \tag{4.3}$$

where $h,k \in B(W)$, and d is a complete metric on $B(W)$. Suppose that the following conditions hold:

$(C1)$: G, F and g are bounded.
$(C2)$: For $x\in W$, $h\in B(W)$, define

$$Kh(x) = \sup_{y\in D}\{g(x,y)+G(x,y,h(\tau(x,y)))\}, \tag{4.4}$$

$$Jh(x) = \sup_{y\in D}\{g(x,y)+F(x,y,h(\tau(x,y)))\}. \tag{4.5}$$

Moreover assume that for every $(x,y)\in W\times D, h,k\in B(W)$ and $t\in W$

$$|G(x,y,h(t))-F(x,y,k(t))| \leq M((h(t),k(t)) \tag{4.6}$$

where

$$\begin{aligned} M((h(t),k(t)) &= a_1d(h(t),k(t))+a_2d(h(t),Kh(t))+a_3d(k(t),Jh(t)) \\ &\quad +a_4d(h(t),Jk(t))+a_5d(k(t),Kh(t)), \end{aligned}$$

where $a_i \geq 0$, $i=1,2,...,5$ with $\sum_{i=1}^5 a_i < 1$.
$(C3)$: There exists a sequence of functions $\{h_n\}\subseteq B(W)$ (boundary of $B(K)$) such that

$$\lim_{n\to\infty} d(h_n, Kh_n) = \lim_{n\to\infty} d(h_n, Jh_n) = 0.$$

Theorem 4.2. *Assume that the conditions $(C1)-(C3)$ are satisfied, then the functional equations (4.1) and (4.2) have a unique, common and bounded solution.*

Proof. Note that $(B(W),d)$ is a complete metric space. By $(C1)$, $J,K : B(W)\to C(W)$. The condition $(C3)$ implies that J and K have $I-$approximate fixed point in the $B(W)$. Let λ be an arbitrary positive number and $h_1,h_2\in B(W)$. Choose $x\in W$ and $y_1,y_2\in D$ such that

$$Kh_j < g(x,y_j)+G(x,y_j,h_j(x_j))+\lambda, \tag{4.7}$$

$$Jh_j < g(x,y_j)+F(x,y_j,h_j(x_j))+\lambda, \tag{4.8}$$

where $x_j = \tau(x,y_j)$, $j=1,2$. By (4.4) and (4.5), it follows that

$$Kh_1 \geq g(x,y_2)+G(x,y_2,h_1(x_2)) \tag{4.9}$$

$$Jh_2 \geq g(x,y_1)+F(x,y_1,h_2(x_1)). \tag{4.10}$$

Now (4.7) and (4.10) imply that

$$\begin{aligned} Kh_1(x)-Jh_2(x) &< G(x,y_1,h_1(x_1))-F(x,y_1,h_2(x_2))+\lambda \\ &\leq |G(x,y_1,h_1(x_1))-F(x,y_1,h_2(x_2))|+\lambda \\ &\leq M((h(t),k(t))+\lambda. \end{aligned} \tag{4.11}$$

From (4.7) and (4.8), we have

$$\begin{array}{rcl} Jh_2(x) - Kh_1(x) & \leq & F(x, y_1, h_2(x_2)) - G(x, y_1, h_1(x_1)) \\ & \leq & |G(x, y_1, h_1(x_1)) - F(x, y_1, h_2(x_2))| \\ & \leq & M((h(t), k(t)). \end{array}$$

Using (4.11) and (4.12), we have

$$|Kh_1(x) - Jh_2(x)| \leq M((h(t), k(t)). \tag{4.12}$$

As above inequality is true for any $x \in W$, and $\lambda > 0$ is taken arbitrary so we obtain

$$d(Kh_1, Jh_2) \leq M((h(t), k(t)). \tag{4.13}$$

Therefore by Corollary (B), the pair (K, J) has a unique common fixed point h^*, that is, $h^*(x)$ is unique, bounded and common solution of (4.1) and (4.2). □

(**III**) : Existence and uniqueness of common solution of system of integral equations.

Now we discuss the application of fixed point theorems we proved in the previous section in solving the system of Volterra type integral equations. Such system is given by the following equations.

$$u(t) = \int_0^t K_1(t, s, u(s))ds + g(t), \tag{4.14}$$

$$w(t) = \int_0^t K_2(t, s, w(s))ds + g(t). \tag{4.15}$$

for $t \in [0, a]$, where $a > 0$. We find the solution of the system (4.14) and (4.15). Let $C_1([0, a], \mathbb{R})$ be the closed subspace of $C([0, a], \mathbb{R})$ all continuous functions defined on $[0, a]$. For $u \in C_1([0, a], \mathbb{R})$ define supremum norm as:

$$\|u\|_\tau = \sup_{t\in[0,a]} \{|u(t)|\, e^{-\tau t}\}$$

where $\tau > 0$ is taken arbitrary. Let $C_1([0, a], \mathbb{R})$ be endowed with the metric

$$d_\tau(u, v) = \sup_{t\in[0,a]} \|u(t) - v(t)\|_\tau \tag{3.17}$$

for all $u, v \in C_1([0, a], \mathbb{R})$. With these setting $C_1([0, a], \mathbb{R}, \|\cdot\|_\tau)$ becomes Banach space.

Now we prove the theorem to ensure the existence of solution of system of integral equations. For more information on such applications we refer the reader to [1, 24].

Theorem 4.3. *Suppose that (i)* $K_1, K_2 : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$ *and* $g : [0, a] \to \mathbb{R}$ *are continuous;*

(ii) Define

$$\begin{array}{rcl} Tu(t) & = & \displaystyle\int_0^t K_1(t, s, u(s))ds + g(t), \\ Su(t) & = & \displaystyle\int_0^t K_2(t, s, u(s))ds + g(t). \end{array}$$

If there exists a $\tau \geq 1$ *such that*

$$\|K_1(t, s, u) - K_2(t, s, v)\|_\tau \leq \tau M(u, v)$$

for all $t, s \in [0, a]$ *and* $u, v \in C_1([0, a], \mathbb{R})$, *where*

$$\begin{aligned} M(u,v) &= a_1\|u(t)-v(t)\|_\tau + a_2\|u(t)-Tu(t)\|_\tau \\ &+ a_3\|v(t)-Sv(t)\|_\tau + a_4\|u(t)-Tv(t)\|_\tau \\ &+ a_5\|v(t)-Su(t)\|_\tau\}. \end{aligned}$$

(iii) there exists a sequence $\{u_n\} \subseteq C_1([0, a], \mathbb{R})$ *such that* $\lim_{n\to\infty} |u_n(t) - Tu_n(t)| = \lim_{n\to\infty} |u_n(t) - Su_n(t)| = 0$. *Then the system of integral equations given in (4.14) and (4.15) has a solution.*

Proof. By assumption (iii) it follows that S and T have $I-$approximate fixed point in the $C_1([0, a], \mathbb{R})$.

$$\begin{aligned} |Tu(t) - Sv(t)| &= \int_0^t |K_1(t,s,u(s) - K_2(t,s,v(s)))|\, ds \\ &= \int_0^t |K_1(t,s,u(s) - K_2(t,s,v(s)))|\, e^{-\tau s} e^{\tau s} ds \\ &\leq \int_0^t \|K_1(t,s,u) - K_2(t,s,v)\|_\tau e^{\tau s} ds \\ &\leq \int_0^t \tau M(u,v) e^{\tau s} ds \leq \tau M(u,v) \int_0^t e^{\tau s} ds \\ &\leq \tau M(u,v) \frac{e^{\tau t}}{\tau}. \end{aligned}$$

This implies

$$|Tu(t) - Sv(t)|\, e^{-\tau t} \leq M(u,v),$$

That is

$$\|Tu(t) - Tv(t)\|_\tau \leq M(u,v)$$

So all the conditions of Corollary 2.8 are satisfied. Hence the system of integral equations given in (4.14) and (4.15) has a unique common solution. □

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