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Research Paper

NUMERICAL SOLUTION OF FOKKER-PLANCK EQUATION USING THE LEAST SQUARES METHOD WITH SATISFIER FUNCTION

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Abstract. In this article, we have solved the Fokker-Planck Equation(FPE) by numerical method. For the approximate solution of this problem, we used of polynomial basis functions and the least squares method. The least squares method together with the satisfier function are used to transform the the FPE to the solution of equation systems. Also, we debate the convergence of the presented technique. Then we consider illustrative examples to represent the applicability and validity of this method.

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1. Introduction and Background

FPE describes phenomena in different fields of natural science such as chemical physics, solid-state physics, quantum optics and circuit theory [8]. In first, this equation was utilized by Fokker and Planck (see [14]) to represent the Brownian motion particles. If a small particle of mass m is immersed in a fluid, the equation of motion for the distribution function $Q(s,t)$ is given by:

$$\frac{\partial Q}{\partial t} = \gamma \frac{\partial (sQ)}{\partial s} + \gamma \frac{KT}{m} \frac{\partial^2 Q}{\partial s^2}. \tag{1.1}$$

Where t is the time, s is the velocity for the Brownian motion of small particle, respectively γ, K and T are the fraction constant, Boltzmann's constant, the temperature of fluid, and [14]. Above equation is one of the simplest type of FPEs.
The general FPEs for the motion of a concentration field $u(x,t)$ of one space variable x at time t has the following form [1, 10, 11, 14, 16]

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial x^2} B(x) \right] u, \tag{1.2}$$

while the initial condition is:

$$u(x,0) = g(x), \quad x \in R, \tag{1.3}$$

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where $u(x,t)$ is unknown, $B(x) > 0$ is the diffusion coefficient, and $A(x) > 0$ is the drift coefficient. The drift and diffusion coefficients may also depend on time i.e.

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right]u. \tag{1.4}$$

Equation (1.1) is seen to be a special case of the Fokker-Planck equation where the drift coefficient is linear and the diffusion coefficient is constant. Equation (1.2) is an equation of motion for the distribution function $u(x,t)$. Mathematically speaking, this equation is a linear second-order partial differential equation of parabolic type. Equation (1.1) is also called forward Kolmogorov equation [1]. The similar partial differential equation is a backward Kolmogorov equation that is [14] in the form:

$$\frac{\partial u}{\partial t} = \left[-A(x,t)\frac{\partial}{\partial x} + B(x,t)\frac{\partial^2}{\partial x^2}\right]u. \tag{1.5}$$

A generalization of equation (1.2) to N variables $x_1, \cdots x_N$ has the following form:

$$\frac{\partial u}{\partial t} = \left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(X) + \sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(X)\right]u, \tag{1.6}$$

where $X = (x_1, \cdots x_N)$ and the initial condition is:

$$u(X,0) = g(X), \quad X \in \mathbb{R}^N. \tag{1.7}$$

The drift vector A_i and diffusion tensor $B_{i,j}$ generally depend on N variables $x_1, \cdots x_N$.
Finding analytical solutions of the Fokker-Planck equation is difficult; especially, if no separation of variables is possible or if the number of variables is large.
various methods of solutions are: numerical integration methods, simulation methods, etc. [14].
There is a more general form of Fokker-Planck equation. Nonlinear Fokker-Planck equation has important applications in various areas such as plasma physics, population dynamics, engineering, polymer physics, biophysics, psychology and marketing [16]. In one variable case the nonlinear Fokker-Planck equation is written in the following form:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)\right]u. \tag{1.8}$$

For N variables $x_1, \cdots x_N$ it has the form:

$$\frac{\partial u}{\partial t} = \left[\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(X,t,u) + \sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(X,t,u)\right]u, \tag{1.9}$$

where $X = (x_1 \cdots x_N)$. It is worth to note that some semi analytic methods are utilized to solve the FPE. For example, by the Adomian decomposition method in [1] FPE is studied. In [16], auther developed for this equation by VIM. Readers for other studies on this model or other similar models the interested can see [6, 7, 12, 13, 17]. Researchers of [20] used a finite difference method [2]-[5] to solve the type of FPEs that in a storage ring was explaining the stochastic dynamics. The influence of electromagnetic models conduce to stochastic differential equations in equivalently to the FPE and 6-dimensional phase space[20].

In this investigation, we transform the problem to a set of algebraic equations system by expanding the unknown function as Legendre polynomials, with unknown coefficients. Recently, the spectral method with Satisfier function has successfully been applied to approximate the solutions of non-homogenous problems [9], [18] and [19]. The used technique is to transform the given problem to the problem of finding optimal solution of a real value function. Unknown functions are expanded with unknown coefficients and polynomial basis functions. Then an algebraic function in terms of unknown coefficients is attained which should be optimized with respect to its variables.
The different parts of this article are: In two section, we are appropriated to the solution of FPEs by the least square technique. We debate the convergence of the technique in section 3. Section 4 includes our numerical findings together with demonstrating the accuracy of the proposed scheme via some numerical examples. Finally, section 5 is conclusion.

2. Satisfier function and Least squares method

in this article, we consider the Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)\right]u, \tag{2.1}$$

with initial condition

$$u(x,0) = g(x) \quad 0 \leqslant x \leqslant 1. \tag{2.2}$$

Let

$$F(u) = \frac{\partial u}{\partial t} - \left[-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)\right]u = 0. \tag{2.3}$$

A least squares approximation to (2.3) is constructed as follows. The approximation u_{nm} is sought in the form of the truncated series

$$u_{nm}(x,t) = \sum_{k=0}^{n}\sum_{l=0}^{m} c_{kl}\psi_{kl}(x,t) + w(x,t) \tag{2.4}$$

where $\psi_{kl}(x,t) = t\phi_k(x)\phi_l(t)$, and $\{\phi_k\}_{k=0}^{\infty}$ are basis function. In relation (2.4), $\psi_{kl}(x,t)$ (for $k = 0,1,...,n, l = 0,1,...,m$), satisfy in homogenous condition and $w(x,t)$ satisfies in nonhomogeneous condition, thus $u_{nm}(x,t)$ satisfies in condition (2.2). In this study, we take $\{\phi_k\}_{k=0}^{\infty}$ the Legendre polynomials in $[0,1]$, and $w(x,0) = g(x)$. The first few Legendre polynomials in $[0,1]$ are given by

$$\begin{aligned}
\phi_0(x) &= 1,\\
\phi_1(x) &= 2x-1,\\
\phi_2(x) &= \frac{1}{2}(3(2x-1)^2-1),\\
\phi_3(x) &= \frac{1}{2}(5(2x-1)^3-3(2x-1)),\\
\phi_4(x) &= \frac{1}{8}(35(2x-1)^4-30(2x-1)^2+3),\\
&\vdots
\end{aligned}$$

Let

$$J(u)=\int_0^T\int_0^1 (F(u))^2 dxdt. \tag{2.5}$$

By the least squares equations, the expansion coefficients c_{kl} are obtained

$$\min_{c_{kl}} J(u_{nm}) = \min_{c_{kl}} \|F(u_{nm})\|^2, \tag{2.6}$$

for $k=0,...,n$, $l=0,...,m$, where

$$\|.\|^2=\int_0^T\int_0^1 (.)^2 dxdt. \tag{2.7}$$

Now, if in (2.6) we get by $J(c_{00},c_{01},\dots,c_{nm})$ the expressions that are minimized, for minimization from the necessary and initial conditions, we should solve the systems that are nonlinear.

$$\frac{\partial J}{\partial c_{kl}}=0, \quad k=0,...,n, \quad l=0,...,m. \tag{2.8}$$

Least Squares equations (2.6) gives a system of $(n+m+2)$ nonlinear equations which can be solved for c_{kl}, $k=0,1,....,n$, $l=0,1,....,m$ using Newton's iterative technique.

3. The convergence

In first, we consider $T=1$. This does not affect the generality of the subject. We debate the convergence of the technique presented in previous section in this section. We represent that the approximate value of μ_{nm} convergence to zero, while n and m increase and

$$\mu_{nm}=\min_{c_{ij}} J(u_{nm}) = \min_{c_{ij}} \|F(u_{nm})\|^2 . \tag{3.1}$$

In the following, we express some lemmas and the required function space.
Let $H=[0,1]\times[0,1]$. and then we suppose the Banach space $(C_1^2(H), \|\ .\ \|_\infty)$ as follows

$$C_1^2(H)=\{u: H\to\mathbb{R} \quad | \quad u, u_{xx}, u_t \in C(H)\},$$

and

$$D(H) = \{h(x,t) \in C_1^2(H) \quad | \quad h(x,0) = 0\}.$$

In Lemma 1, it has been shown that polynomials of the metric space $D(H)$ are dens in $D(H)$ space.

Lemma 1: Consider $q(x,t) \in D(H)$. Sequence of polynomial functions $\{l_{mn}(x,t)\}_{m,n\in N} \subset D(H)$ such that $l_{mn} \to q$ with respect to $\| \,.\, \|_\infty$ exists in $D(H)$ space.

Proof: In first, we suppose $q(x,t) \in D(H)$, then $\frac{\partial q(x,t)}{\partial t} \in C(H)$. Now consider

$$q(x,t) = q(x,0) + \int_0^t \frac{\partial q(x,s)}{\partial s} ds,$$

there exists sequence of polynomials $\{k_{mn}\}_{m,n} \in N$ such that $k_{mn} \to \frac{\partial q}{\partial t}$ with respect to $\| \,.\, \|_\infty$. This is according to Weierstrass approximation theorem . Let

$$p_{mn}(x,t) = q(x,0) + \int_0^t k_{mn}(x,s)ds.$$

Note that $p_{mn}(x,t)$ is a polynomial with degree depend on n and m. Besides, it has the following properties

$$p_{nm}(x,0) = q(x,0) = 0, \tag{3.2}$$

and $p_{mn} \to h$, with $\|.\|_\infty$.

■

Consider equations (2.1) and (2.2) . In addition, let $G^{mn}(H)$ be as follows:

$$G^{mn}(H) = \{l_{mn} + f \quad | \quad l_{mn} \in D(H) \bigcap < \{\phi_i(x)\}_{i=0}^m \times \{\phi_j(t)\}_{j=0}^n >\},$$

$< \{\phi_i(x)\}_{i=0}^m >$ is the Banach space that this space generated by polynomials of degree at most m . Therefore $G^{mn}(H)$ is a metric subspace of $C_1^2(H)$.

Lemma 2: Suppose $u(x,t)$ function be the solution of the Eq. (2.1). if $m, n \to \infty$ with respect to $\| \,.\, \|_\infty$. , function $u^* \in G^{mn}(H)$ exists that $u^* \to u$.

Proof: With conditions $u(x,0) = g(x)$, suppose u be the solution of problem (2.1). Also suppose $q(x,t) := u(x,t) - g(x)$, obviously $q(x,t) \in D(H)$. There exists a sequence of polynomial basis functions $\{l_{mn}(x,t)\}_{m,n\in N} \subset D(H)$ such that $l_{mn} \to h$ with respect to $\| \,.\, \|_\infty$. This subject obtain according to Lemma 1

According to $u^*(x,t) = l_{mn}(x,t) + g(x)$, we have $u^* \in G^{mn}(H)$ and $u^* \to u$ with respect to $\| \,.\, \|_\infty$. ■

Now suppose J in (2.5) as an operator $J : (C_1^2(H), \| \,.\, \|) \to \mathbb{R}$. Above lemma 3 represent that the functional J is continuous on its domain. Theorem 1 is very important and key to prove the lemma and the next theorem.

Theorem 1: If B mapping be continuous of a metric space X into a metric space Y that X and Y have compact space. Then B mapping has uniformly continuous.

Proof: [15].

Lemma 3: On the Banach space $(C_1^2(H), \| \,.\, \|_\infty)$, the J is continuous functional.

Proof: We represent that functional $J : C_1^2(H) \to R$ is continuous , where

$$J(u) = \int_0^1 \int_0^1 B^2 dx dt.$$

Let $\varepsilon > 0$ and $u^* \in C_1^2(H)$. We consider

$$I = H \times [-M - d, M + d] \times [-M - d, M + d] \times [-M - d, M + d]$$

and $d > 0$ where

$$M = \max\{\|u^*\|_\infty, \|u^*_{xx}\|_\infty, \|u^*_t\|_\infty\}.$$

Obviously we have

$$Y^* := (x, t, u^*(x, t), u^*_{xx}(x, t), u^*_t(x, t)) \in I. \tag{3.3}$$

$\gamma > 0$ is given. Let $\|u - u^*\| < \delta$ and $\delta > 0$. For small enough value of δ, we have

$$Y := (x, t, u(x, t), u_{xx}(x, t), u_t(x, t)) \in I. \tag{3.4}$$

Since B is continuous mapping on I compact metric space with respect to all its arguments, with the help of Theorem 1, B is uniformly continuous on I. So if $\delta > 0$ be sufficiently small, then $|Y^* - Y| < \gamma$ implies that $|B(Y^*) - B(Y)| < \varepsilon$. Thus, $|J(u) - J(u^*)| < \varepsilon$. ■
Now, we can represent the convergence of the technique.

Theorem 2: Consider μ_{mn} be the minimum of the functional J on $G^{mn}(H)$. Then:

$$\lim_{m,n\to\infty} \mu_{mn} = 0.$$

Proof: Suppose $u \in C_1^2(H)$ be the solution of equation (2.1). Hence, $J(u) = 0$. With the help of Lemma 2, there exists $u_{mn} \in G^{mn}(H)$ such that $\lim_{m,n\to\infty} u_{mn} = u$. Since the minimum function and the J are continuous functional, therefore we have

$$\lim_{m,n\to\infty} \mu_{mn} = \lim_{m,n\to\infty} minJ(u_{mn}) = minJ(\lim_{m,n\to\infty} u_{m,n}) = J(u) = 0.$$

■

4. Illustrative Examples

We use the technique that presented in Section 2 to solve the following examples.

4.1. **Example 1.** We consider equation (1.2), (1.3) with:[7, 8]

$$g(x) = x, \quad x \in [0, 1].$$

Let in Eq. (1.2), we consider $A(x) = -1$, and $B(x) = 1$. The exact solution is $u(x, t) = x + t$. By the technique presented in section 2 with $m = n = 1$, we get the following values of c_{kl}s:

$$c_{00} = 1, c_{01} = 0, c_{10} = 0, c_{11} = 0,$$

and $\mu_{nm} = 0$. Thus from (2.4) we have

$$u_{nm}(x, t) = x + t.$$

Which is the exact solution.

4.2. **Example 2.** In this exmplae, we consider equation (1.2), with $A(x) = x$, $B(x) = \frac{x^2}{2}$ and $g(x) = x$ [7, 8] . The exact solution is $u(x, t) = xe^t$. Using the present method in Section 2 we achieve the following values of c_{ij}s, for $n = 1$ and different values of m in approximation (2.4)

$$m = 2: \quad c_{00} = 0.659505, c_{01} = 0.176129, c_{10} = 0.659211, c_{11} = 0.177011,$$
$$c_{02} = 0.0234729, c_{12} = 0.0228298,$$

$$m = 3: \quad c_{00} = 0.658926, c_{01} = 0.178068, c_{10} = 0.658934, c_{11} = 0.178044,$$
$$c_{02} = 0.0203988, c_{12} = 0.0204415, c_{03} = 0.00174825, c_{13} = 0.00172112,$$

$$m = 4: \quad c_{00} = 0.658952, c_{01} = 0.177989, c_{10} = 0.658952, c_{11} = 0.17799,$$
$$c_{02} = 0.0205326, c_{12} = 0.0205315, c_{03} = 0.00156732, c_{13} = 0.00156898,$$
$$c_{04} = 0.0000995086, c_{14} = 0.0000985175.$$

In the following table the values of minimum μ_{nm} for different values of approximations are denoted. It is obvious that with increase in the number of n, m basis functions , the approximate value μ_{nm} converges to zero.

n	m	μ_{nm}
1	2	3.63895×10^{-6}
1	3	3.63895×10^{-8}
1	4	9.16871×10^{-11}

The following table represents the absolute error using the procedure proposed in the section 2 for $m = 4, n = 1$.

Absolute error with n=1, m=4 for Example 2.

t	x=0.2	x=0.4	x=0.6	x=0.8	x=1.0
0.01	1.19898E-7	1.97124E-7	2.74351E-7	3.51576E-7	4.28802E-7
0.05	3.10534E-7	4.87321E-7	6.64108E-7	8.40895E-7	1.01768E-6
0.10	1.65711E-7	2.09041E-7	2.52371E-7	2.95701E-7	3.39031E-7
0.15	1.07986E-7	2.55929E-7	4.03873E-7	5.51816E-7	6.99761E-7
1.00	3.00371E-9	1.19279E-9	6.18126E-10	2.42904E-9	4.23995E-9

4.3. **Example 3.** In this example, we consider the backward Kolmogorov equation (1.5) [7, 8] with diffusion and drift coefficients given by:

$$A(x,t) = -(x+1),$$

$$B(x,t) = x^2 e^t.$$

The initial condition in (1.3) be given by:

$$g(x) = x+1,$$

. The exact solution is $u(x,t) = e^t(x+1)$.

The values of approximate minimum μ_{nm}, for different number of basis functions n, m, are demonstrated in the following table.

n	m	μ_{nm}
1	2	5.52151×10^{-5}
1	3	2.3303×10^{-7}
1	4	6.05615×10^{-10}

The following table shows the absolute error using the procedure proposed in the section 2 for $n = 1, m = 4$.

Absolute error with n=1, m=4 for Example 3.

t	x=0.2	x=0.4	x=0.6	x=0.8	x=1.0
0.01	6.98206E-7	6.74692E-7	6.51179E-7	6.27665E-7	6.04152E-7
0.05	1.76917E-6	1.62712E-6	1.48506E-6	1.34301E-6	1.20095E-6
0.10	8.40025E-7	5.84412E-7	3.28798E-7	7.31848E-8	1.82429E-7
0.15	8.25439E-7	1.08358E-6	1.34171E-6	1.59985E-6	1.85799E-6
1.00	5.1126E-9	7.81422E-9	1.05158E-8	1.32175E-8	1.59191E-8

4.4. **Example 4.** Consider[7, 8] the nonlinear Fokker-Planck equation (1.8) with:

$$A(x,t,u) = \frac{7}{2}u,$$

$$B(x,t,u) = xu,$$

and

$$g(x) = x.$$

The exact solution is $u(x,t) = \frac{x}{t+1}$. Using the present method in Section 2 we achieve the following values of c_{ij}s, for $n = 1, m = 2$ in approximation (2.4)

$$c_{00} = -0.344824, c_{01} = 0.114111, c_{10} = -0.345492,$$

$$c_{11} = 0.115604, c_{02} = -0.019282, c_{12} = -0.020134,$$

and

$$\mu_{nm} = 5.24079 \times 10^{-5}.$$

The approximate solution of $u(x,t)$ obtained with $n = 1, m = 2$ at $t = 0.5$ is plotted in Figure 1 in comparison with the exact solution. Since the error is very low, the numerical solution is coincide to the exact solution.

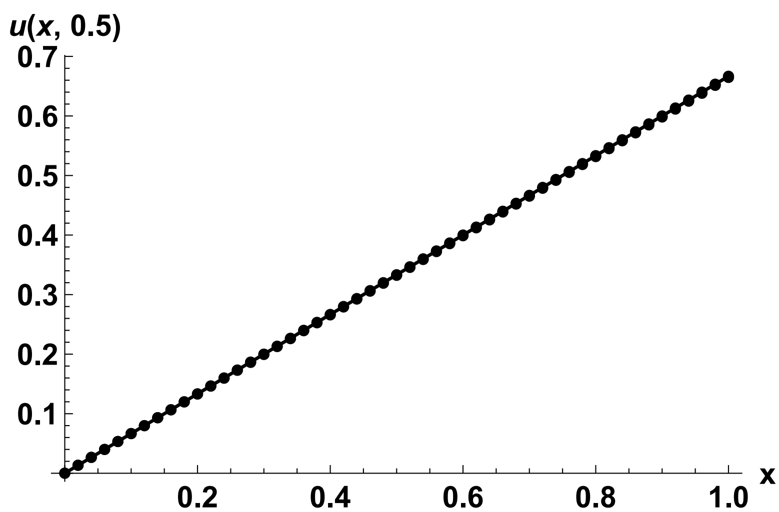

Fig.1. approximate solution and Exact(—) of $u(x, 0.5)$ ($\bullet\bullet\bullet$).

5. Conclusion

In this manuscript, the satisfier function in Least squares method was successfully used for solving the FPEs. The choice of satisfier and basis functions provide great flexibility with which to impose initial conditions. Moreover, only a small number of bases are requiered to obtain a satisfactory result. The convergence of the technique has been extensively debated and illustrative examples to display applicability and validity of the new method are included.

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