Mathematical Analysis
& Convex Optimization

Vol. 4 (2023), No. 1, 97–117
https:\\ maco.lu.ac.ir
DOI: 10.22034/maco.4.1.9

Research Paper

TO NUMERICAL EXPLORE A FRACTIONAL IMPLICIT Q-DIFFERENTIAL EQUATIONS WITH HILFER TYPE AND VIA NONLOCAL CONDITIONS

MOHAMMAD ESMAEL SAMEI* AND ALIREZA HATAMI

Abstract. This paper tries to show that there is only one solution for problem of fractional q-differential equations with Hilfer type, and it does so by using a particular method known as Schaefer's fixed point theorem and the Banach contraction principle. After that, we create a integral type of the problem for nonlocal condition. Next, we show that Ulam stability is true. The Gröwnwall rule for singular kernels of the equations helps to show our findings are correct. We confirm our findings by giving a few practical examples.

MSC(2010): 34A08; 26A33; 34B15.
Keywords: implicit q-differential equation, Hilfer fractional q-derivative, nonlinear analysis, existence, Ulam stability.

1. Introduction

Kilbas, Hilfer, Podlubny and some other potential mathematicians have developed the subject of fractional calculus well [19, 28, 31]. Researchers are using a type of differential and integral equations with fractional order in different parts of engineering, math, physics, and bioengineering [1, 2, 5, 14, 17]. There are many ways to talk about fractional integrals and derivatives. But, the two most common ways are called Riemann-Liouville and Caputo. Recently, Hilfer has introduced a generalized form of the Riemann-Liouville fractional derivative [19]. We suggest that you read a bunch of papers to see how they recently used the Hilfer fractional derivative [15, 20, 41].

In recent research works, several researchers studied fractional implicit differential equations ($\mathbb{FIDE}$s) [4, 9, 16, 36, 37]. Sousa *et al.* investigated the Ulam-Hyers-Rassias (U-H-S) stability for $\mathbb{FIDE}$s using the γ-Hilfer operator [35]. Vivek *et al.* discussed about the existence solutions and U-H stability results for pantograph $\mathbb{FD}$s with Hilfer fractional derivative [38, 39]. The U-stability analysis is very useful in many applications, such as numerical analysis, optimization, etc., where finding the exact solution is quite difficult [23, 40, 42]. Wang *et al.* discussed the existence of solutions to nonlocal initial value problem of Hilfer type $\mathbb{FDE}$ as form

$$\begin{cases} \mathcal{D}^{r,\gamma}\beta(s) = \psi\left(s, \beta(s)\right), & s \in \mathrm{J} := [a, \tau_0],\, a \geq 0, \\ \mathcal{I}^{1-\ell}\beta(a) = \sum_{i=1}^{m} a_i \beta(v_i), & r \leq \ell = r + \gamma - r\gamma < 1,\, v_i \in \mathrm{J}, \end{cases}$$

Date: Received: November 16, 2022, Accepted: March 1, 2023.
*Corresponding author.

where $0 < r < 1$, $0 \leq \gamma \leq 1$ [41]. In 2020, Harikrishnan *et al.*, investigated the pantograph equations of the form

$$\begin{cases} {}^{\rho}\mathcal{D}^{r,\gamma}\beta(s) = \psi\left(s, \beta(s), \beta(ks), {}^{\rho}\mathcal{D}^{r,\gamma}\beta(ks)\right), & s \in \mathrm{J}, \\ {}^{\rho}\mathcal{I}^{1-\ell}\beta(a) = \beta_a, & r \leq \ell < 1, \end{cases}$$

where ${}^{\rho}\mathcal{D}^{r,\gamma}$ the Hilfer-Katugampola fractional derivative of order $0 < r < 1$, type $0 \leq \gamma \leq 1$, with respect to $\rho > 0$ and ${}^{\rho}\mathcal{I}^{1-\ell}$ is fractional integral order $1 - \ell$ such that $\ell = r + \gamma - r\gamma$, $\psi : \mathrm{J} \times \mathbb{R}^3$ is a given continuous function and $0 < \ell < 1$ [18]. The theory of fractional quantum differential equations has become an area of research investigation in the last years, see for example [12, 24, 29] and the references cited therein. Many scholars pay attention to differential equations involving fractional q-calculus [3, 6, 10, 25]. There have been some monographs dealing with the existence and Ulam-stability of solutions for fractional q-differential equations by the use of some well-known fixed point theorems [6, 11, 21, 22, 27, 30, 33].

The goal of this paper is to research an equation that includes a type of fractional quantum derivative called a Hilfer fractional q-derivative, and has nonlocal condition ($\mathrm{H}\mathbb{F}\mathbb{I}q - \mathbb{D}\mathbb{E}$):

$$\begin{cases} \mathcal{D}_q^{r,\gamma}\beta(s) = \psi\left(s, \beta(s), \mathcal{D}_q^{r,\gamma}\beta(s)\right), & s \in \mathrm{J}, \\ \mathcal{I}_q^{1-\ell}\beta(0) = \sum_{i=1}^{m} a_i \beta(v_i), & r \leq \ell = r + \gamma - r\gamma < 1,\ v_i \in \mathrm{J}, \end{cases} \tag{1.1}$$

where $0 < r < 1$, $0 \leq \gamma \leq 1$ are order and type of Hilfer fractional q-derivative, $\psi : \mathrm{J} \times \mathbb{R}^2 \to \mathbb{R}$ is a given continuous function, $(1 - \ell)$ is order of left-sided Riemann-Liouville fractional q-integral, a_i are real numbers and v_i, $i = 1, 2, \ldots, m$ are prefixed points satisfying $0 < v_1 \leq v_2 \leq \cdots \leq v_m < \tau_0$.

This document is laid out like this. Section 2 has important ideas about the Hilfer fractional $q-$derivative. In Sections 3 and 4, we show our main outcome by using Schaefer's theorem for finding a fixed point and we talk about checking if something is stable, respectively. Section 5 has some things to show.

2. Preliminaries

Let $\mathbb{T}_{\mathrm{s}_0} = \{0\} \cup \left\{\mathrm{s} : \mathrm{s} = \mathrm{s}_0 q^{\aleph},\ 0 < q < 1\right\}$, for $\aleph \in \mathbb{N}$, $\mathrm{s}_0 \in \mathbb{R}$ [10]. If there is no confusion concerning s_0 we shall denote $\mathbb{T}_{\mathrm{s}_0}$ by $\mathbb{T}$. Define $[r]_q = (1 - q^r)(1 - q)^{-1}$ for $r \in \mathbb{R}$ [25]. The q-factorial function $(s - \nu)_q^{(\overline{n})}$ for s, $\nu \in \mathbb{R}$ and $\overline{n} \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, is defined by ([3]),

$$(s - \nu)_q^{(\overline{n})} = \begin{cases} \prod_{\mathrm{k}=0}^{\overline{n}-1}(s - \nu q^{\mathrm{k}}), & \overline{n} \in \mathbb{N}, \\ 1, & \overline{n} = 0, \end{cases} \tag{2.1}$$

and for $r \in \mathbb{R}$, we have

$$(s - \nu)_q^{(r)} = s^r \prod_{k=0}^{\infty} \frac{\mathrm{v} - \nu q^k}{s - \nu q^{r+k}}. \tag{2.2}$$

For more details for the function, we recommend see [6, 7, 13, 34]. Note that $s^{(r)} = s^r$ whenever $\nu = 0$. The q-Gamma function is given by $\Gamma_q(r) = (1-q)^{1-r}(1-q)_q^{(r-1)}$, Note that, $\Gamma_q(r + 1) = [r]_q \Gamma_q(r)$ and q-Beta function is defined by $B_q(\ell, r) = \frac{\Gamma_q(\ell)\Gamma_q(r)}{\Gamma_q(\ell+r)}$ [7, Lemma 1].

The higher order q-derivative and the q-integral of a function $\beta : \mathbb{T} \to \mathbb{R}$ are defined by

$$\mathcal{D}_q\beta(s) = \left(\tfrac{\mathrm{d}}{\mathrm{d}s}\right)_q \beta(s) = \tfrac{\beta(qs)-\beta(s)}{(q-1)s}, \qquad \forall\, s \in \mathbb{T} \setminus \{0\}, \tag{2.3}$$

$\mathcal{D}_q\beta(0) = \lim_{s\to 0} \mathcal{D}_q\beta(s)$, $\mathcal{D}_q^{\overline{n}}\beta(s) = \mathcal{D}_q\left(\mathcal{D}_q^{\overline{n}-1}\beta(s)\right)$, for all $\overline{n} \geq 1$ and

$$\mathcal{I}_q\beta(s) = \int_0^s \beta(\eta)\,\mathrm{d}_q\eta = s(1-q)\sum_{k=0}^{\infty} q^k\beta(q^k s), \qquad 0 \leq s \leq \tau_0, \tag{2.4}$$

provided the series is absolutely converges, respectively [3]. If $s_1 \in [0, \tau_0]$, then

$$\int_{s_1}^{\tau_0} \beta(\nu)\,\mathrm{d}_q\nu = \mathcal{I}_q\beta(\tau_0) - \mathcal{I}_q\beta(s_1) = (1-q)\sum_{k=0}^{\infty} q^k\left[\tau_0\beta(q^k\tau_0) - s_1\beta(q^k s_1)\right]. \tag{2.5}$$

In [3], operator $\mathcal{I}_q^{\overline{n}}$ is given by $\mathcal{I}_q^0\beta(s) = \beta(s)$ and $\mathcal{I}_q^{\overline{n}}\beta(s) = \mathcal{I}_q(\mathcal{I}_q^{\overline{n}-1}\beta(s))$, where $\overline{n} \geq 1$, $\beta \in C(\mathrm{J})$. It has been proved that $\mathcal{D}_q\left(\mathcal{I}_q\beta(s)\right) = \beta(s)$, $\mathcal{I}_q(\mathcal{D}_q\beta(s)) = \beta(s) - \beta(0)$, whenever the function β is continuous at $s = 0$ [3]. The fractional Riemann-Liouville type q-integral of the function is defined by

$$\mathcal{I}_q^r\beta(s) = \int_0^s \frac{(s-\nu)_q^{(r-1)}}{\Gamma_q(r)}\beta(\nu)\,\mathrm{d}_q\nu, \qquad \mathcal{I}_q^0\beta(s) = \beta(s), \tag{2.6}$$

$\forall\, s \in \mathrm{J}$ and $r > 0$ [6, 13]. The Caputo fractional q-derivative of the function β is defined by

$$^{\mathrm{C}}\mathcal{D}_q^r\beta(s) = \mathcal{I}_q^{r-[r]}\left(\mathcal{D}_q^{[r]}\beta(s)\right) = \int_0^s \frac{(s-\nu)_q^{(r-[r]-1)}}{\Gamma_q\left(r-[r]\right)}\,\mathcal{D}_q^{[r]}\beta(\nu)\,\mathrm{d}_q\nu, \tag{2.7}$$

for all $s \in \mathrm{J}$ and $r > 0$ [13, 32]. It has been proved that $\mathcal{I}_q^{r_1}\left(\mathcal{I}_q^{r_2}\beta(s)\right) = \mathcal{I}_q^{r_1+r_2}\beta(s)$ and $^{\mathrm{C}}\mathcal{D}_q^r\left(\mathcal{I}_q^r\beta(s)\right) = \beta(s)$, where $r_1, r_2 \geq 0$ [13]. See [26, 28, 34] for numerical Algorithms.

Definition 2.1. [19] The left-sided Hilfer fractional q-derivative of order $0 < r < 1$ and $0 \leq \gamma \leq 1$ of function $\beta(s)$ is defined by

$$\mathcal{D}_q^{r,\gamma}\beta(s) = \mathcal{I}_q^{\gamma(1-r)}\,D\left(\mathcal{I}_q^{(1-\gamma)(1-r)}\beta(s)\right), \qquad D := \tfrac{\mathrm{d}}{\mathrm{d}s}.$$

Remark 2.2. (see [19])

(1) The operator $\mathcal{D}_q^{r,\gamma}$ also can be written as

$$\mathcal{D}_q^{r,\gamma} = \mathcal{I}_q^{\gamma(1-r)}\,D\left(\mathcal{I}_q^{(1-\gamma)(1-r)}\beta(s)\right) = \mathcal{I}_q^{\gamma(1-r)}\left(\mathcal{D}_q^{\ell}\beta(s)\right),$$

with $\ell = r + \gamma - r\gamma$.

(2) The left-sided Riemann-Liouville and Caputo fractional derivative can be presented as $\mathcal{D}_q^r = \mathcal{D}_q^{r,0}$ and $^{\mathrm{C}}\mathcal{D}_q^r\beta(s) := \mathcal{I}_q^{1-r}D\beta(s)$ whenever $\gamma = 0$ and $\gamma = 1$ respectively.

Lemma 2.3. *[28, Property 2.1] If $r > 0$ and $\gamma > 0$, there exist*

$$\mathcal{I}_q^r\beta^{\gamma-1}(s) = \frac{\Gamma_q(\gamma)}{\Gamma_q(r+\gamma)}s^{r+\gamma-1}, \quad \mathcal{D}_q^r\beta^{r-1}(s) = 0,\ (0 < r < 1).$$

3. Main Results

We consider the Banach (B) and Lebesgue-integrable functions spaces, $C(\mathrm{J})$ and $L^1(\mathrm{J})$ with the norms $\|\beta\|_C = \max\{|\beta(s)| : s \in \mathrm{J}\}$ and

$$\|\beta\|_1 = \int_0^{\tau_0} |\beta(\lambda)|\, \mathrm{d}\lambda,$$

respectively. Also, we define the weighted space by

$$C_\ell(\mathrm{J}) = \Big\{\beta : (0, \tau_0] \to \mathbb{R} : s^\ell \beta(s) \in C(\mathrm{J}), 0 \le \ell < 1\Big\}.$$

Obviously, $C_\ell(\mathrm{J})$ is the B space with the norm $\|\beta\|_{C_\ell} = \|s^\ell \beta(s)\|_C$. Meanwhile,

$$C_\ell^n(\mathrm{J}) = \Big\{\beta \in C^{n-1}(\mathrm{J}) : \beta^{(n)} \in C_\ell(\mathrm{J})\Big\},$$

is the B-space with the norm

$$\|\beta\|_{C_\ell^n} = \sum_{i=0}^{n-1} \left\|\beta^{(k)}\right\|_C + \left\|\beta^{(n)}\right\|_{C_\ell}, \qquad n \in \mathbb{N}.$$

Moreover, $C_\ell^0(\mathrm{J}) = C_\ell(\mathrm{J})$.

Lemma 3.1. *[28, Lemma 2.3] If $r > 0$, $\gamma > 0$, and $\beta \in L^1(\mathrm{J})$, for $s \in \mathrm{J}$ there exist the following properties $\mathcal{I}_q^r\left(\mathcal{I}_q^\gamma \beta(s)\right) = \mathcal{I}_q^{r+\gamma}\beta(s)$ and $\mathcal{D}_q^r\left(\mathcal{I}_q^r \beta(s)\right) = \beta(s)$. In particular, if β belongs to $C_\ell(\mathrm{J})$ or $C(\mathrm{J})$, then these equalities hold at each s belongs to $(0, \tau_0]$ or J, respectively.*

Lemma 3.2. *[28, Lemma 2.5] Let $0 < r < 1$, $0 \le \ell < 1$. If $\beta \in C_\ell(\mathrm{J})$ and $\mathcal{I}_q^{1-r}\beta \in C_\ell^1(\mathrm{J})$, then*

$$\mathcal{I}_q^r\left(\mathcal{D}_q^r \beta(s)\right) = \beta(s) - \frac{\mathcal{I}_q^{1-r}\beta(0)}{\Gamma_q(r)} s^{r-1}, \qquad \forall\, s \in \mathrm{J}.$$

Lemma 3.3. *[20, Lemma 13] For $0 \le \ell < 1$ and $\beta \in C_\ell(\mathrm{J})$, then $\mathcal{I}_q^r \beta(0) := \lim_{s\to 0^+} \mathcal{I}_q^r \beta(s) = 0$ for $0 \le \ell < r$.*

Lemma 3.4. *[20, Lemma 20] Let $r > 0$, $\gamma > 0$, and $\ell = r + \gamma - r\gamma$. If*

$$\beta \in C_{1-\ell}^\ell(\mathrm{J}) = \Big\{\beta \in C_{1-\ell}(\mathrm{J}) : \mathcal{D}_q^\ell \beta \in C_{1-\ell}(\mathrm{J})\Big\},$$

then $\mathcal{I}_q^\ell\big(\mathcal{D}_q^\ell \beta(s)\big) = \mathcal{I}_q^r\left(\mathcal{D}_q^{r,\gamma}\beta(s)\right)$ and $\mathcal{D}_q^\ell\big(\mathcal{I}_q^r \beta(s)\big) = \mathcal{D}_q^{\gamma(1-r)}\beta(s)$.

Lemma 3.5. *[20, Lemma 21] Let $\beta \in L^1(\mathrm{J})$ and $\mathcal{D}_q^{\gamma(1-r)}\beta \in L^1(\mathrm{J})$ existed, then*

$$\mathcal{D}_q^{r,\gamma}\left(\mathcal{I}_q^r \beta(s)\right) = \mathcal{I}_q^{\gamma(1-r)}\left(\mathcal{D}_q^{\gamma(1-r)}\beta(s)\right).$$

Lemma 3.6. *[20, Theorem 23] Let $\psi : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ be a function such that $\psi \in C_{1-\ell}(\mathrm{J})$ for any $\beta \in C_{1-\ell}(\mathrm{J})$. A function $\beta \in C_{1-\ell}^\ell(\mathrm{J})$ is a solution of q-fractional $\mathbb{IVP}$*

$$\begin{cases} \mathcal{D}_q^{r,\gamma}\beta(s) = \psi(s, \beta(s)), & 0 < r < 1,\ 0 \le \gamma \le 1, \\ \mathcal{I}_q^{1-\ell}\beta(0) = \beta_0, & \ell = r + \gamma - r\gamma, \end{cases}$$

iff β satisfies the following q-integral equation:

$$\beta(s) = \frac{\beta_0 s^{\ell-1}}{\Gamma_q(\ell)} + \int_0^s \frac{(s-\nu)_q^{(r-1)}}{\Gamma_q(r)}\, \psi(\nu, \beta(\nu))\, \mathrm{d}_q\nu.$$

From Lemma 3.6, one can be established an important equivalent q-integral equation for problem (1.1) which in it, we used some ideas from [41, Lemma 2.12], as form

$$\beta(s) = \frac{\Xi s^{\ell-1}}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i - \nu)_q^{(r-1)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu + \int_0^s \frac{(s-\nu)_q^{(r-1)}}{\Gamma_q(r)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu, \tag{3.1}$$

where

$$\Xi = \left[\Gamma_q(\ell) - \sum_{i=1}^{m} a_i\, v_i^{\ell-1}\right]^{-1}, \qquad \Gamma_q(\ell) \neq \sum_{i=1}^{m} a_i v_i^{\ell-1}, \tag{3.2}$$

and $\tilde{\psi}_\beta(s) = \psi(s, \beta(s), \mathcal{D}_q^{r,\gamma}\beta(s))$.

The existence of solutions for Problem (1.1) is discussed in the sequel.

Lemma 3.7. *Assume that $\psi \in C(\mathrm{J} \times \mathbb{R}^2)$. A function $\beta \in C_{1-\ell}^{\ell}(\mathrm{J})$ is a solution of the problem (1.1) iff β satisfies Eq. (3.1).*

Proof. We expressed a solution of problem (1.1) by employing Lemma 3.6, by

$$\beta(s) = \frac{\mathcal{I}_q^{1-\ell}\beta(0)}{\Gamma_q(\ell)} s^{\ell-1} + \int_0^s \frac{(s-\nu)_q^{(r-1)}}{\Gamma_q(r)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu. \tag{3.3}$$

By substituting $s = v_i$ into the equation and multiplying a_i to both sides, we have

$$a_i\beta(v_i) = \frac{\mathcal{I}_q^{1-\ell}\beta(0)}{\Gamma_q(\ell)} a_i v_i^{\ell-1} + a_i \int_0^{v_i} \frac{(v_i-\nu)_q^{(r-1)}}{\Gamma_q(r)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu, \tag{3.4}$$

Thus, The nonlocal condition implies that

$$\mathcal{I}_q^{1-\ell}\beta(0) = \sum_{i=1}^{m} a_i\beta(v_i) = \frac{\mathcal{I}_q^{1-\ell}\beta(0)}{\Gamma_q(\ell)} \sum_{i=1}^{m} a_i\, v_i^{\ell-1} + \sum_{i=1}^{m} a_i \int_0^{s_i} \frac{(v_i-\nu)_q^{(r-1)}}{\Gamma_q(r)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu,$$

indeed

$$\mathcal{I}_q^{1-\ell}\beta(0) = \frac{\Gamma_q(\ell)}{\Gamma_q(r)} \Xi \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i-\nu)_q^{(r-1)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu. \tag{3.5}$$

Eqs. (3.5) to (3.3) imply Eq. (3.1). Furthermore, β is a solution of the Eq. (3.1), when β is a solution of (1.1). Applying $\mathcal{I}_q^{1-\ell}$ to both sides of (3.1), we have

$$\mathcal{I}_q^{1-\ell}\beta(s) = \mathcal{I}_q^{1-\ell} s^{\ell-1} \frac{\Xi}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i-\nu)_q^{(r-1)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu + \mathcal{I}_q^{1-\ell}\left(\mathcal{I}_q^{r}\tilde{\psi}_\beta(s)\right).$$

Using the Lemmas 2.3 and 3.1,

$$\mathcal{I}_q^{1-\ell}\beta(s) = \frac{\Gamma_q(\ell)}{\Gamma_q(r)} \Xi \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i-\nu)_q^{(r-1)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu + \mathcal{I}_q^{1-\gamma(1-r)}\tilde{\psi}_\beta(s).$$

Since $1 - \ell < 1 - \gamma(1-r)$, Lemma 3.3 can be used when taking the limit as $s \to 0$,

$$\mathcal{I}_q^{1-\ell}\beta(0) = \frac{\Gamma_q(\ell)}{\Gamma_q(r)} \Xi \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i-\nu)_q^{(r-1)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu. \tag{3.6}$$

Substituting $s = v_i$ into (3.1), we obtain

$$\beta(v_i) = \frac{\Xi}{\Gamma_q(r)} v_i^{\ell-1} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i - \nu)_q^{(r-1)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu + \int_0^{v_i} \frac{(v_i - \nu)_q^{(r-1)}}{\Gamma_q(r)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu.$$

Then, we derive

$$\begin{aligned}
\sum_{i=1}^{m} a_i \beta(v_i) &= \frac{\Xi}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i - \nu)_q^{(r-1)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu \sum_{i=1}^{m} a_i\, v_i^{\ell-1} \\
&\quad + \sum_{i=1}^{m} a_i \int_0^{v_i} \frac{(v_i - \nu)_q^{(r-1)}}{\Gamma_q(r)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu \\
&= \left(1 + \Xi \sum_{i=1}^{m} a_i\, v_i^{\ell-1}\right) \sum_{i=1}^{m} a_i \int_0^{v_i} \frac{(v_i - \nu)_q^{(r-1)}}{\Gamma_q(r)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu \\
&= \frac{\Gamma_q(\ell)}{\Gamma_q(r)} \Xi \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i - \nu)_q^{(r-1)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu,
\end{aligned}$$

that is

$$\sum_{i=1}^{m} a_i \beta(v_i) = \frac{\Gamma_q(\ell)}{\Gamma_q(r)} \Xi \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i - \nu)_q^{(r-1)} \tilde{\psi}_\beta(\nu)\, \mathrm{d}_q\nu. \tag{3.7}$$

It follows (3.6) and (3.7) that $\mathcal{I}_q^{1-\ell}\beta(0) = \sum_{i=1}^{m} a_i \beta(v_i)$. Now by applying $\mathcal{D}_q^{\ell}$ to both sides of (3.1), it follows from Lemmas 2.3 and 3.4 that

$$\mathcal{D}_q^{\ell}\beta(s) = \mathcal{D}_q^{\gamma(1-r)} \tilde{\psi}_\beta(s) = \mathcal{D}_q^{\gamma(1-r)} \psi\left(s, \beta(s), \mathcal{D}_q^{r,\gamma}\beta(s)\right). \tag{3.8}$$

$\beta \in C_{1-\ell}^{\ell}(\mathrm{J})$, implies that $\mathcal{D}_q^{\ell}\beta \in C_{1-\ell}(\mathrm{J})$, and $\mathcal{D}_q^{\gamma(1-r)}\psi = D\mathcal{I}_q^{1-\gamma(1-r)}\psi \in C_{1-\ell}(\mathrm{J})$. Furthermore, from $\psi \in C_{1-\ell}(\mathrm{J})$, we have $\mathcal{I}_q^{1-\gamma(1-r)}\psi \in C_{1-\ell}(\mathrm{J})$. Thus $\mathcal{I}_q^{1-\gamma(1-r)}\psi \in C_{1-\ell}^{1}(\mathrm{J})$. Hence, ψ and $\mathcal{I}_q^{1-\gamma(1-r)}\psi$ satisfy the conditions of Lemma 3.2. Next, by applying $\mathcal{I}_q^{\gamma(1-r)}$ to both sides of (3.8) and Lemma 3.2, we obtain

$$\mathcal{D}_q^{r,\gamma}\beta(s) = \tilde{\psi}_\beta(s) - \frac{\mathcal{I}_q^{1-\gamma(1-r)} \tilde{\psi}_\beta(0)}{\Gamma_q(\gamma(1-r))} s^{\gamma(1-r)-1},$$

where $\mathcal{I}_q^{\gamma(1-r)}\tilde{\psi}_\beta(0) = 0$ is implied by Lemma 3.3. Hence, it reduces to $\mathcal{D}_q^{r,\gamma}\beta(s) = \tilde{\psi}_\beta(s)$. The results are proved completely. $\square$

Theorem 3.8. *the problem* (1.1) *has at least one solution in*

$$C_{1-\ell}^{\ell}(\mathrm{J}) \subset C_{1-\ell}^{r,\gamma}(\mathrm{J}) = \left\{\beta \in C_{1-\ell}(\mathrm{J}) \,:\, \mathcal{D}_q^{r,\gamma}\beta \in C_{1-\ell}(\mathrm{J})\right\},$$

whenever

(H1) *There exist* $\overline{\beta}_\imath \in C_{1-\ell}(\mathrm{J})$ *with* $\overline{\beta}_\imath^{\,*} = \sup_{s\in \mathrm{J}} \overline{\beta}_\imath(s) < 1$, $\imath = 0, 1, 2$, *such that*

$$|\psi(s, \beta_1, \beta_2)| \leq \overline{\beta}_0(s) + \overline{\beta}_1(s)\,|\beta_1| + \overline{\beta}_2(s)\,|\beta_2|\,, \qquad s \in \mathrm{J},\ \beta_1, \beta_2 \in \mathbb{R}.$$

Proof. Consider the operator $\mathcal{P} : C_{1-\ell}(\mathrm{J}) \to C_{1-\ell}(\mathrm{J})$.

$$(\mathcal{P}\beta)(s) = \frac{\Xi\, s^{\ell-1}}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i - \lambda)_q^{(r-1)} \tilde{\psi}_\beta(\lambda)\, \mathrm{d}_q\lambda + \int_0^{s} \frac{\tilde{\psi}_\beta(\lambda)}{\Gamma_q(r)} (s - \lambda)_q^{(r-1)}\, \mathrm{d}_q\lambda. \tag{3.9}$$

Step 1: Let β_n be a sequence such that $\beta_n \to \beta \in C_{1-\ell}(\mathrm{J})$. Then for each $s \in \mathrm{J}$,

$$\begin{aligned}
&\left|\big((\mathcal{P}\beta_n)(s) - (\mathcal{P}\beta)(s)\big)s^{1-\ell}\right| \\
&\quad \le \frac{|\Xi|}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i - \lambda)_q^{(r-1)} \left|\tilde{\psi}_{\beta_n}(\lambda) - \tilde{\psi}_{\beta}(\lambda)\right| \mathrm{d}_q\lambda \\
&\qquad + \frac{s^{1-\ell}}{\Gamma_q(r)} \int_0^{s} (s - \lambda)_q^{(r-1)} \left|\tilde{\psi}_{\beta_n}(\lambda) - \tilde{\psi}_{\beta}(\lambda)\right| \mathrm{d}_q\lambda \\
&\quad \le \frac{|\Xi|\, B_q(\ell, r) \sum_{i=1}^{m} a_i (v_i)^{r+\ell-1}}{\Gamma_q(r)} \left\|\tilde{\psi}_{\beta_n}(\cdot) - \tilde{\psi}_{\beta_n}(\cdot)\right\|_{C_{1-\ell}} \\
&\qquad + \frac{\tau_0^r B_q(\ell, r)}{\Gamma_q(r)} \left\|\tilde{\psi}_{\beta_n}(\cdot) - \tilde{\psi}_{\beta}(\cdot)\right\|_{C_{1-\ell}} \\
&\quad \le \frac{B_q(\ell, r)}{\Gamma_q(r)} \left(|\Xi| \sum_{i=1}^{m} a_i (v_i)^{r+\ell-1} + \tau_0^r \right) \left\|\tilde{\psi}_{\beta_n}(\cdot) - \tilde{\psi}_{\beta}(\cdot)\right\|_{C_{1-\ell}},
\end{aligned}$$

where we use the formula

$$\begin{aligned}
\int_a^s (s - \lambda)_q^{(r-1)} |\beta(\lambda)|\, \mathrm{d}_q\lambda &\le \left(\int_a^s (s - \lambda)_q^{(r-1)} (\lambda - a)^{\ell-1}\, \mathrm{d}_q\lambda \right) \|\beta\|_{C_{1-\ell}} \\
&= (s - a)_q^{(r+\ell-1)} B_q(\ell, r) \|\beta\|_{C_{1-\ell}}.
\end{aligned}$$

Continiouty $\tilde{\psi}_\beta$ implies that $\|\mathcal{P}\beta_n - \mathcal{P}\beta\|_{C_{1-\ell}} \to 0$ as $n \to \infty$. Thus, $\mathcal{P}$ is continuous.

Step 2: It is enough to show that for $\eta > 0$, there exists a positive constant l such that $\beta \in B_\eta = \{\beta \in C_{1-\ell}(\mathrm{J}) : \|\beta\| \le \eta\}$, we have $\|\mathcal{P}(\beta)\|_{C_{1-\ell}} \le l$, and $\left|s^{1-\ell}(\mathcal{P}\beta)(s)\right| \le \mathscr{A}_1 + \mathscr{A}_2$, where

$$\begin{aligned}
\mathscr{A}_1 &= \frac{|\Xi|}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i - \lambda)_q^{(r-1)} \left|\tilde{\psi}_\beta(\lambda)\right| \mathrm{d}_q\lambda, \\
\mathscr{A}_2 &= \frac{s^{1-\ell}}{\Gamma_q(r)} \int_0^{s} (s - \lambda)_q^{(r-1)} \left|\tilde{\psi}_\beta(\lambda)\right| \mathrm{d}_q\lambda, \qquad (3.10)
\end{aligned}$$

and by (H1),

$$\begin{aligned}
\left|\tilde{\psi}_\beta(s)\right| &= \left|\psi\left(s, \beta(s), \tilde{\psi}_\beta(s)\right)\right| \le \overline{\beta}_0(s) + \overline{\beta}_1(s)\,|\beta(s)| + \overline{\beta}_2(s) \left|\tilde{\psi}_\beta(s)\right| \\
&\le \overline{\beta}_0^{\,*} + \overline{\beta}_1^{\,*} |\beta(s)| + \overline{\beta}_2^{\,*} \left|\tilde{\psi}_\beta(s)\right| \le \frac{\overline{\beta}_0^{\,*} + \overline{\beta}_1^{\,*} |\beta(s)|}{1 - \overline{\beta}_2^{\,*}}. \qquad (3.11)
\end{aligned}$$

We estimate $\mathscr{A}_1, \mathscr{A}_2$ terms separately, we have

$$\begin{aligned}
\mathscr{A}_1 &= \frac{|\Xi|}{(1-\overline{\beta}_2^{\,*})} \sum_{i=1}^{m} a_i \left(\frac{\overline{\beta}_0^{\,*} v_i^r}{\Gamma_q(r+1)} + \frac{\overline{\beta}_1^{\,*} v_i^{r+\ell-1}}{\Gamma_q(r)} B_q(\ell, r) \|\beta\|_{C_{1-\ell}} \right), \\
\mathscr{A}_2 &= \frac{1}{1 - \overline{\beta}_2^{\,*}} \left(\frac{\overline{\beta}_0^{\,*} \tau_0^{r-\ell+1}}{\Gamma_q(r+1)} + \frac{\overline{\beta}_1^{\,*} \tau_0^r}{\Gamma_q(r)} B_q(\ell, r) \|\beta\|_{C_{1-\ell}} \right). \qquad (3.12)
\end{aligned}$$

Thus,

$$\left|s^{1-\ell}\,(\mathcal{P}\beta)(s)\right| \leq \frac{\overline{\beta}_0^{\,*}}{(1-\overline{\beta}_2^{\,*})\Gamma_q(r+1)}\left(|\Xi|\sum_{i=1}^{m} a_i v_i^r + \tau_0^{r+\ell-1}\right) + \frac{\overline{\beta}_1^{\,*}}{(1-\overline{\beta}_2^{\,*})\Gamma_q(r)}\left(|\Xi|\sum_{i=1}^{m} a_i (v_i)^{r+\ell-1} + \tau_0^r\right) B_q(\ell,r)\,\|\beta\|_{C_{1-\ell}} := L.$$

So, $\mathcal{P}$ maps bounded sets into bounded sets in $C_{1-\ell}(\mathrm{J})$.

Step 3: Let $s_1, s_2 \in \mathrm{J}$, $s_2 \leq s_1$, B_η be a bounded set of $C_{1-\ell}(\mathrm{J})$ as in Step 2, and let $\beta \in B_\eta$. Then

$$\begin{aligned}\left|s_1^{1-\ell}(\mathcal{P}\beta)(s_1)\right. & \left. - s_2^{1-\ell}(\mathcal{P}\beta)(s_2)\right| \\ & \leq \left|\frac{s_1^{1-\ell}}{\Gamma_q(r)}\int_0^{s_1}(s_1-\lambda)_q^{(r-1)}\tilde{\psi}_\beta(\lambda)\,\mathrm{d}_q\lambda - \frac{s_2^{1-\ell}}{\Gamma_q(r)}\int_0^{s_2}(s_2-\lambda)_q^{(r-1)}\tilde{\psi}_\beta(\lambda)\,\mathrm{d}_q\lambda\right| \\ & \leq \left|\frac{1}{\Gamma_q(r)}\int_0^{s_1}\left[s_1^{1-\ell}(s_1-\lambda)_q^{(r-1)} - s_2^{1-\ell}(s_2-\lambda)_q^{(r-1)}\right]\tilde{\psi}_\beta(\lambda)\,\mathrm{d}_q\lambda\right| \\ & \quad + \left|\frac{s_2^{1-\ell}}{\Gamma_q(r)}\int_{s_1}^{s_2}(s_2-\lambda)_q^{(r-1)}\tilde{\psi}_\beta(\lambda)\,\mathrm{d}_q\lambda\right|.\end{aligned}$$

As $s_1 \to s_2$, the right hand side of the above inequality tends to zero. This yields, $\mathcal{P}$ maps bounded sets into equicontinuous set of $C_{1-\ell}(\mathrm{J})$. As a consequence of Step 1 to 3, together with Arzelá-Ascoli theorem, we can conclude that $\mathcal{P} : C_{1-\ell}(\mathrm{J}) \to C_{1-\ell}(\mathrm{J})$ is completely continuous.

Step 4: We consider the set of all $\beta \in C_{1-\ell}(\mathrm{J})$ with this feature that $\beta = \delta(\mathcal{P}\beta)$, here $0 < \delta < 1$ and denote by Υ. Thus, for $\beta \in \Upsilon$, we have

$$\beta(s) = \delta\left[\frac{|\Xi|\ s^{\ell-1}}{\Gamma_q(r)}\sum_{i=1}^{m} a_i\int_0^{v_i}(v_i-\lambda)_q^{(r-1)}\tilde{\psi}_\beta(\lambda) + \int_0^s \frac{\tilde{\psi}_\beta(\lambda)}{\Gamma_q(r)}(s-\lambda)_q^{(r-1)}\,\mathrm{d}_q\lambda\right], \qquad \forall s \in \mathrm{J}.$$

This implies by (H2) that for each $s \in \mathrm{J}$, we have

$$\left|s^{1-\ell}\beta(s)\right| \leq \left|s^{1-\ell}\,(\mathcal{P}\beta)(s)\right| \leq \frac{\overline{\beta}_0^{*}}{(1-\overline{\beta}_2^{*})\Gamma_q(r+1)}\left(|\Xi|\sum_{i=1}^{m} a_i\, v_i^r + \tau_0^{r+\ell-1}\right) + \frac{\overline{\beta}_1^{\,*}}{\left(1-\overline{\beta}_2^{\,*}\right)\Gamma_q(r)}\left(|\Xi|\sum_{i=1}^{m} a_i\, v_i^{r+\ell-1} + \tau_0^r\right) B_q(\ell,r)\,\|\beta\|_{C_{1-\ell}} := R.$$

This shows that the set Υ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that $\mathcal{P}$ has a fixed point which is a solution of problem (1.1). The proof is completed. □

Theorem 3.9. *Assume that*

(H2) *Let $\psi : \mathrm{J} \times \mathbb{R}^2 \to \mathbb{R}$ be a function such that $\psi \in C_{1-\ell}^{\gamma(1-r)}(\mathrm{J})$ for any $\beta \in C_{1-\ell}^{\ell}(\mathrm{J})$ and there exist positive constants $K_1, K_2 > 0$ such that*

$$\left|\psi\left(s, \beta_1, \beta_2\right) - \psi\left(s, \acute{\beta}_1, \acute{\beta}_2\right)\right| \leq \sum_{i=1}^{2} K_i \left|\beta_i - \acute{\beta}_i\right|, \qquad \forall\, \beta_i, \acute{\beta}_i \in \mathbb{R},\ s \in \mathrm{J}.$$

If

$$\left(\frac{K_1}{1-K_2}\right)\frac{B_q(\ell, r)}{\Gamma_q(r)}\left(|\Xi| \sum_{i=1}^{m} a_i v_i^{r+\ell-1} + \tau_0^r\right) < 1, \tag{3.13}$$

then the system (1.1) has a unique solution.

Proof. Let the operator $\mathcal{P} : C_{1-\ell}(\mathrm{J}) \to C_{1-\ell}(\mathrm{J})$.

$$(\mathcal{P}\beta)(s) = \frac{|\Xi|\ s^{\ell-1}}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i - \lambda)_q^{(r-1)} \tilde{\psi}_\beta(\lambda)\, \mathrm{d}_q\lambda + \int_0^s \frac{\tilde{\psi}_\beta(\lambda)}{\Gamma_q(r)} (s-\lambda)_q^{(r-1)}\, \mathrm{d}_q\lambda.$$

Lemma 3.7 shows that the fixed points of $\mathcal{P}$ are solutions of system (1.1). Let $\beta_1, \beta_2 \in C_{1-\ell}(\mathrm{J})$ and $s \in \mathrm{J}$, then we have

$$\begin{aligned}\left|((\mathcal{P}\beta_1)(s)\ - (\mathcal{P}\beta_2)(s))\, s^{1-\ell}\right| &\leq \frac{|\Xi|}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i - \lambda)_q^{(r-1)} \left|\tilde{\psi}_{\beta_1}(\lambda) - \tilde{\psi}_{\beta_2}(\lambda)\right| \mathrm{d}_q\lambda \\ &\quad + \frac{s^{1-\ell}}{\Gamma_q(r)} \int_0^s (s-\lambda)_q^{(r-1)} \left|\tilde{\psi}_{\beta_1}(\lambda) - \tilde{\psi}_{\beta_2}(\lambda)\right| \mathrm{d}_q\lambda,\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\left|\tilde{\psi}_{\beta_1}(s) - \tilde{\psi}_{\beta_2}(s)\right| &\leq K_1 \left|\beta_1(s) - \beta_2(s)\right| + K_2 \left|\tilde{\psi}_{\beta_1}(s) - \tilde{\psi}_{\beta_2}(s)\right| \\ &\leq \frac{K_1}{1-K_2} \left|\beta_1(s) - \beta_2(s)\right|.\end{aligned} \tag{3.15}$$

By replacing (3.15) in the inequality (3.14), we get

$$\begin{aligned}&\left|((\mathcal{P}\beta_1)(s)\ - (\mathcal{P}\beta_2)(s))\, s^{1-\ell}\right| \\ &\qquad \leq \frac{|\Xi|}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \left(\frac{K_1}{(1-K_2)} B_q(\ell, r)\, v_i^{r+\ell-1} \left\|\beta_1 - \beta_2\right\|_{C_{1-\ell}}\right) \\ &\qquad\quad + \frac{s^{1-\ell}\, s^{r+\ell-1}}{\Gamma_q(r)} \left(\frac{K_1}{1-K_2}\right) B_q(\ell, r) \left\|\beta_1 - \beta_2\right\|_{C_{1-\ell}} \\ &\qquad \leq \left(\frac{K_1}{1-K_2}\right) \frac{B_q(\ell, r)}{\Gamma_q(r)} \left(|\Xi| \sum_{i=1}^{m} a_i\, v_i^{r+\ell-1} + \tau_0^r\right) \left\|\beta_1 - \beta_2\right\|_{C_{1-\ell}}.\end{aligned}$$

Hence,

$$\left\|(\mathcal{P}\beta_1) - (\mathcal{P}\beta_2)\right\|_{C_{1-\ell}} \leq \left(\frac{K_1\, B_q(\ell, r)}{(1-K_2)\, \Gamma_q(r)}\right)\left(|\Xi| \sum_{i=1}^{m} a_i\, v_i^{r+\ell-1} + \tau_0^r\right) \left\|\beta_1 - \beta_2\right\|_{C_{1-\ell}}.$$

From (3.13), it follows that $\mathcal{P}$ has a unique fixed point which is solution of problem (1.1). □

4. **Stability Analysis**

In this section, we prove four different types of U-stability results for problem (1.1). A function $\acute{\beta} \in C^{\ell}_{1-\ell}(\mathrm{J})$ is a solution of the inequality

$$\left|\mathcal{D}_q^{r,\gamma}\acute{\beta}(s) - \psi\left(s, \acute{\beta}(s), \mathcal{D}_q^{r,\gamma}\acute{\beta}(s)\right)\right| \leq \epsilon, \quad s \in \mathrm{J}, \tag{4.1}$$

iff there exist a function $g \in C^{\ell}_{1-\ell}(\mathrm{J})$ such that
(i) $| \ g(s) \ |\leq \epsilon, s \in \mathrm{J}$;
(ii) $\mathcal{D}_q^{r,\gamma}\acute{\beta}(s) = \psi(s, \acute{\beta}(s), \mathcal{D}_q^{r,\gamma}\acute{\beta}(s)) + g(s)$, for each $s \in \mathrm{J}$.

Definition 4.1. [43] The problem (1.1) is U-H, generalized U-H stable, and U-H-R, generalized U-H-R stable with respect to $\varphi \in C_{1-\ell}(\mathrm{J})$, if there exists

- a real number $C_\psi > 0$ such that for each $\epsilon > 0$ and for each solution $\acute{\beta} \in C^{\ell}_{1-\ell}(\mathrm{J})$ of inequality (4.1) there exists a solution $\beta \in C^{\ell}_{1-\ell}(\mathrm{J})$ of problem (1.1) with
$$\left|\acute{\beta}(s) - \beta(s)\right| \leq C_\psi \epsilon;$$
- $\varphi_\psi \in C\left([0,\infty),[0,\infty)\right)$, $\varphi_\psi(0) = 0$ such that for each solution $\acute{\beta} \in C^{\ell}_{1-\ell}(\mathrm{J})$ of the inequality (4.1), there exists a solution $\beta \in C^{\ell}_{1-\ell}(\mathrm{J})$ of problem (1.1) with
$$\left|\acute{\beta}(s) - \beta(s)\right| \leq \varphi_\psi \epsilon;$$
- a real number $C_\psi > 0$ such that for each $\epsilon > 0$ and for each solution $\acute{\beta} \in C^{\ell}_{1-\ell}(\mathrm{J})$ of the inequality
$$\left|\mathcal{D}_q^{r,\gamma}\acute{\beta}(s) - \psi\left(s, \acute{\beta}(s), \mathcal{D}_q^{r,\gamma}\acute{\beta}(s)\right)\right| \leq \epsilon\varphi(s),$$
there exists a solution $\beta \in C^{\ell}_{1-\ell}(\mathrm{J})$ of problem (1.1) with $\left|\acute{\beta}(s) - \beta(s)\right| \leq C_\psi \epsilon \varphi(s)$;
- a real number $C_{\psi,\varphi} > 0$ such that for each solution $\acute{\beta} \in C^{\ell}_{1-\ell}(\mathrm{J})$ of the inequality
$$\left|\mathcal{D}_q^{r,\gamma}\acute{\beta}(s) - \psi\left(s, \acute{\beta}(s), \mathcal{D}_q^{r,\gamma}\acute{\beta}(s)\right)\right| \leq \varphi(s),$$
there exists a solution $\beta \in C^{\ell}_{1-\ell}(\mathrm{J})$ of problem (1.1) with $\left|\acute{\beta}(s) - \beta(s)\right| \leq C_{\psi,\varphi}\varphi(s)$;

for $s \in \mathrm{J}$, respectively.

Lemma 4.2. *If a function $\acute{\beta} \in C^{\ell}_{1-\ell}(\mathrm{J})$ is a solution of the inequality (4.1), then $\acute{\beta}$ is a solution of the integral inequality*

$$\left|\acute{\beta}(s) - \mathscr{A}_{\acute{\beta}} - \int_0^s \frac{(s-\lambda)^{r-1}}{\Gamma_q(r)}\tilde{\psi}_{\acute{\beta}}(\nu)\,\mathrm{d}_q\nu\right| \leq \left(\frac{|\Xi|\,(ma)\tau_0^{\ell+r-1}}{\Gamma_q(r+1)} + \frac{\tau_0^r}{\Gamma_q(r+1)}\right)\epsilon, \tag{4.2}$$

where,

$$\mathscr{A}_{\acute{\beta}} = \frac{\Xi s^{\ell-1}}{\Gamma_q(r)}\sum_{i=1}^m a_i \int_1^{v_i}(v_i-\nu)_q^{(r-1)}\tilde{\psi}_{\acute{\beta}}(\nu)\,\mathrm{d}_q\nu. \tag{4.3}$$

Proof. By explanations at the beginning of this section, we have $\mathcal{D}_q^{r,\gamma}\acute{\beta}(s) = \tilde{\psi}_{\acute{\beta}}(s) + g(s)$. Then

$$\acute{\beta}(s) = \frac{\Xi\, s^{\ell-1}}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \left(\int_0^{v_i} (v_i - \lambda)_q^{(r-1)} \tilde{\psi}_{\acute{\beta}}(\lambda)\, \mathrm{d}_q\lambda + \int_0^{v_i} (v_i - \lambda)_q^{(r-1)} g(\lambda)\, \mathrm{d}_q\lambda \right)$$
$$+ \int_0^s \frac{\tilde{\psi}_{\acute{\beta}}(\lambda)}{\Gamma_q(r)} (s-\lambda)_q^{(r-1)}\, \mathrm{d}_q\lambda + \int_0^s \frac{(s-\lambda)_q^{(r-1)}}{\Gamma_q(r)} g(\lambda)\, \mathrm{d}_q\lambda.$$

From this it follows that

$$\left| \acute{\beta}(s) - \mathscr{A}_{\acute{\beta}} - \int_0^s \frac{(s-\lambda)_q^{(r-1)}}{\Gamma_q(r)} \tilde{\psi}_{\acute{\beta}}(\lambda)\, \mathrm{d}_q\lambda \right|$$
$$= \left| \frac{\Xi\, s^{\ell-1}}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i - \lambda)_q^{(r-1)} g(\lambda)\, \mathrm{d}_q\lambda + \int_0^s \frac{(s-\lambda)_q^{(r-1)}}{\Gamma_q(r)} g(\lambda)\, \mathrm{d}_q\lambda \right|$$
$$\leq \frac{|\Xi|\ s^{\ell-1}}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i - \lambda)_q^{(r-1)} \left|g(\lambda)\right|\, \mathrm{d}_q\lambda + \int_0^s \frac{(s-\lambda)_q^{(r-1)}}{\Gamma_q(r)} \left|g(\lambda)\right|\, \mathrm{d}_q\lambda$$
$$\leq \left(\frac{|\Xi|\, ma\tau_0^{\ell+r-1}}{\Gamma_q(r+1)} + \frac{\tau_0^r}{\Gamma_q(r+1)} \right)\epsilon,$$

where $a = \max\{a_i : i = 1, 2, \dots, m\}$. □

Lemma 4.3. *If a function $\acute{\beta} \in C_{1-\ell}^{\ell}(\mathrm{J})$ is a solution of the inequality (4.1), then $\acute{\beta}$ is a solution of the integral inequality*

$$\left| \acute{\beta}(s) - \mathscr{A}_{\acute{\beta}} - \int_0^s \frac{(s-\lambda)_q^{(r-1)}}{\Gamma_q(r)} \tilde{\psi}_{\beta}(\lambda)\, \mathrm{d}_q\lambda \right| \leq \left(\Xi\, s^{\ell-1}(ma) + 1 \right) \epsilon C_{\varphi}\varphi(s), \tag{4.4}$$

where, $A_{\acute{\beta}}$ is defined by (4.3).

Proof. The proof of the theorem directly follows from Lemma 4.2. □

The next Lemma 4.4, is generalization of Gronwall's lemma for singular kernals.

Lemma 4.4. *[8, Lemma 3.4] Let $\beta : \mathrm{J} \to [0, \infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on J and there are constant $a > 0$ such that*

$$\beta(s) \leq g(s) + a \int_0^s \frac{\beta(\lambda)}{(s-\lambda)_q^{(r)}}\, \mathrm{d}_q\lambda, \qquad 0 < r < 1.$$

Then there exists a constant K_r such that

$$\beta(s) \leq g(s) + K_r\, a \int_0^s \frac{g(\lambda)}{(s-\lambda)_q^{(r)}}\, \mathrm{d}_q\lambda, \qquad \forall\, s \in \mathrm{J}.$$

We ready to prove our stability results for problem (1.1).

Theorem 4.5. *If the hypothesis* (H2) *and* (3.13) *are satisfied, then the problem* (1.1) *is* U-H *stable.*

Proof. Let $\epsilon > 0$, $\acute{\beta} \in C_{1-\ell}^{\ell}(\mathrm{J})$ be a function which satisfies the inequality:

$$\left| \mathcal{D}_q^{r,\gamma}\acute{\beta}(s) - \psi\left(s, \acute{\beta}(s), \mathcal{D}_q^{r,\gamma}\acute{\beta}(s)\right) \right| \leq \epsilon, \qquad s \in \mathrm{J}, \tag{4.5}$$

and $\beta \in C^{\ell}_{1-\ell}(\mathrm{J})$ the unique solution of the following implicit differential equation

$$\begin{cases} \mathcal{D}_q^{r,\gamma}\beta(s) = \psi\left(s, \beta(s), \mathcal{D}_q^{r,\gamma}\beta(s)\right), & s \in \mathrm{J}, \\ \mathcal{I}_q^{1-\ell}\beta(0) = \mathcal{I}_q^{1-\ell}\acute{\beta}(0) = \sum_{i=1}^{m} a_i \beta(v_i), & v_i \in \mathrm{J},\ \ell = r+\gamma-r\gamma, \end{cases}$$

where $0<r<1$, $0 \leq \gamma \leq 1$. Using Lemma 3.7, we obtain

$$\beta(s) = \mathscr{A}_\beta + \int_0^s \frac{(s-\lambda)_q^{(r-1)}}{\Gamma_q(r)} \tilde{\psi}_\beta(\lambda)\,\mathrm{d}_q\lambda. \tag{4.6}$$

where

$$\mathscr{A}_\beta = \frac{|\Xi|\ s^{\ell-1}}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i-\lambda)_q^{(r-1)} \tilde{\psi}_\beta(\lambda)\,\mathrm{d}_q\lambda.$$

On the other hand, if $\beta(v_i) = \acute{\beta}(v_i)$, and $\mathcal{I}_q^{1-\ell}\beta(0) = \mathcal{I}_q^{1-\ell}\acute{\beta}(0)$, then $A_\beta = A_{\acute{\beta}}$. Indeed,

$$\begin{aligned} \left|\mathscr{A}_\beta - \mathscr{A}_{\acute{\beta}}\right| &\leq \frac{|\Xi|\ s^{\ell-1}}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i-\lambda)_q^{(r-1)} \left|\tilde{\psi}_\beta(\lambda) - \tilde{\psi}_{\acute{\beta}}(\lambda))\right| \mathrm{d}_q\lambda \\ &\leq \frac{|\Xi|\ s^{\ell-1}}{\Gamma_q(r)} \sum_{i=1}^{m} a_i \int_0^{v_i} (v_i-\lambda)_q^{(r-1)} \left(\frac{K_1}{1-K_2}\right) \left|\beta(\lambda) - \acute{\beta}(\lambda)\right| \mathrm{d}_q\lambda \\ &\leq \frac{K_1\,|\Xi|\ s^{\ell-1}}{(1-K_2)\,\Gamma_q(r)} \sum_{i=1}^{m} a_i \mathcal{I}_q^r \left|\beta(v_i) - \acute{\beta}(v_i)\right| = 0. \end{aligned}$$

Thus, $\mathscr{A}_\beta = \mathscr{A}_{\acute{\beta}}$. Then, we obtain afin Eq. (4.6). By integration of the inequality (4.5) and applying Lemma 4.2, we obtain

$$\left|\acute{\beta}(s) - \mathscr{A}_{\acute{\beta}} - \int_0^s \frac{(s-\lambda)_q^{(r-1)}}{\Gamma_q(r)} \tilde{\psi}_{\acute{\beta}}(\lambda)\,\mathrm{d}_q\lambda\right| \leq \left(\frac{|\Xi|\,m\,a\,\tau_0^{\ell+r-1}}{\Gamma_q(r+1)} + \frac{\tau_0^r}{\Gamma_q(r+1)}\right)\epsilon, \tag{4.7}$$

and so, by using (4.7), we have

$$\begin{aligned} \left|\acute{\beta}(s) - \beta(s)\right| &\leq \left|\acute{\beta}(s) - \mathscr{A}_{\acute{\beta}} - \int_0^s \frac{(s-\lambda)_q^{(r-1)}}{\Gamma_q(r)} \tilde{\psi}_{\acute{\beta}}(\lambda)\,\mathrm{d}_q\lambda\right| \\ &\quad + \int_0^s \frac{(s-\lambda)_q^{(r-1)}}{\Gamma_q(r)} \left|\tilde{\psi}_{\acute{\beta}}(\lambda) - \tilde{\psi}_\beta(\lambda)\right| \mathrm{d}_q\lambda \\ &\leq \left|\acute{\beta}(s) - \mathscr{A}_{\acute{\beta}} - \int_0^s \frac{(s-\lambda)_q^{(r-1)}}{\Gamma_q(r)} \tilde{\psi}_{\acute{\beta}}(\lambda)\,\mathrm{d}_q\lambda\right| \\ &\quad + \frac{K_1}{(1-K_2)\,\Gamma_q(r)} \int_0^s (s-\lambda)_q^{(r-1)} \left|\acute{\beta}(\lambda) - \beta(\lambda)\right| \mathrm{d}_q\lambda, \\ &\leq \left(\frac{|\Xi|\,ma\tau_0^{\ell+r-1}}{\Gamma_q(r+1)} + \frac{\tau_0^r}{\Gamma_q(r+1)}\right)\epsilon \\ &\quad + \frac{K_1}{(1-K_2)\,\Gamma_q(r)} \int_0^s (s-\lambda)_q^{(r-1)} \left|\acute{\beta}(\lambda) - \beta(\lambda)\right| \mathrm{d}_q\lambda, \end{aligned}$$

for $s \in \mathrm{J}$, and to apply Lemma 4.4, we obtain

$$\left|\acute{\beta}(s)-\beta(s)\right| \leq \left(|\Xi|\, ma\tau_0^{\ell+r-1}+\tau_0^r\right)\left[1+\frac{\nu K_1\,\tau_0^r}{(1-K_2)\Gamma_q(r+1)}\right]\frac{\epsilon}{\Gamma_q(r+1)} = C_\psi \epsilon,$$

where $\nu=\nu(r)$ is a constant, which completes the proof of the theorem. Moreover, if we set $\psi(\epsilon)=C_\psi\epsilon; \psi(0)=0$, then the problem (1.1) is generalized U-H stable. □

Theorem 4.6. *Assume that* (H2),

(H3) *There exists an increasing function* $\varphi \in C_{1-\ell}(\mathrm{J})$ *and there exists* $C_\varphi>0$ *such that* $\mathcal{I}_q^r\varphi(s) \leq C_\varphi\varphi(s)$ *for* $s\in\mathrm{J}$,

and (3.13) *are satisfied, then the problem* (1.1) *is* U-H-R *stable.*

Proof. Let $\epsilon>0$ and let $\acute{\beta}\in C_{1-\ell}^{\ell}(\mathrm{J})$ be a function which satisfies the inequality:

$$\left|\mathcal{D}_q^{r,\gamma}\acute{\beta}(s)-\psi\left(s,\acute{\beta}(s),\mathcal{D}_q^{r,\gamma}\acute{\beta}(\lambda)\right)\right| \leq \epsilon\varphi(s), \qquad s\in\mathrm{J}, \tag{4.8}$$

and let $\beta\in C_{1-\ell}^{\ell}(\mathrm{J})$ the unique solution of the following implicit differential equation

$$\begin{cases} \mathcal{D}_q^{r,\gamma}\beta(s)=\psi\left(s,\beta(s),\mathcal{D}_q^{r,\gamma}\beta(s)\right), & s\in\mathrm{J},\\ \mathcal{I}_q^{1-\ell}\beta(0)=\mathcal{I}_q^{1-\ell}\acute{\beta}(0)=\displaystyle\sum_{i=1}^m a_i\beta(v_i), & v_i\in\mathrm{J},\ \ell=r+\gamma-r\gamma, \end{cases}$$

where $0<r<1$, $0\leq\gamma\leq 1$. Using Lemma 3.7, we get Eq. (4.6). By integration of (4.8) and applying Lemma 4.3, we obtain

$$\left|\acute{\beta}(s)-\mathscr{A}_{\acute{\beta}}-\int_0^s\frac{(s-\lambda)_q^{(r-1)}}{\Gamma_q(r)}\tilde{\psi}_{\acute{\beta}}(\lambda)\,\mathrm{d}_q\lambda\right| \leq \left(|\Xi|\ s^{\ell-1}ma+1\right)\epsilon C_\varphi\varphi(s).$$

On the other hand, by using Eq. (4.7) and Lemma 4.4, we obtain

$$\begin{aligned}\left|\acute{\beta}(s)-\beta(s)\right| &\leq \left|\acute{\beta}(s)-\mathscr{A}_{\acute{\beta}}-\int_0^s\frac{(s-\lambda)_q^{(r-1)}}{\Gamma_q(r)}\tilde{\psi}_{\acute{\beta}}(\lambda)\,\mathrm{d}_q\lambda\right|\\ &\quad+\frac{K_1}{(1-K_2)\,\Gamma_q(r)}\int_0^s(s-\lambda)_q^{(r-1)}\left|\acute{\beta}(\lambda)-\beta(\lambda)\right|\,\mathrm{d}_q\lambda\\ &\leq \left(|\Xi|\,s^{\ell-1}ma+1\right)\epsilon C_\varphi\varphi(s)\\ &\quad+\frac{K_1}{(1-K_2)\,\Gamma_q(r)}\int_0^s(s-\lambda)_q^{(r-1)}\left|\acute{\beta}(\lambda)-\beta(\lambda)\right|\,\mathrm{d}_q\lambda\\ &\leq\left[\left(|\Xi|\,s^{\ell-1}ma+1\right)\left(1+\frac{K_1\,\nu C_\varphi}{1-K_2}\right)C_\varphi\right]\epsilon\varphi(s),\end{aligned}$$

where $\nu=\nu(r)$ is a constant, then $\left|\acute{\beta}(s)-\beta(s)\right|\leq C_\psi\epsilon\varphi(s)$, which completes the proof of the theorem. □

5. **Illustrative Examples**

This section contains some examples to illustrate the usefulness of our main results.

Example 5.1. Consider the $\mathbb{HFI}q-\mathbb{DE}$

$$\begin{cases} \mathcal{D}_q^{1/2,2/3}\beta(s) = \dfrac{\sqrt{7}+|\beta(s)|+\left|\mathcal{D}_q^{1/2,2/3}\beta(s)\right|}{48e^{s+3.4}\left(1+|\beta(s)|+\left|\mathcal{D}_q^{1/2,2/3}\beta(s)\right|\right)}, & s\in \mathrm{J}=(1,2],\\ \mathcal{D}_q^{1-\ell}\beta(1) = 3\beta\left(\frac{6}{5}\right)+2\beta\left(\frac{3}{2}\right)+\sqrt{5}\beta\left(\frac{7}{4}\right), \end{cases} \tag{5.1}$$

for different values of

$$q\in\left\{\tfrac{\ln(2)}{5},\ \tfrac{1}{2},\ \tfrac{\sqrt[5]{2^7}}{3}\right\}\subset(0,1).$$

Here, for $\beta,\acute{\beta}\in[0,\infty)$,

$$\psi(s,\beta,\acute{\beta}) = \tfrac{\sqrt{7}+|\beta|+|\acute{\beta}|}{48e^{s+3.4}\left(1+|\beta|+|\acute{\beta}|\right)}, \qquad s\in \mathrm{J},$$

and $r=\frac{1}{2}\in(0,1)$, $\gamma=\frac{2}{3}\in[0,1]$, $\ell=r+\gamma-r\gamma=\frac{1}{2}+\frac{2}{3}-\frac{1}{2}\left(\frac{2}{3}\right)=\frac{5}{6}$, $a_1=3$, $a_2=2$, $a_3=\sqrt{5}$, $v_1=\frac{6}{5}\in \mathrm{J}$, $v_2=\frac{3}{2}\in \mathrm{J}$, $v_3=\frac{7}{4}\in \mathrm{J}$. Using this data in Eq. (3.2), we have

$$\begin{aligned}\Xi &= \left[\Gamma_q(\ell)-\sum_{i=1}^{m}a_i\,v_i^{\ell-1}\right]^{-1} = \left[\Gamma_q\left(\tfrac{5}{6}\right)-\left(3\left(\tfrac{6}{5}\right)^{5/6-1}+2\left(\tfrac{3}{2}\right)^{5/6-1}\right)\right]^{-1}\\ &\simeq \begin{cases} -0.17341, & q=\frac{\ln(2)}{5},\\ -0.17477, & q=\frac{1}{2},\\ -0.17560, & q=\frac{\sqrt[5]{2^7}}{3},\end{cases}\end{aligned}$$

Clearly, ψ is continuous and condition (H1) is satisfied with

$$|\psi(s,\beta_1,\beta_2)|\leq\frac{1}{48e^{s+3}}\left(\sqrt{7}+|\beta_1|+|\beta_2|\right),$$

and $\overline{\beta}_0(s)=\frac{\sqrt{7}}{48e^{s+3.4}}$, $\overline{\beta}_0{}^*(s)=\frac{\sqrt{7}}{48e^{4.4}}$, $\overline{\beta}_1(s)=\overline{\beta}_2(s)=\frac{1}{48e^{s+3.4}}$, $\overline{\beta}_1{}^*(s)=\overline{\beta}_2{}^*(s)=\frac{1}{48e^{4.4}}$, for $s\in \mathrm{J}$. In addition, for $\beta_i,\acute{\beta}_i\in[0,\infty)$, $i=1,2$ and $s\in \mathrm{J}$, we have

$$\left|\psi\big(s,\beta_1,\beta_2\big)-\psi\big(s,\acute{\beta}_1,\acute{\beta}_2\big)\right|\leq\frac{1}{48e^{4.4}}\left(\left|\beta_1-\acute{\beta}_1\right|+\left|\beta_2-\acute{\beta}_2\right|\right).$$

Thus, condition (H2) is satisfied with $K_1=K_2=\frac{1}{48e^{4.4}}$. We see that (3.9) holds with

$$|\Xi|\approx 0.17341,\ 0.17477,\ 0.17560,$$

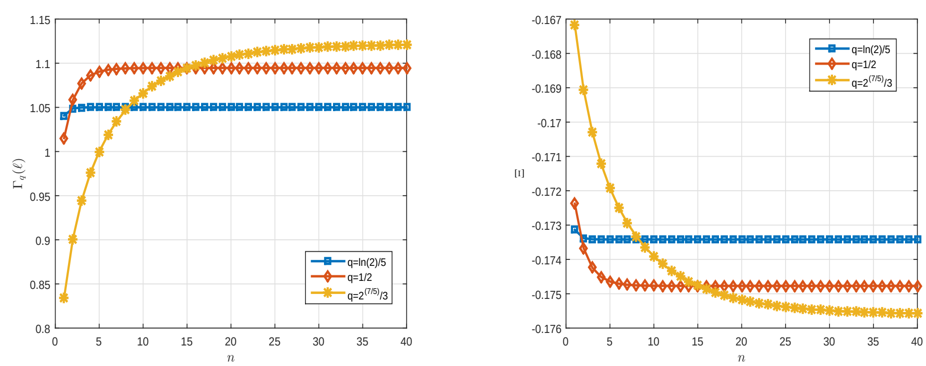
when $q=\frac{\ln(2)}{5},\ \frac{1}{2},\ \frac{\sqrt[5]{2^7}}{3}$, respectively. Table 1 shows the numerical results of $\Gamma_q(\ell)$, Ξ and $\widehat{\Lambda}$ for problem (5.1) with given values for $0<q<1$. One can see 2D plot of these variables for different cases of q in Figures 1a, 1b and 2 respectively. Now, by employing Eq. (3.13), we obtain

$$\begin{aligned}\widehat{\Lambda} &= \left(\tfrac{K_1}{1-K_2}\right)\tfrac{B_q(\ell,r)}{\Gamma_q(r)}\left(|\Xi|\sum_{i=1}^{m}a_iv_i^{r+\ell-1}+\tau_0^r\right)\\ &= \left(\tfrac{1}{48e^{4.4}-1}\right)\frac{\Gamma_q\left(\frac{5}{6}\right)}{\Gamma_q\left(\frac{1}{2}+\frac{5}{6}\right)}\left(|\Xi|\sum_{i=1}^{m}a_iv_i^{1/2+5/6-1}+2^{1/2}\right)\simeq\left\{\begin{array}{ll} 0.0010, & q=\frac{\ln(2)}{5},\\ 0.0012, & q=\frac{1}{2},\\ 0.0014, & q=\frac{\sqrt[5]{2^7}}{3},\end{array}\right\}<1.\end{aligned}$$

So it follows from Theorem 3.9 that the problem (5.1) has at least one solution on J.

TABLE 1. Numerical results of $\Gamma_q(\ell)$, Ξ and $\widehat{\Lambda}$ for Problem (5.1) for three cases of q in Example 5.1.

	$q=\frac{\ln(2)}{5}$			$q=\frac{1}{2}$			$\frac{\sqrt[5]{2^7}}{3}$		
n	$\Gamma_q(\ell)$	Ξ	$\widehat{\Lambda}$	$\Gamma_q(\ell)$	Ξ	$\widehat{\Lambda}$	$\Gamma_q(\ell)$	Ξ	$\widehat{\Lambda}$
1	1.0408	−0.1731	0.0009	1.0152	−0.1724	0.0008	0.8344	−0.1672	0.0003
2	1.0488	−0.1734	0.0010	1.0584	−0.1737	0.0010	0.9009	−0.1690	0.0005
3	1.0499	−0.1734	0.0010	1.0773	−0.1742	0.0011	0.9444	−0.1703	0.0006
4	1.0500	−0.1734	0.0010	1.0861	−0.1745	0.0012	0.9759	−0.1712	0.0007
5	1.0500	−0.1734	0.0010	1.0905	−0.1746	0.0012	0.9999	−0.1719	0.0008
6	1.0501	−0.1734	0.0010	1.0926	−0.1747	0.0012	1.0189	−0.1725	0.0009
7	1.0501	−0.1734	0.0010	1.0936	−0.1747	0.0012	1.0343	−0.1729	0.0009
8	1.0501	−0.1734	0.0010	1.0942	−0.1748	0.0012	1.0469	−0.1733	0.0010
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
14	1.0501	−0.1734	0.0010	1.0947	−0.1748	0.0012	1.0903	−0.1746	0.0012
15	1.0501	−0.1734	0.0010	1.0947	−0.1748	0.0012	1.0944	−0.1748	0.0012
16	1.0501	−0.1734	0.0010	1.0947	−0.1748	0.0012	1.0979	−0.1749	0.0013
17	1.0501	−0.1734	0.0010	1.0947	−0.1748	0.0012	1.1009	−0.1750	0.0013
18	1.0501	−0.1734	0.0010	1.0947	−0.1748	0.0012	1.1035	−0.1750	0.0013
19	1.0501	−0.1734	0.0010	1.0947	−0.1748	0.0012	1.1058	−0.1751	0.0013
20	1.0501	−0.1734	0.0010	1.0947	−0.1748	0.0012	1.1078	−0.1752	0.0013
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
25	1.0501	−0.1734	0.0010	1.0947	−0.1748	0.0012	1.1146	−0.1754	0.0014
26	1.0501	−0.1734	0.0010	1.0947	−0.1748	0.0012	1.1155	−0.1754	0.0014
27	1.0501	−0.1734	0.0010	1.0947	−0.1748	0.0012	1.1163	−0.1754	0.0014
28	1.0501	−0.1734	0.0010	1.0947	−0.1748	0.0012	1.1169	−0.1755	0.0014
29	1.0501	−0.1734	0.0010	1.0947	−0.1748	0.0012	1.1175	−0.1755	0.0014
30	1.0501	−0.1734	0.0010	1.0947	−0.1748	0.0012	1.1181	−0.1755	0.0014



(A) $\Gamma_q(\ell)$ (B) Ξ

FIGURE 1. Graphical representation of $\Gamma_q(\ell)$ and Ξ for Problem (5.1) for different values of q in Example 5.1.

In the next example, we show our results for changes in fractional order r.

Example 5.2. In the problem (5.1), by choosing

$$r \in \left\{\tfrac{1}{8},\, \tfrac{1}{2},\, \tfrac{8}{9}\right\} \subset (0,1),$$

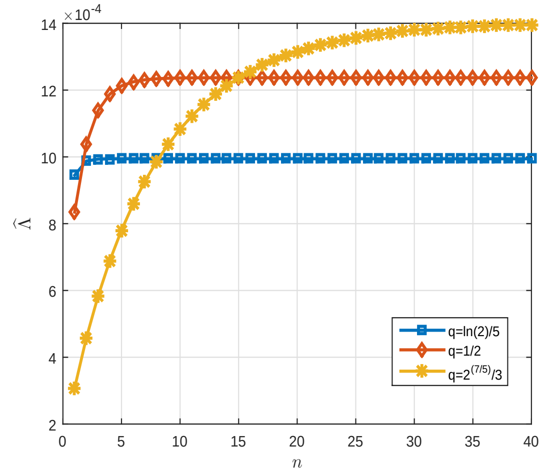


FIGURE 2. Graphical representation of $\widehat{\Lambda}$ for Problem (5.1) with three different values for q in Example 5.1.

$q = \frac{3}{7} \in (0,1)$, $\gamma = \frac{2}{3} \in [0,1]$,

$$\ell = r + \gamma - r\gamma = \begin{cases} \frac{17}{24}, & r = \frac{1}{8}, \\ \frac{5}{6}, & r = \frac{1}{2}, \\ \frac{26}{27}, & r = \frac{8}{9}, \end{cases} \qquad 1 - \ell = \begin{cases} \frac{7}{24}, & r = \frac{1}{8}, \\ \frac{1}{6}, & r = \frac{1}{2}, \\ \frac{1}{27}, & r = \frac{8}{9}, \end{cases}$$

we consider the HFI$q - \mathbb{DE}$:

$$\begin{cases} \mathcal{D}_{3/7}^{r,2/3} \beta(s) = \dfrac{\sqrt{7} + |\beta(s)| + \left|\mathcal{D}_{3/7}^{r,2/3}\beta(s)\right|}{48 e^{s+3.4}\left(1 + |\beta(s)| + \left|\mathcal{D}_{3/7}^{r,2/3}\beta(s)\right|\right)}, & s \in \mathrm{J} = (1,2], \\ \mathcal{D}_{3/7}^{1-\ell}\beta(1) = 3\beta\left(\frac{6}{5}\right) + 2\beta\left(\frac{3}{2}\right) + \sqrt{5}\beta\left(\frac{7}{4}\right). \end{cases} \tag{5.2}$$

By placing this data in Eq. (3.2), we have

$$\Xi = \left[\Gamma_{3/7}(\ell) - \sum_{i=1}^{m} a_i\, v_i^{\ell-1}\right]^{-1} = \left[\Gamma_{3/7}\left(\tfrac{5}{6}\right) - \left(3\left(\tfrac{6}{5}\right)^{5/6-1} + 2\left(\tfrac{3}{2}\right)^{5/6-1}\right)\right]^{-1}$$
$$\simeq \begin{cases} -0.1878, & r = \frac{1}{8}, \\ -0.1746, & r = \frac{1}{2}, \\ -0.1633, & r = \frac{8}{9}, \end{cases}$$

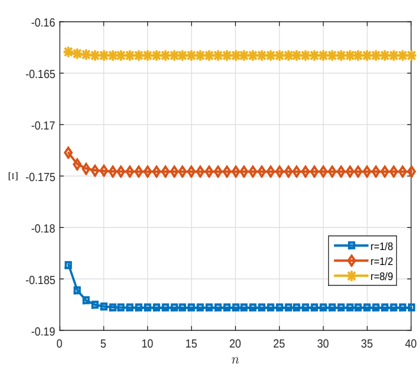
Table 2 shows the numerical results of $\Gamma_{3/7}(\ell)$, Ξ and $\widehat{\Lambda}$ for problem (5.1) with given values for $0 < r < 1$. One can see 2D plot of these variables for different cases of r in Figures 3a, 3b, respectively. In Example 5.1, we show that the condition (H2) is satisfied with $K_1 = \frac{1}{e^2}$, $K_2 = \frac{1}{4}$. In addition, by applying Eq. (3.13), we obtain

$$\widehat{\Lambda} = \left(\frac{K_1}{1-K_2}\right)\frac{B_{3/7}(\ell, r)}{\Gamma_{3/7}(r)}\left(|\Xi|\sum_{i=1}^{m} q_i v_i^{r+\ell-1} + \tau_0^r\right) \simeq \left\{\begin{matrix} 0.0030, & r = \frac{1}{8}, \\ 0.0012, & r = \frac{1}{2}, \\ 0.0010, & r = \frac{8}{9}, \end{matrix}\right\} < 1.$$

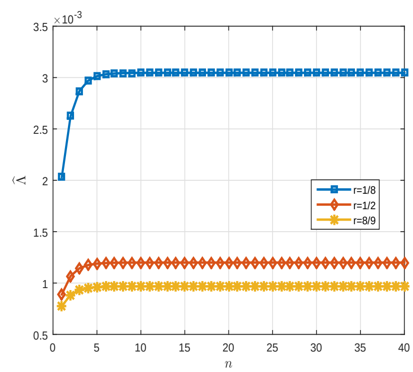
It follows from Theorem 3.9 that the problem (5.2) has a unique solution on J. For $s \in \mathrm{J}$,

TABLE 2. Numerical results of $\Gamma_q(\ell)$, Ξ and $\widehat{\Lambda}$ for Problem (5.2) for three cases of r in Example 5.2.

	$r=\frac{1}{8}$			$r=\frac{1}{2}$			$r=\frac{8}{9}$		
n	$\Gamma_q(\ell)$	Ξ	$\widehat{\Lambda}$	$\Gamma_q(\ell)$	Ξ	$\widehat{\Lambda}$	$\Gamma_q(\ell)$	Ξ	$\widehat{\Lambda}$
1	1.0755	−0.1836	0.0020	1.0279	−0.1728	0.0009	1.0035	−0.1630	0.0008
2	1.1480	−0.1861	0.0026	1.0642	−0.1738	0.0011	1.0107	−0.1631	0.0009
3	1.1761	−0.1871	0.0029	1.0782	−0.1743	0.0011	1.0135	−0.1632	0.0009
4	1.1878	−0.1875	0.0030	1.0839	−0.1744	0.0012	1.0146	−0.1633	0.0010
5	1.1927	−0.1877	0.0030	1.0863	−0.1745	0.0012	1.0151	−0.1633	0.0010
6	1.1947	−0.1878	0.0030	1.0873	−0.1745	0.0012	1.0153	−0.1633	0.0010
7	1.1956	−0.1878	0.0030	1.0878	−0.1746	0.0012	1.0154	−0.1633	0.0010
8	1.1960	−0.1878	0.0030	1.0880	−0.1746	0.0012	1.0154	−0.1633	0.0010
9	1.1962	−0.1878	0.0030	1.0881	−0.1746	0.0012	1.0154	−0.1633	0.0010
10	1.1963	−0.1878	0.0030	1.0881	−0.1746	0.0012	1.0154	−0.1633	0.0010
11	1.1963	−0.1878	0.0030	1.0881	−0.1746	0.0012	1.0154	−0.1633	0.0010
12	1.1963	−0.1878	0.0030	1.0881	−0.1746	0.0012	1.0154	−0.1633	0.0010
13	1.1963	−0.1878	0.0030	1.0881	−0.1746	0.0012	1.0154	−0.1633	0.0010



(A) Ξ



(B) $\widehat{\Lambda}$

FIGURE 3. Graphical representation of Ξ and $\widehat{\Lambda}$ for Problem (5.2) with different values of r in Example 5.2.

let $\varphi(s) = s$. Since

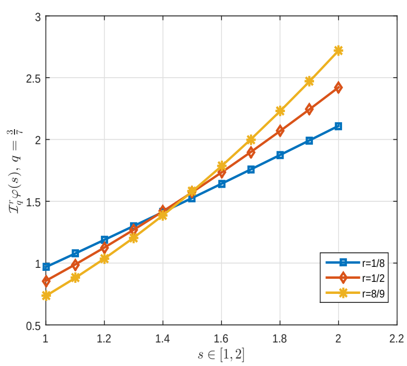
$$\begin{aligned} \mathcal{I}_{3/7}^{r}\varphi(s) &= \frac{1}{\Gamma_{3/7}(r)}\int_1^s (s-\lambda)_{3/7}^{(r-1)}\,\lambda\,\mathrm{d}_{3/7}\lambda \\ &\simeq \begin{cases} 2.1103, & r=\frac{1}{8},\\ 2.4221, & r=\frac{1}{2},\\ 2.7192, & r=\frac{8}{9}, \end{cases} \leq \begin{cases} 5.8557, & r=\frac{1}{8},\\ 5.2126, & r=\frac{1}{2},\\ 4.2735, & r=\frac{8}{9}, \end{cases} \simeq \frac{s\,\tau_0}{r\,\Gamma_{3/7}(r)} = C_\varphi\varphi(s), \end{aligned} \tag{5.3}$$

condition (H3) is satisfied with $C_\varphi = \frac{1}{r\,\Gamma_{3/7}(r)}\tau_0$. Table 3 shows these numerical results.

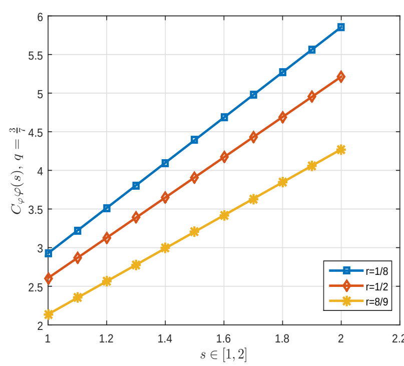
One can see 2D plot of variables Ξ and $\widehat{\Lambda}$ for different cases of r in Figures 3a and 3b respectively. One can see 2D plot of variables $\mathcal{I}_q^r\varphi(s)$ and $C_\varphi\varphi(s)$ for different cases of r in Figures 4a and 4b respectively. Furthermore, Figures 5a, 5b and 5c compare the variables

TABLE 3. Numerical results of $\mathcal{I}^{r}_{3/7}\varphi(s)$, $\Gamma_q(r)$ and $C_\varphi\varphi(s)$ for Problem (5.2) for three cases of r in Example 5.2.

	$r=\frac{1}{8}$			$r=\frac{1}{2}$			$r=\frac{8}{9}$		
s	$\mathcal{I}^{r}_{3/7}\varphi(s)$	$\Gamma_{3/7}(r)$	$C_\varphi\varphi(s)$	$\mathcal{I}^{r}_{3/7}\varphi(s)$	$\Gamma_{3/7}(r)$	$C_\varphi\varphi(s)$	$\mathcal{I}^{r}_{3/7}\varphi(s)$	$\Gamma_{3/7}(r)$	$C_\varphi\varphi(s)$
1.00	0.9676	5.4648	2.9278	0.8563	1.5347	2.6063	0.7342	1.0530	2.1367
1.10	1.0771	5.4648	3.2206	0.9879	1.5347	2.8669	0.8791	1.0530	2.3504
1.20	1.1878	5.4648	3.5134	1.1257	1.5347	3.1276	1.0361	1.0530	2.5641
1.30	1.2998	5.4648	3.8062	1.2693	1.5347	3.3882	1.2052	1.0530	2.7778
1.40	1.4128	5.4648	4.0990	1.4185	1.5347	3.6488	1.3863	1.0530	2.9914
1.50	1.5268	5.4648	4.3918	1.5732	1.5347	3.9094	1.5792	1.0530	3.2051
1.60	1.6418	5.4648	4.6846	1.7331	1.5347	4.1701	1.7840	1.0530	3.4188
1.70	1.7577	5.4648	4.9773	1.8981	1.5347	4.4307	2.0004	1.0530	3.6324
1.80	1.8744	5.4648	5.2701	2.0680	1.5347	4.6913	2.2285	1.0530	3.8461
1.90	1.9919	5.4648	5.5629	2.2427	1.5347	4.9520	2.4681	1.0530	4.0598
2.00	2.1103	5.4648	5.8557	2.4221	1.5347	5.2126	2.7192	1.0530	4.2735



(A) $\mathcal{I}^{r}_{q}\varphi(s)$



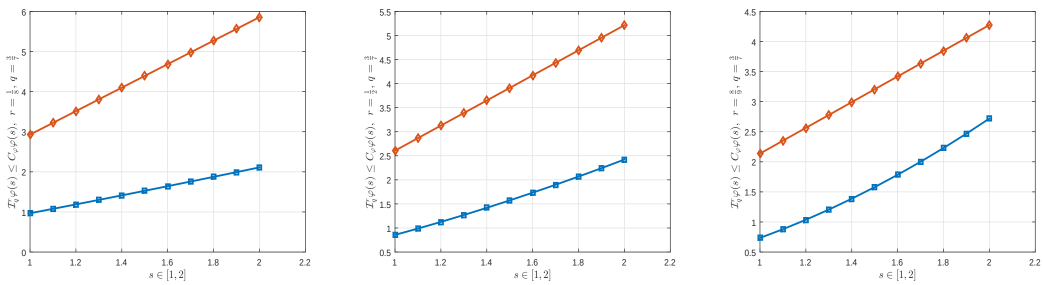
(B) $C_\varphi\varphi(s)$

FIGURE 4. Graphical representation of $\mathcal{I}^{r}_{q}\varphi(s)$ and $C_\varphi\varphi(s)$ for problem (5.2) with different values of r for $s \in \mathrm{J}$ in Example 5.2.

for different cases of r and $s \in \mathrm{J}$ in Problem (5.2) and show that the inequality (5.3) is established. It follows from Theorem 4.6 that the problem (5.2) is U-H-R stable.

Conclusion

The fractional quantum derivative called a Hilfer fractional q-derivative involving Riemann-Liouville and Caputo type fractional q-derivatives, $\mathbb{HFI}q-\mathbb{DE}$, has been investigated in this work in details. The investigation of this particular equation provides us with a powerful tool in modeling most scientific phenomena without the need to remove most parameters which have an essential role in the physical interpretation of the studied phenomena. The problem (1.1) has been studied under some bondary conditions. The stability analysis show a benefit results about the proble. Two examples have been provided to support our results' validity and applicability in fields of physics and engineering.



(A) $r = \frac{1}{8}$ (B) $r = \frac{1}{2}$ (C) $r = \frac{8}{9}$

FIGURE 5. Compare two variables $\mathcal{I}_q^r \varphi(s)$ and $C_\varphi \varphi(s)$ for problem (5.2) when $q = \frac{3}{7}$ and (A) $r = \frac{1}{8}$, (B) $r = \frac{1}{2}$, (C) $r = \frac{8}{9}$.

Acknowledgments

The authors was supported by Bu-Ali Sina University.

References

[1] S. Abbas, M. Benchohra, and M. Abdalla Darwish. Asymptotic stability for implicit Hilfer fractional differential equations. *Panamerican Mathematical Journal*, 27:40–52, 2017.

[2] S. Abbas, M. Benchohra, J. E. Lazreg, and Y. Zhou. A survey on Hadamard and Hilfer fractional differential equations: Analysis and stability. *Chaos, Solitons & Fractals*, 102:47–71, 2017.

[3] C. R. Adams. The general theory of a class of linear partial q-difference equations. *Transactions of the American Mathematical Society*, 26:283–312, 1924.

[4] W. Albarakati, M. Benchohra, and S. Bouriah. Existence and stability results for nonlinear implicit fractional differential equations with delay and impulses. *Differential Equations & Applications*, 2:273–293, 2016.

[5] Pari Amiri and M. E. Samei. Existence of urysohn and atangana-baleanu fractional integral inclusion systems solutions via common fixed point of multi-valued operators. *Chaos, Solitons & Fractals*, 165(2):112822, 2022.

[6] M. H. Annaby and Z. S. Mansour. *q-Fractional Calculus and Equations*. Springer Heidelberg, Cambridge, 2012.

[7] F. Atici and P. W. Eloe. Fractional q-calculus on a time scale. *Journal of Nonlinear Mathematical Physics*, 14(3):341–352, 2007.

[8] M. Benchohra, J. Henderson, S. K. Ntouyas, and A. Ouahab. Existence results for fractional order functional differential equations. *Journal of Mathematical Analysis and Applications*, 338:1340–1350, 2008.

[9] S. P. Bhairat and M. E. Samei. Nonexistence of global solutions for a Hilfer-Katugampola fractional differential problem. *Partial Differential Equations in Applied Mathematics*, 7:100495, 2023.

[10] M. Bohner and A. Peterson. *Dynamic Equations on Time Scales*. Birkhäuser, Boston, 2001.

[11] R. I. Butt, T. Abdeljawad, M. A. Alqudah, and M. U. Rehman. Ulam stability of Caputo q-fractional delay difference equation: q-fractional gronwall inequality approach. *Journal of Inequalities and Applications*, 2019:30, 2019.

[12] M. El-Shahed and F. M. Al-Askar. On the existence and uniqueness of solutions for q-fractional boundary value problem. *International Journal of Mathematical Analysis*, 5(33):1619–1630, 2011.

[13] R. A. C. Ferreira. Nontrivials solutions for fractional q-difference boundary value problems. *Electronic journal of qualitative theory of differential equations*, 70:1–101, 2010.

[14] K. M. Furati, M. D. Kassim, and N. E. Tatar. Existence and uniqueness for a problem involving Hilfer fractional derivative. *Computers & Mathematics with Applications*, 64(6):1616–1626, 2012.

[15] K. M. Furati, M. D. Kassim, and N. E. Tatar. Non-existence of global solutions for a differential equation involving Hilfer fractional derivative. *Electronic Journal of Differential Equations*, 2013(235):1–10, 2013.

[16] Faouzi Haddouchi, M. E. Samei, and Shahram Rezapour. Study of a sequential ψ-Hilfer fractional integro-differential equations with nonlocal BCs. *Journal of Pseudo-Differential Operators and Applications*, 14:61, 2023.

[17] S. N. Hajiseyedazizi, M. E. Samei, J. Alzabut, and Y. Chu. On multi-step methods for singular fractional q-integro-differential equations. *Open Mathematics*, 19:1378–1405, 2021.

[18] S. Harikrishnan, E. M. Elsayed, and K. Kanagarajan. Existence theory and stability analysis of nonlinear neutral pantograph equations via Hilfer-Katugampola fractional derivative. *Journal of Advances in Applied & Computational Mathematics*, 7:1–7, 2020.

[19] R. Hilfer. *Application of Fractional Calculus in Physics.* World Scientific, Singapore, 1999.

[20] R. Hilfer, Y. Luchko, and Ž. Tomovski. Operational method for the solution of fractional differential equations with generalized Riemann-Lioville fractional derivative. *Fractional calculus and Applications Analysis*, 12(3):299–318, 2009.

[21] M. Houas, F. Martínez, M. E. Samei, and M. K. A. Kaabar. Uniqueness and Ulam-Hyers-Rassias stability results for sequential fractional pantograph q-differential equations. *Journal of Inequalities and Applications*, 2022:93, 2022.

[22] Mohamed Houas and M. E. Samei. Existence and stability of solutions for linear and nonlinear damping of q-fractional Duffing-Rayleigh problem. *Mediterranean Journal of Mathematics*, 20:148, 2023.

[23] R. W. Ibrahim. Generalized Ulam-Hyers stability for fractional differential equations. *International Journal of Mathematics*, 23(05):1250056, 2012.

[24] G. Ismail and H. H. Abu-zinadah. Analytic approximations to a non-linear third order jerk equations via modified global error minimization method. *Journal of King Saud University - Science*, 33(1), 2021.

[25] F. H. Jackson. q-difference equations. *American Journal of Mathematics*, 32:305–314, 1910.

[26] V. Kac and P. Cheung. *Quantum Calculus.* Universitext, Springer, New York, 2002.

[27] V. Kalvandi and M. E. Samei. New stability results for a sum-type fractional q-integro-differential equation. *Journal of Advanced Mathematical Studies*, 12(2):201–209, 2019.

[28] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations.* Elsevier Science B.V, Amsterdam, 2006.

[29] X. Li, Z. Han, and S. Sun. Existence of positive solutions of nonlinear fractional q-difference equation with parameter. *Advances in Difference Equations*, 2013:260, 2013.

[30] S. Liang and M. E. Samei. New approach to solutions of a class of singular fractional q-differential problem via quantum calculus. *Advances in Difference Equations*, 2020:14, 2020.

[31] I. Podlubny. *Fractional differential equations.* Academic Press, San Diego, 1999.

[32] P. M. Rajković, S. D. Marinković, and M. S. Stanković. Fractional integrals and derivatives in q-calculus. *Applicable Analysis and Discrete Mathematics*, 1:311–323, 2007.

[33] M. E. Samei and A. Fathipour. Existence and stability results for a class of nonlinear fractional q-integro-differential equation. *International Journal of Nonlinear Analysis and Applications*, 14(1):1–19, 2023.

[34] M. E. Samei, H. Zanganeh, and S. M. Aydogan. Investigation of a class of the singular fractional integro-differential quantum equations with multi-step methods. *Journal of Mathematical Extension*, 17(1):1–545, 2021.

[35] J. V. D. C. Sousa and E. C. D. Oliveira. On the Ulam-Hyers-Rassias stability for nonlinear fractional differential equations using the ψ-Hilfer operator. *Journal of Fixed Point Theory and Applications*, 20:96, 2018.

[36] S. T. Sutar and K. D. Kucche. Global existence and uniqueness for implicit differential equation of arbitrary order. *Fractional Differential Calculus*, 5(2):199–208, 2015.

[37] H. L. Tidke and R. P. Mahajan. Existence and uniqueness of nonlinear implicit fractional differential equation with Riemann-Liouville derivative. *American Journal of Computational and Applied Mathematics*, 7(2):46–50, 2017.

[38] D. Vivek, K. Kanagarajan, and E. M. Elsayed. Some existence and stability results for Hilfer-fractional implicit differential equations with nonlocal conditions. *Mediterranean Journal of Mathematics*, 15:15, 2018.

[39] D. Vivek, K. Kanagarajan, and S. Sivasundaram. Dynamics and stability of pantograph equations via Hilfer fractional derivative. *Nonlinear Studies*, 23(4):685–698, 2016.

[40] J. Wang, L. Lv, and Y. Zhou. Ulam stability and data dependence for fractional differential equations with Caputo derivative. *Electronic Journal of Qualitative Theory of Differential Equations*, 63:1–10, 2011.

[41] J. Wang and Y. Zhang. Nonlocal initial value problems for differential equations with Hilfer fractional derivative. *Applied Mathematics and Computation*, 266:850–859, 2015.

[42] J. R. Wang, L. L. Lv, and Y. Zhou. New concepts and results in stability of fractional differential equations. *Communications on Nonlinear Science and Numerical Simulations*, 17(6):2530–2538, 2012.

[43] A. Zada, S. Ali, and Y. Li. Ulam stability for a class of implicit fractional differential equations with non instantaneous integral impulses and boundary condition. *Advances in Difference Equations*, 317:1–26, 2017.

(Mohammad Esmael Samei) Department of Mathematics, Faculty of Basic Science, Bu-Ali Sina University, Hamedan 65178-38695, Iran.
Email address: mesamei@basu.ac.ir
Email address: mesamei@gmail.com

(Alireza Hatami) Department of Mathematics, Faculty of Basic Science, Bu-Ali Sina University, Hamedan 65178-38695, Iran.
Email address: alirezahatami4816@gmail.com