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Research Paper

DENSITY OF BALANCED CONVEX-POLYNOMIALS IN $L^p(\mu)$

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ABSTRACT. A bounded linear operator T on a locally convex space X is balanced convex-cyclic if there exists a vector $x \in X$ such that the balanced convex hull of $orb(T,x)$ is dense in X.A balanced convex-polynomial is a balanced convex combination of monomials $\{1, z, z^2, z^3, \dots\}$.In this paper we prove that the balanced convex-polynomials are dense in $L^p(\mu)$ when $\mu([-1,1]) = 0$. Our results are used to characterize which multiplication operators on various real Banach spaces are balanced convex-cyclic. Also, it is shown for certain multiplication operators that every nonempty closed invariant balanced convex-set is a closed invariant subspace.

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1. Introduction and Background

Let X be a locally convex space and let $B(X)$ denote the algebra of all continuous linear operators on X and μ be a compactly supported positive regular Borel measure on $\mathbb{R}$. Convex-cyclic operators were introduced by Rezaei in [4]. If S be a subset of the vector space X then the balanced hull of S, that is shown by $ba(S)$, is defined in [1].

We will say that S is a balanced set whenever $S = ba(S)$. The balanced convex hull of S, that is shown by $bc(S)$, is defined in [1]. Let $\mathcal{BCP}$ denote the balanced convex hull of the set of monomials $\Big\{1, z, z^2, z^3, \dots\Big\}$ within the vector space of polynomials.Thus,

$$\mathcal{BCP} = \Big\{\sum_{k=0}^{n} a_k z^k : a_k \in \mathbb{C}, \sum_{k=0}^{n} |a_k| \leq 1\Big\}.$$

The elements of $\mathcal{BCP}$ are called *balanced convex − polynomials*. Hence the balanced convex hull of an orbit is

$$bc\Big(orb(T,x))\Big) = \Big\{p(T)x : p \in \mathcal{BCP}\Big\}.$$

Definition 1.1. An operator $T \in B(X)$ is called balanced convex-cyclic if there exists a vector $x \in X$ such that the balanced convex hull of $orb(T,x)$ is dense in X and such a vector x is said to be a balanced convex-cyclic vector for T.

Balanced convex-cyclic operators were studied in [1]. In this paper we will determine when the balanced convex-polynomials are dense in $L^p(\mu)$. Also we will show that the nonempty closed invariant balanced convex-sets for the operator M_x of multiplication by x on $L^2{}_{\mathbb{R}}(\mu)$ are the same as the closed invariant subspaces of M_x if and only if $\mu([-1,1]) = 0$. We consider the

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balanced convex-cyclicity of multiplication operator M_x on $L^2{}_{\mathbb{R}}(\mu)$ as an one of our results. In the next section we will investigate the density of balanced convex-set.

2. Basic theorems and preliminaries

Proposition 2.1. *(Balanced convex-polynomials)*

1. *A polynomial p is a balanced convex -polynomial if and only if*
$$\sum_{k=0}^{n} \frac{1}{k!}|p^{(k)}(0)| \leq 1.$$
2. *If p is a balanced convex-polynomial, then $p(\overline{D}) \subseteq \overline{D}$.*
3. *If $p(z)$ is a convex-polynomial, then $p(e^{i\theta}z)$ is a balanced convex-polynomial.*
4. *The composition and multiplication of balanced convex-polynomials are balanced convex-polynomial.*

Theorem 2.2. *(Dense Hahn-Banach Criterion) Let X be a locally convex space over the real or complex numbers and let S be a nonempty subset of X. Then the balanced convex hull of S is dense in X if and only if for every nonzero continuous linear functional f on X we have that*
$$\sup |f(S)| = +\infty.$$

Proof. See ([1], propositin2.2). □

Corollary 2.3. *(Hahn-Banach characterization of balanced convex-cyclicity) Let $T \in L(X)$ and $x \in X$. Then the following are equivalent:*

1. *The vector x is a balanced convex-cyclic vector for T .*
2. *For every nonzero bounded linear functional f on X we have*
$$\sup_{n} \Big| f(T^n x) \Big| = +\infty.$$
3. *For every nonzero bounded linear functional f on X we have*
$$\sup_{p \in \mathcal{BCP}} \left\{ \Big| f(p(T)x) \Big| \right\} = +\infty.$$

Proof. See ([1], Theorem2.3). □

We will use the above result to determine when the balanced convex-polynomials are dense in certain function spaces. If $S \subseteq \mathbb{R}$, then let $M_c(S)$ be the space of all real compactly supported regular Borel measures carried by S with finite total variation.

Corollary 2.4. *The followings hold:*

1. *For $\mu \in {M_c}^{+}(\mathbb{R})$ and $1 \leq p < \infty$, we have that the balanced convex-polynomials are dense in $L^p(\mu)$ if and only if for every nonzero $f \in L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$, we have*
$$\sup_{n \geq 1} \Big| \int x^n f(x) d\mu \Big| = +\infty.$$
2. *If $\mu \in {M_c}^{+}(\mathbb{R})$,then the balanced convex-polynomials are weak$^* -$ dense in $L^{\infty}(\mu)$ if and only if for every nonzero $f \in L^1(\mu)$ we have $\sup_{n \geq 1} \Big| \int x^n f(x) d\mu \Big| = +\infty$.*

Proof. This follows from ([3], Corollary1) and Theorem(2.2). □

Proposition 2.5. *If $b > a \geq 1$ and $f \in L^1(a,b)$ and $f(x) \leq \delta < 0$ for a.e. $x \in (a,b)$, then*

$$\int_a^b x^n f(x)dx \to -\infty \ as \ n \to \infty.$$

Proof. Since $f(x) \leq \delta < 0$ on (a,b) and $x^n \geq 0$, then $x^n f(x) \leq \delta x^n$ on (a,b).So we have

$$\int_a^b x^n f(x)dx \ \leq \delta \int_a^b x^n dx = \frac{\delta}{n+1}(b^{n+1} - a^{n+1}) \to -\infty \ as \ n \to \infty.$$

□

Corollary 2.6. *If $a < b \leq -1$ or $b > a \geq 1$ and $f \in L^1(a,b)$, then*

$$|\int_a^b x^n f(x)dx| \to +\infty \ as \ n \to +\infty.$$

Proof. This follows from ([3], Propsition7) and Proposition(2.5). □

Theorem 2.7. *If μ is a finite real measure with compact support in $(-\infty,-1] \cup [1,+\infty)$ that is not supported on sets $\{-1\}$ and $\{1\}$, then $\sup\limits_{n\geq 1} |\int x^n d\mu| = +\infty$.*

Proof. If *support*(μ) lies in $(-\infty,-1]$, then it is proved in ([3],Theorem14). Suppose that *support*(μ) lies in $[1,+\infty)$. Let $F: \mathbb{R} \to \mathbb{R}$ by $F(x) = \mu([x,+\infty))$. Since μ is compact support we may choose $b > 1$ such that μ is the zero measure on $[b,+\infty)$, thus $F(b) = \mu([b,+\infty)) = 0$ and $\mu(\{b\}) = 0$. μ is not the zero measure on $(1,b)$. Thus F is not almost everywhere equal to zero on $(1,b)$. Since x^n is continuous we may use integration by parts to give

$$\begin{aligned}
\int x^n d\mu &= \int_{[1,b]} x^n d\mu \\
&= \int_{\{1,b\}} x^n d\mu + \int_{(1,b)} x^n d\mu \\
&= \mu(\{1\}) + b^n \mu(\{b\}) + \int_1^b x^n dF(x) \\
&= \mu(\{1\}) + b^n F(b) - F(1) - \int_1^b n x^{n-1} F(x)dx \\
&= \mu(\{1\}) + 0 - F(1) - \int_1^b n x^{n-1} F(x)dx.
\end{aligned}$$

Since $F \in L^\infty(1,b)$ and F is nonzero on a set of positive Lebesgue measure in $(1,b)$ then, by Corollary(2.6) we have $\sup\limits_n |\int x^n d\mu| = \sup\limits_n |\int n x^{n-1} F(x)dx| = +\infty$. Suppose that *support*$(\mu)$ lies in $(-\infty,-1] \cup [1,+\infty)$. Let $F: \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \begin{cases} \mu([x,+\infty)), & x \geq 1 \\ \mu((-\infty,x]), & x \leq -1 \\ 0, & o.w. \end{cases}$$

Since μ is compact support we may choose an $a < -1$ and a $b > 1$ such that $\mu\Big((-\infty,a]\Big) = 0$ and $\mu\Big([b,+\infty)\Big) = 0$, so $\mu\Big((a,-1) \cup (1,b)\Big) \neq 0$. Thus $F(a) = 0, F(b) = 0,$ $\mu(\{a\}) = 0$ and $\mu(\{b\}) = 0$. μ is not the zero measure on $(a,-1)$ and $(1,b)$. Thus F is not almost everywhere equal to zero on $(a,-1)\cup(1,b)$. Since x^n is continuous we may use integration by parts to give

$$\begin{aligned}\int x^n d\mu &= a^n\mu(\{a\})+(-1)^n\mu(\{-1\})+\mu(\{1\})+b^n\mu(\{b\})+\int_a^{-1}x^n dF+\int_1^b x^n dF\\ &=(-1)^n\mu(\{-1\})+\mu(\{1\})+(-1)^nF(-1)-a^nF(a)\\ &\quad-\int_a^{-1}nx^{n-1}F(x)dx+b^nF(b)-F(1)-\int_1^b nx^{n-1}F(x)dx\\ &=(-1)^n\mu(\{-1\})+\mu(\{1\})+(-1)^nF(-1)-F(1)\\ &\quad-\int_a^{-1}nx^{n-1}F(x)dx-\int_1^b nx^{n-1}F(x)dx.\end{aligned}$$

Since $F\in L^\infty(a,-1)$ and $F\in L^\infty(1,b)$ and F is nonzero on a set of positive Lebesgue measure so we have $\inf\limits_{n\geq1}\int_a^{-1}nx^{n-1}F(x)dx=-\infty$, thus $\inf\limits_{n\geq1}(-\int_a^{-1}nx^{n-1}F(x)dx)=k$. Also we have $\sup\limits_{n\geq1}\int_1^b nx^{n-1}F(x)dx=+\infty$, thus $\inf\limits_{n\geq1}(-\int_1^b nx^{n-1}F(x)dx)=-\infty$. Hence we have $\inf\limits_{n\geq1}\int x^n d\mu=-\infty$ and $\sup\limits_{n\geq1}|\int x^n d\mu|=+\infty$. □

Corollary 2.8. *Let μ be a finite positive regular Borel measure with compact support on the real line, the followings are equivalent:*

1. *The balanced convex-polynomials are weak$^*-$dense in $L^\infty(\mu)$.*
2. *The balanced convex-polynomials are dense in $L^p(\mu)$ for all $1\leq p<\infty$.*
3. *For any $f\in L^p(\mu)$ with $|f|>0$ μ a.e. the set $\Big\{p(x)f(x):p\in\mathcal{BCP}\Big\}$ is dense in $L^p(\mu)$ when $1\leq p<\infty$ and weak$^*-$dense in $L^\infty(\mu)$.*
4. *$\mu([-1,1])=0$.*

Proof. This follows from Corollary(2.4) and Theorem(2.7). □

3. Balanced convex-cyclic multiplication operators and invariant balanced convex-sets

Let $L^2{}_{\mathbb{R}}(\mu)$ be the real Hilbert space of all real-valued Lebesgue measurable functions that are square integrable with respect to μ.(See [2]). Also let $M_{x,\mu}$ denote the operator of multiplication by the independent variable x acting on $L^2{}_{\mathbb{R}}(\mu)$.

In this section we determine when $M_{x,\mu}$ is balanced convex-cyclic and characterize its balanced convex-cyclic vectors. We also determine the invariant balanced convex-sets for $M_{x,\mu}$ where $M_{x,\mu}$ is balanced convex-cyclic.

The next theorem will charactrize the balanced convex-cyclicity of the multiplication operator $M_{x,\mu}$.

Theorem 3.1. *Let μ is a positive finite regular Borel measure with compact support in $\mathbb{R}$ and $M_{x,\mu}$ is the multiplication operator on $L^2{}_{\mathbb{R}}(\mu)$, then we have:*

1. *$M_{x,\mu}$ is balanced convex-cyclic if and only if $\mu([-1,1])=0$.*
2. *If $M_{x,\mu}$ is balanced convex-cyclic, then the balanced convex-cyclic vectors for $M_{x,\mu}$ are the same as its cyclic vectors.*

Proof. 1. If $M_{x,\mu}$ is balanced convex-cyclic, then from item 2 of proposition(2.1), we have $\mu([-1,1]) = 0$. Conversely, if $\mu([-1,1]) = 0$, then it follows from item 2 of corollary(2.8) that the balanced convex-polynomials are dense in $L^2{}_{\mathbb{R}}(\mu)$, which implies that $M_{x,\mu}$ is balanced convex-cyclic and the function $f(x) = 1$ is a balanced convex-cyclic vector for $M_{x,\mu}$.

2. Recall that the cyclic vector for $M_{x,\mu}$ are those functions $f \in L^2{}_{\mathbb{R}}(\mu)$ that satisfy $|f| > 0$ μ *a.e.* So, if f is a balanced convex-cyclic vector for $M_{x,\mu}$, then it is also a cyclic vector and thus must satisfy $|f| > 0$ μ *a.e.* Conversely, suppose that $M_{x,\mu}$ is balanced convex-cyclic and $f \in L^2{}_{\mathbb{R}}(\mu)$ satisfies $|f| > 0$ μ *a.e.* Since $M_{x,\mu}$ is balanced convex-cyclic we know from 1 that $\mu([-1,1]) = 0$ and since $|f| > 0$ μ *a.e.*, corollary(2.8) item 3 says exactly that f is a balanced convex-cyclic vector for $M_{x,\mu}$. □

Now we characterize the invariant closed balanced convex-sets for the operator $M_{x,\mu}$ when $\mu([-1,1]) = 0$.

Theorem 3.2. *If $\mu \in {M_c}^+(\mathbb{R})$, then the followings are equivalent:*

1. *The nonempty closed invariant balanced convex-sets for the multiplication operator $M_{x,\mu}$ on $L^2{}_{\mathbb{R}}(\mu)$ are the same as the closed invariant subspaces for $M_{x,\mu}$.*
2. *$\mu([-1,1]) = 0$.*

Proof. (1) $\Rightarrow$ (2) To show that $\mu([-1,1]) = 0$, suppose that $\mu([-1,1]) > 0$. We will obtain a contradiction. Let $S = \{f \in L^2{}_{\mathbb{R}}(\mu) : |f(x)| \leq 1 \ for \ \mu \ a.e. \ x \in [-1,1]\}$. S is a closed balanced convex-set in $L^2{}_{\mathbb{R}}(\mu)$ that is invariant under $M_{x,\mu}$. By our assumption that $\mu([-1,1]) > 0$, S is nonzero. Also by assumption 1, S is a subspace. But clearly S is not a subspace unless it is the zero subspace.This gives the contradiction and so $\mu([-1,1]) = 0$.

(2) $\Rightarrow$ (1) Recall that a nonempty balanced convex-set is a subspace if and only if it is closed under scalar multiplication. So, it suffices to show that it is invariant under multiplication by nonzero scalars. So, let $A \neq \{0\}$ be a closed balanced convex-set in $L^2{}_{\mathbb{R}}(\mu)$ that is invariant under $M_{x,\mu}$. Let $f \in A \diagdown \{0\}$, and let $r \in \mathbb{R} \diagdown \{0\}$. Let $\vartheta = |f|^2 d\mu$ and consider the measure ϑ and the space $L^2{}_{\mathbb{R}}(\vartheta)$. Since $\mu([-1,1]) = 0$, it follows that $\vartheta([-1,1]) = 0$, and by item 4 of corollary(2.8), we know that the balanced convex-polynomials are dense in $L^2{}_{\mathbb{R}}(\vartheta)$. Hence there are nonzero balanced convex-polynomials $\{p_n\}_{n=1}^{\infty}$ such that $p_n \to r$ in $L^2{}_{\mathbb{R}}(\vartheta)$. So $\int |p_n f - rf|^2 d\mu = \int |p_n - r|^2 |f|^2 d\mu = \int |p_n - r|^2 d\vartheta \to 0$ as $n \to \infty$ and so $p_n f \to rf$ in $L^2{}_{\mathbb{R}}(\mu)$. Now since A is balanced convex and invariant under $M_{x,\mu}$ and $f \in A$ so $p_n f \in A$ for every n and $p_n f \to rf$ in $L^2{}_{\mathbb{R}}(\mu)$. Since A is a closed set, $rf \in A$. Thus A is invariant under scalar multiplication. Hence A is a subspace. □

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