Mathematical Analysis
& Convex Optimization
Vol. 4 (2023), No. 1, 45–59
https:\\ maco.lu.ac.ir
DOI: 10.22034/maco.4.1.4
Research Paper

A SURVEY ON EXISTENCE OF A SOLUTION TO FRACTIONAL DIFFERENCE BOUNDARY VALUE PROBLEM WITH $|u|^{p-2}u$ TERM

MOHSEN KHALEGHI MOGHADAM*

Abstract. In this paper, we deal with the existence of a non-trivial solution for the following fractional discrete boundary-value problem for any $k \in [1,T]_{\mathbb{N}_0}$

$$\begin{cases} {}_{T+1}\nabla_k^\alpha \left({}_k\nabla_0^\alpha(u(k))\right) + {}_k\nabla_0^\alpha \left({}_{T+1}\nabla_k^\alpha(u(k))\right) + \phi_p(u(k)) = \lambda f(k,u(k)), \\ u(0) = u(T+1) = 0, \end{cases}$$

where $0 < \alpha < 1$ and ${}_k\nabla_0^\alpha$ is the left nabla discrete fractional difference and ${}_{T+1}\nabla_k^\alpha$ is the right nabla discrete fractional difference $f : [1,T]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\lambda > 0$ is a parameter and ϕ_p is the so called p-Laplacian operator defined as $\phi_p(s) = |s|^{p-2}s$ and $1 < p < +\infty$. The technical method is variational approach for differentiable functionals. Several examples are included to illustrate the main results.

MSC(2010): 26A33; 39A10; 39A27.
Keywords: Discrete fractional calculus, Discrete nonlinear boundary value problem, Non trivial solution, Variational methods, Critical point theory.

1. Introduction

Initial value problems in discrete fractional calculus considered in [4]. The first concepts of fractional nabla differences traces back to the works of Gray and Zhang [21]. Discrete fractional calculus with the nabla operator studied in [5]. In [6] authors studied two-point boundary value problems for finite fractional difference equations. This kind of problems play a fundamental role in different fields of research, for example in biological, Atici and Şengül introduced and solved Gompertz fractional difference equation for tumor growth models [7].

We refer the reader to the new monograph [26] that works for differential and integral equations and systems and for many theoretical and applied problems in mathematics, mathematical physics, probability and statistics, applied computer science and numerical methods. Also we refer the reader to the recent monograph on the introduction to fractional nabla calculus [15]. Another well-known monograph is [24] that is devoted to the systematic and comprehensive exposition of classical and modern results in the theory of fractional integrals and derivatives and their applications. It is well known that variational methods is an important tool to deal with the problems for differential and difference equations. Variational methods for dealing with fractional difference equations with boundary value conditions have appeared in [14, 23]. More, recently, in [13, 17, 18] by starting from the seminal papers [8, 9], the existence and multiplicity of solutions for nonlinear discrete boundary value problems

Date: Received: February 5, 2023, Accepted: June 20, 2023.
*Corresponding author.

have been investigated by adopting variational methods.

There seems to be increasing interest in the existence of solutions to boundary value problems for finite difference equations with fractional difference operator during the last two decades. In last decades, some researchers investigated q-fractional difference equations. Later, q-fractional boundary value problems considered by many researchers; see for instance, [25] and references therein.

The other important tool in the study of nonlinear difference equations is fixed point methods; see, for instance, [16] and references therein. Morse theory is also other tool in the study of nonlinear fractional differential equations [19].

The aim of this paper is to establish the existence of non-trivial solution for the following discrete boundary-value problem

(1.1)
$$\begin{cases} {}_{T+1}\nabla_k^{\alpha}\left({}_k\nabla_0^{\alpha}(u(k))\right) + {}_k\nabla_0^{\alpha}\left({}_{T+1}\nabla_k^{\alpha}(u(k))\right) + \phi_p(u(k)) = \lambda f(k,u(k)), \quad k \in [1,T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases}$$

where $0 < \alpha < 1$ and ${}_k\nabla_0^{\alpha}$ is the left nabla discrete fractional difference and ${}_{T+1}\nabla_k^{\alpha}$ is the right nabla discrete fractional difference and $\nabla u(k) = u(k) - u(k-1)$ is the backward difference operator $f : [1,T]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\lambda > 0$ is a parameter and $T \geq 2$ is fixed positive integer and ϕ_p is the so called p-Laplacian operator defined as $\phi_p(s) = |s|^{p-2}s$ and $1 < p < +\infty$ and $\mathbb{N}_1 = \{1, 2, 3, \cdots\}$ and ${}_T\mathbb{N} = \{\cdots T-2, T-1, T\}$ and $[1,T]_{\mathbb{N}_0}$ is the discrete set $\{1, 2, \cdots, T-1, T\} = \mathbb{N}_1 \bigcap {}_T\mathbb{N}$.

The term $|u|^{p-2}u$ in (1.1) and other nonlinear difference equations plays a fundamental role in the modeling of many phenomena [20].

In this paper, based on a local minimum theorem (Theorem 2.4) due to Bonanno [10], we ensure an exact interval of the parameter λ, in which the problem (1.1) admits at least a non-trivial solution. As an example, here, we point out the following special case of our main results.

Theorem 1.1. *Let $h : [1,T] \to \mathbb{R}$ be a positive and essentially bounded function and $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function and*

$$\lim_{d \to 0^+} \frac{g(d)}{d} = +\infty, \qquad \lim_{c \to +\infty} \frac{g(c)}{c} = 0.$$

Then for any

$$\lambda \in \left]0, +\infty\right[,$$

the problem

$$\begin{cases} {}_{T+1}\nabla_k^{\alpha}\left({}_k\nabla_0^{\alpha}(u(k))\right) + {}_k\nabla_0^{\alpha}\left({}_{T+1}\nabla_k^{\alpha}(u(k))\right) = \lambda h(k) g(u(k)) - u(k), \quad k \in [1,T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.2}$$

has at least one non-trivial solution in the space $\{u : [0, T+1] \to \mathbb{R} : u(0) = u(T+1) = 0\}$.

The rest of this paper is arranged as follows. In section 2, we provide some basic definitions and preliminary results and fundamental functional and lemma and main tool (Theorem 2.4) and in Section 3, we provide our auxiliary inequalities. In Section 4, we provide our main results that contains several theorems and proof the special case of main result (Theorem 1.1) finally, we illustrate the results by giving examples.

2. Preliminaries

The following definitions will be helpful to our discuss.

Definition 2.1. [3] **(i)** Let m be a natural number, then the m rising factorial of t is written as

$$t^{\overline{m}} = \prod_{k=0}^{m-1}(t+k), \qquad t^{\overline{0}} = 1. \tag{2.1}$$

(ii) For any real number, the α rising function is increasing on $\mathbb{N}_0$ and

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad \text{such that} \quad t \in \mathbb{R}\backslash\{\cdots,-2,-1,0\}, \quad 0^{\overline{\alpha}} = 0. \tag{2.2}$$

Definition 2.2. Let f be defined on $\mathbb{N}_{a-1}\bigcap {}_{b+1}\mathbb{N}$, $a<b$, $\alpha \in (0,1)$, then the nabla discrete new (left Gerasimov-Caputo) fractional difference is defined by

$$\left({}^{C}_{k}\nabla^{\alpha}_{a-1}f\right)(k) = \frac{1}{\Gamma(1-\alpha)}\sum_{s=a}^{k}\nabla_s f(s)(k-\rho(s))^{\overline{-\alpha}}, \quad k \in \mathbb{N}_a, \tag{2.3}$$

and the right Gerasimov-Caputo one by

$$\left({}^{C}_{b+1}\nabla^{\alpha}_{k}f\right)(k) = \frac{1}{\Gamma(1-\alpha)}\sum_{s=k}^{b}(-\Delta_s f)(s)(s-\rho(k))^{\overline{-\alpha}}, \quad k \in {}_b\mathbb{N}, \tag{2.4}$$

and in the left Riemann-Liouville sense by

$$\left({}^{R}_{k}\nabla^{\alpha}_{a-1}f\right)(k) = \frac{1}{\Gamma(1-\alpha)}\nabla_k\sum_{s=a}^{k}f(s)(k-\rho(s))^{\overline{-\alpha}}, \quad k \in \mathbb{N}_a, \tag{2.5}$$

$$= \frac{1}{\Gamma(-\alpha)}\sum_{s=a}^{k}f(s)(k-\rho(s))^{\overline{-\alpha-1}}, \quad k \in \mathbb{N}_a, \tag{2.6}$$

and the right Riemann-Liouville one by

$$\left({}^{R}_{b+1}\nabla^{\alpha}_{k}f\right)(k) = \frac{1}{\Gamma(1-\alpha)}(-\Delta_k)\sum_{s=k}^{b}f(s)(s-\rho(k))^{\overline{-\alpha}}, \quad k \in {}_b\mathbb{N}, \tag{2.7}$$

$$= \frac{1}{\Gamma(-\alpha)}\sum_{s=k}^{b}f(s)(s-\rho(k))^{\overline{-\alpha-1}}, \quad k \in {}_b\mathbb{N}, \tag{2.8}$$

where $\rho(k) = k-1$ be the backward jump operator.

For example, Let $f(k)=1$ be defined on $\mathbb{N}_{a-1}\bigcap {}_{b+1}\mathbb{N}$, therefore from (2.3) and (2.4), we have [1]

$${}^{C}_{b+1}\nabla^{\alpha}_{k}1 = {}^{C}_{k}\nabla^{\alpha}_{a-1}1 = 0, \qquad k \in \mathbb{N}_a\bigcap {}_b\mathbb{N}. \tag{2.9}$$

The relation between the nabla left and right Gerasimov-Caputo and Riemann-Liouville fractional differences are as follow:

$$\left({}^{C}_{k}\nabla^{\alpha}_{a-1}f\right)(k) = \left({}^{R}_{k}\nabla^{\alpha}_{a-1}f\right)(k) - \frac{(k-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}f(a-1), \tag{2.10}$$

$$\left({}_{b+1}^{C}\nabla_{k}^{\alpha}f\right)(k)=\left({}_{b+1}^{R}\nabla_{k}^{\alpha}f\right)(k)-\frac{(b+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}f(b+1). \tag{2.11}$$

Thus by (2.9), (2.10) and (2.11), we have for any $k\in\mathbb{N}_a\bigcap{}_b\mathbb{N}$,

$$ {}_{b+1}^{R}\nabla_{k}^{\alpha}1=\frac{(b+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)},\qquad {}_{k}^{R}\nabla_{a-1}^{\alpha}1=\frac{(k-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}. \tag{2.12}$$

Regarding the domains of the fractional type differences we observe:

(i) The nabla left fractional difference ${}_k\nabla_{a-1}^{\alpha}$ maps functions defined on ${}_{a-1}\mathbb{N}$ to functions defined on ${}_a\mathbb{N}$.

(ii) The nabla right fractional difference ${}_{b+1}\nabla_{k}^{\alpha}$ maps functions defined on ${}_{b+1}\mathbb{N}$ to functions defined on ${}_b\mathbb{N}$.

As in [12] one can show that, for $\alpha\to 0$, one has ${}_k\nabla_a^{\alpha}(f(k))\to f(t)$ and for $\alpha\to 1$, one has ${}_k\nabla_a^{\alpha}(f(k))\to\nabla f(t)$. We note that the nabla Riemann-Liouville and Gerasimov-Caputo fractional differences, for $0<\alpha<1$, coincide when f vanishes at the end points that is $f(a-1)=0=f(b+1)$ [1]. Indeed, when $0<\alpha<1$, those conclude from (2.10) and (2.11). So, for convenience, from now on we will use the symbol ∇^{α} instead of ${}^{R}\nabla^{\alpha}$ or ${}^{C}\nabla^{\alpha}$.
Now we present summation by parts formula in new discrete fractional calculus.

Theorem 2.3. *(* [2, Theorem 4.4] *Integration by parts for fractional difference) For functions f and g defined on $\mathbb{N}_a\bigcap{}_b\mathbb{N}$, $a\equiv b$ (mod 1), and $0<\alpha<1$, one has*

$$\sum_{k=a}^{b}f(k)\left({}_k\nabla_{a-1}^{\alpha}g\right)(k)=\sum_{k=a}^{b}g(k)\left({}_{b+1}\nabla_{k}^{\alpha}f\right)(k). \tag{2.13}$$

Similarly,

$$\sum_{k=a}^{b}f(k)\left({}_{b+1}\nabla_{k}^{\alpha}g\right)(k)=\sum_{k=a}^{b}g(k)\left({}_k\nabla_{a-1}^{\alpha}f\right)(k). \tag{2.14}$$

Our main tool is a local minimum theorem due to Bonanno (see [10, Theorem 5.1]), which is recalled below (see also [10, Proposition 2.1]). Such a result is more general than [22, Theorem 2.5] since the critical point, surely, is not zero.

First, for given $\Phi,\ \Psi: X\to\mathbb{R}$, we defined the following functions

$$\beta(r_1,r_2)=\inf_{v\in\Phi^{-1}(]r_1,r_2[)}\frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)}\Psi(u)-\Psi(v)}{r_2-\Phi(v)}, \tag{2.15}$$

and

$$\rho(r_1,r_2)=\sup_{v\in\Phi^{-1}(]r_1,r_2[)}\frac{\Psi(v)-\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{\Phi(v)-r_1}, \tag{2.16}$$

for all $r_1,r_2\in\mathbb{R}$, with $r_1<r_2$.

Theorem 2.4. *([10, Theorem 5.1]) Let X be a reflexive real Banach space, $\Phi: X\to\mathbb{R}$ a sequentially weakly semicontinuous coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* and $\Psi: X\to\mathbb{R}$ a continuously*

Gâteaux differentiable functional whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ *and assume that there are* $r_1, r_2 \in \mathbb{R}$*, with* $r_1 < r_2$*, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2),$$

where β *and* ρ *are given by (2.15) and (2.16). Then, for each*

$$\lambda \in \Lambda = \left] \frac{1}{\rho(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)} \right[,$$

there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ *such that* $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]r_1, r_2[)$ *and* $I'_\lambda(u_{0,\lambda}) = 0$*.*

We refer to the paper [11] in which Theorem 2.4 has been successfully employed to the existence of at least one non-trivial solution for two-point boundary value problems.

In order to give the variational formulation of the problem (1.1), let us define the finite $T-$dimensional Banach space

$$W := \{u : [0, T+1]_{\mathbb{N}_0} \to \mathbb{R} : u(0) = u(T+1) = 0\},$$

which is equipped with the norm

$$\|u\| := \left(\sum_{k=1}^{T} |u(k)|^2 \right)^{\frac{1}{2}}.$$

Let $\Phi : W \to \mathbb{R}$ be the functional

$$\Phi(u) := \frac{1}{2} \sum_{k=1}^{T} | \left({}_k\nabla_0^\alpha u\right)(k)|^2 + | \left({}_{T+1}\nabla_k^\alpha u\right)(k)|^2 + \frac{1}{p} \sum_{k=1}^{T} |u(k)|^p. \tag{2.17}$$

An easy computation ensures that Φ turns out to be of class C^1 on W and Gateaux differentiable with

$$\begin{aligned} \Phi'(u)(v) &= \sum_{k=1}^{T} \left({}_k\nabla_0^\alpha(u(k))\right)\left({}_k\nabla_0^\alpha v(k)\right) + \left({}_{T+1}\nabla_k^\alpha(u(k))\right)\left({}_{T+1}\nabla_k^\alpha v(k)\right) \\ &+ \sum_{k=1}^{T} |u(k)|^{p-2} u(k) v(k), \end{aligned}$$

for all $u, v \in W$. To study the problem (1.1), for every $\lambda > 0$, we consider the functional $I_\lambda : W \to \mathbb{R}$ defined by

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u), \qquad \Psi(u) := \sum_{k=1}^{T} F(k, u(k)), \tag{2.18}$$

where $F(k, u) = \int_0^u f(k, t)dt$.

Lemma 2.5. *The function* u *be a critical point of* I_λ *in* W*, iff* u *be a solution of the problem* (1.1)*.*

Proof. First, let $\overline{u}$ be a critical point of I_λ in W. Then by previous argument for all $v \in W$, $I'_\lambda(\overline{u})(v) = 0$ and $\overline{u}(0) = \overline{u}(T+1) = v(0) = v(T+1) = 0$. We applying the summation by parts formulas (2.13) and (2.14) in Theorem 2.3. Thus, by selecting $f(k) = \left({}_k\nabla_{a-1}^\alpha(\overline{u}(k))\right)$

and $g(k) = v(k)$ defined on $\mathbb{N}_1 \bigcap {}_T\mathbb{N}$ in (2.13) and selecting $f(k) = \left({}_{T+1}\nabla_k^\alpha(\overline{u}(k))\right)$ and $g(k) = v(k)$ defined on $\mathbb{N}_1 \bigcap {}_T\mathbb{N}$ in (2.14), one has

$$\begin{aligned}
0 &= I'_\lambda(\overline{u})(v) \\
&= \sum_{k=1}^{T} \left({}_k\nabla_0^\alpha(\overline{u}(k))\right)\left({}_k\nabla_0^\alpha v(k)\right) + \left({}_{T+1}\nabla_k^\alpha(\overline{u}(k))\right)\left({}_{T+1}\nabla_k^\alpha v(k)\right) \\
&+ \sum_{k=1}^{T} |u(k)|^{p-2}u(k)v(k) - \lambda \sum_{k=a}^{b} \left[f(k,\overline{u}(k))\right] v(k) \\
&= \sum_{k=1}^{T} v(k)\left({}_{T+1}\nabla_k^\alpha\left({}_k\nabla_0^\alpha(\overline{u}(k))\right)\right) + \sum_{k=1}^{T} v(k)\left({}_k\nabla_0^\alpha\left({}_{T+1}\nabla_k^\alpha(\overline{u}(k))\right)\right) \\
&+ \sum_{k=1}^{T} |u(k)|^{p-2}u(k)v(k) - \lambda \sum_{k=1}^{T} \left[f(k,\overline{u}(k))\right] v(k) \\
&= \sum_{k=1}^{T} v(k)\left\{\left({}_{T+1}\nabla_k^\alpha\left({}_k\nabla_0^\alpha(\overline{u}(k))\right)\right) + \left({}_k\nabla_0^\alpha\left({}_{T+1}\nabla_k^\alpha(\overline{u}(k))\right)\right)\right\} \\
&+ \sum_{k=1}^{T} |u(k)|^{p-2}u(k)v(k) - \lambda \sum_{k=1}^{T} \left[f(k,\overline{u}(k))\right] v(k).
\end{aligned}$$

Bearing in mind $v \in W$ is arbitrary, one get that

$$\left({}_{T+1}\nabla_k^\alpha\left({}_k\nabla_0^\alpha(\overline{u}(k))\right)\right) + \left({}_k\nabla_0^\alpha\left({}_{T+1}\nabla_k^\alpha(\overline{u}(k))\right)\right) + |u(k)|^{p-2}u(k) - \lambda f(k,\overline{u}(k)) = 0,$$

for every $k \in [1,T]_{\mathbb{N}_0}$. Therefore, $\overline{u}$ is a solution of (1.1). Since $\overline{u}$ be arbitrary, we conclude that every critical point of the functional I_λ in W, is a solution of the problem (1.1). On the other hand, if $\overline{u}$ be a solution of (1.1), then the vice versa holds and the proof is completed. □

3. Auxiliary inequalities

Now we provide some inequalities used throughout the paper, which hold on the space W. In the sequel, we will use the following inequality.

Lemma 3.1. *For every $u \in W$, we have*

$$\|u\|_\infty := \max_{k\in[1,T]} |u(k)| \leq \|u\|. \tag{3.1}$$

Proof. It is clear. □

Lemma 3.2. *For every $u \in W$, we have*

$$\frac{1}{p}(T+1)^{\frac{p(2-p)}{4}}\|u\|^p \leq \Phi(u) \leq 2T(T+1)\|u\|^2 + \frac{1}{p}(T+1)^{\frac{(2-p)}{2}}\|u\|^p. \tag{3.2}$$

Proof. Notice that for any positive real number α, the $-\alpha$ rising function of t, that is $t^{\overline{-\alpha}}$ is decreasing on $\mathbb{N}_0$. Indeed,

$$(t+1)^{\overline{-\alpha}} = \frac{\Gamma(t+1-\alpha)}{\Gamma(t+1)} = \frac{(t-\alpha)\Gamma(t-\alpha)}{t\Gamma(t)} < \frac{\Gamma(t-\alpha)}{\Gamma(t)} = (t)^{\overline{-\alpha}}.$$

Also, for any $k \in [1,T]_{\mathbb{N}}$, one can conclude that $(k-\rho(s))^{\overline{-\alpha}} \geq 0$, for any $s = 1,2,3,\cdots,k$, and $(s-\rho(k))^{\overline{-\alpha}} \geq 0$, for any $s = k, k+1, k+2, k+3, \cdots, T$. By using the discrete Hölder inequality, $(T+1)^{\frac{p(2-p)}{4}}\|u\|^p \leq \sum_{k=1}^{T}|u(k)|^p \leq (T+1)^{\frac{2-p}{2}}\|u\|^p, \forall u \in W$ and $p > 2$. Apply the definitions (2.3) and (2.4) and make use of lower and upper values of $t^{\overline{-\alpha}}$ on its range to obtain

$$\begin{aligned}
\Phi(u) &- \frac{1}{p}\sum_{k=1}^{T}|u(k)|^p \\
&= \frac{1}{2}\sum_{k=1}^{T}|\left({}_k\nabla_0^\alpha u\right)(k)|^2 + |\left({}_{T+1}\nabla_k^\alpha u\right)(k)|^2 \\
&= \frac{1}{2}\sum_{k=1}^{T}\Big(|\frac{1}{\Gamma(1-\alpha)}\sum_{s=1}^{k}\nabla_s u(s)(k-\rho(s))^{\overline{-\alpha}}|^2 \\
&+ |\frac{1}{\Gamma(1-\alpha)}\sum_{s=k}^{T}(-\Delta_s u(s))(s-\rho(k))^{\overline{-\alpha}}|^2\Big) \\
&\leq \frac{1}{2}\sum_{k=1}^{T}\Big(\{\frac{1}{\Gamma(1-\alpha)}\sum_{s=1}^{k}|\nabla_s u(s)|(k-\rho(s))^{\overline{-\alpha}}\}^2 \\
&+ \{\frac{1}{\Gamma(1-\alpha)}\sum_{s=k}^{T}|-\Delta_s u(s)|(s-\rho(k))^{\overline{-\alpha}}\}^2\Big) \\
&\leq \frac{1}{2}\sum_{k=1}^{T}\Big(\{\frac{1}{\Gamma(1-\alpha)}\sum_{s=1}^{k}|\nabla_s u(s)|(1)^{\overline{-\alpha}}|\}^2 \\
&+ \{\frac{1}{\Gamma(1-\alpha)}\sum_{s=k}^{T}|-\Delta_s u(s)|(1)^{\overline{-\alpha}}\}^2\Big) \\
&= \frac{1}{2}\sum_{k=1}^{T}\Big(\{\sum_{s=1}^{k}|\nabla_s u(s)|\}^2 + \{\sum_{s=k}^{T}|-\Delta_s u(s)|\}^2\Big) \\
&= \frac{1}{2}\sum_{k=1}^{T}\Big(\{\sum_{s=1}^{k}|u(s)-u(s-1)|\}^2 + \{\sum_{s=k}^{T}|u(s)-u(s+1)|\}^2\Big)
\end{aligned}$$

so

$$\begin{aligned}
\Phi(u) &- \frac{1}{p}\sum_{k=1}^{T}|u(k)|^p \\
&\leq \frac{1}{2}\sum_{k=1}^{T}\Big(\{\sum_{s=1}^{k}|u(s)-u(s-1)|\}^2+\{\sum_{s=k}^{T}|u(s)-u(s+1)|\}^2\Big) \\
&\leq \frac{1}{2}\sum_{k=1}^{T}\Big(\{\sum_{s=1}^{k}|u(s)|+|u(s-1)|\}^2+\{\sum_{s=k}^{T}|u(s)|+|u(s+1)|\}^2\Big) \\
&= \frac{1}{2}\sum_{k=1}^{T}\Big(\{\sum_{s=1}^{k}|u(s)|+\sum_{s=1}^{k}|u(s-1)|\}^2+\{\sum_{s=k}^{T}|u(s)|+\sum_{s=k}^{T}|u(s+1)|\}^2\Big) \\
&= \frac{1}{2}\sum_{k=1}^{T}\Big(\{\sum_{s=1}^{k}|u(s)|+u(0)+\sum_{s=2}^{k}|u(s-1)|\}^2 \\
&+ \{\sum_{s=k}^{T}|u(s)|+u(T+1)+\sum_{s=k}^{T-1}|u(s+1)|\}^2\Big) \\
&= \frac{1}{2}\sum_{k=1}^{T}\Big(\Big\{\sum_{s=1}^{k}|u(s)|+\sum_{s=1}^{k-1}|u(s)|\Big\}^2+\Big\{\sum_{s=k}^{T}|u(s)|+\sum_{s=k+1}^{T}|u(s)|\Big\}^2\Big) \\
&\leq \frac{1}{2}\sum_{k=1}^{T}\Big(\sum_{s=1}^{k}|u(s)|+\sum_{s=k+1}^{T}|u(s)|+\sum_{s=1}^{k-1}|u(s)|+\sum_{s=k}^{T}|u(s)|\Big)^2 \\
&= \frac{1}{2}\sum_{k=1}^{T}\Big(\sum_{s=1}^{T}|u(s)|+\sum_{s=1}^{T}|u(s)|\Big)^2 \\
&= 2\sum_{k=1}^{T}\Big(\Big\{\sum_{s=1}^{T}|u(s)|\Big\}^2\Big) \\
&= 2(T+1)\Big(\sum_{s=1}^{T}|u(s)|\Big)^2 \\
&\leq 2T(T+1)\sum_{s=1}^{T}|u(s)|^2,
\end{aligned}$$

thus

$$\begin{aligned}
\Phi(u) &\leq 2T(T+1)\sum_{s=1}^{T}|u(s)|^2+\frac{1}{p}\sum_{k=1}^{T}|u(k)|^p \\
&\leq 2T(T+1)\sum_{s=1}^{T}|u(s)|^2+\frac{1}{p}(T+1)^{\frac{(2-p)}{2}}\Big[\sum_{k=1}^{T}|u(k)|^2\Big]^{\frac{p}{2}} \\
&\leq 2T(T+1)\|u\|^2+\frac{1}{p}(T+1)^{\frac{(2-p)}{2}}\|u\|^p,
\end{aligned}$$

on the other hand,

$$\begin{aligned}\Phi(u) &\geq \frac{1}{p}\sum_{k=1}^{T}|u(k)|^p \\ &\geq \frac{1}{p}(T+1)^{\frac{p(2-p)}{4}}\|u\|^p.\end{aligned}$$

□

4. Main Results

First, let us introduce a function for convenience. For given two non-negative constants c and d, put

$$a_d(c) := \frac{\sum_{k=1}^{T}\max_{|\xi|\leq c}F(k,\xi) - \sum_{k=1}^{T}F(k,d)}{\frac{(c)^p}{p(T+1)^{\frac{p(p-2)}{4}}} - \frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 - \frac{Td^p}{p}}.$$

We state our main result as follows.

Theorem 4.1. *Assume that there exist a non-negative constant c_1 and two positive constants c_2 and d with*

(A0) $\frac{(c_1)^p}{p(T+1)^{\frac{p(p-2)}{4}}} < \frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 + \frac{Td^p}{p} < \frac{(c_2)^p}{p(T+1)^{\frac{p(p-2)}{4}}}$,

such that

(A1) $a_d(c_2) < a_d(c_1)$.

Then for any $\lambda \in]\frac{1}{a_d(c_1)}, \frac{1}{a_d(c_2)}[$ the problem (1.1) has at least one non-trivial solution $u_0 \in W$.

Proof. Our aim is to apply Theorem 2.4 to our problem. To this end, take $X = W$, and put Φ, Ψ and I_λ as in (2.17) and (2.18). We know Φ is a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and Ψ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. By similar arguing in [11], put

$$\bar{v}(k) = \begin{cases} d & k \in [1,T]_{\mathbb{N}_0}, \\ 0 & k = 0, T+1, \end{cases}$$

$r_1 = \frac{(c_1)^p}{p(T+1)^{\frac{p(p-2)}{4}}}$ and $r_2 = \frac{(c_2)^p}{p(T+1)^{\frac{p(p-2)}{4}}}$. Clearly $\bar{v} \in W$. Since $\bar{v}$ vanishes at the end points that is $\bar{v}(0) = 0 = \bar{v}(T+1)$, thus its nabla Riemann-Liouville and Gerasimov-Caputo fractional differences coincide, hence for any $k \in \mathbb{N}_1 \bigcap {}_T\mathbb{N}$

$$\left({}_{T+1}\nabla_k^\alpha \bar{v}\right)(k) = \left({}_{T+1}^{R}\nabla_k^\alpha \bar{v}\right)(k) = \left({}_{T+1}^{C}\nabla_k^\alpha \bar{v}\right)(k) = \frac{d(T+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)},$$

$$\left({}_k\nabla_0^\alpha \bar{v}\right)(k) = \left({}_k^{R}\nabla_0^\alpha \bar{v}\right)(k) = \left({}_k^{C}\nabla_0^\alpha \bar{v}\right)(k) = \frac{d(k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}.$$

So, we have

$$\begin{aligned}\Phi(\bar{v}) &= \frac{1}{2}\sum_{k=1}^{T} |\left({}_{k}\nabla_{0}^{\alpha}\bar{v}\right)(k)|^2 + |\left({}_{T+1}\nabla_{k}^{\alpha}\bar{v}\right)(k)|^2 + \frac{1}{p}\sum_{k=1}^{T}|\bar{v}(k)|^p \\ &= \frac{1}{2}\sum_{k=1}^{T} |\frac{d(k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}|^2 + |\frac{d(T+1-k)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}|^2 + \frac{Td^p}{p} \\ &= \frac{d^2}{2\left(\Gamma(1-\alpha)\right)^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2 + |(T+1-k)^{\overline{-\alpha}}|^2 + \frac{Td^p}{p} \\ &= \frac{d^2}{\left(\Gamma(1-\alpha)\right)^2}\sum_{k=1}^{T}|(k)^{\overline{-\alpha}}|^2 + \frac{Td^p}{p} \\ &= \frac{d^2}{\left(\Gamma(1-\alpha)\right)^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 + \frac{Td^p}{p},\end{aligned}$$

and

$$\Psi(\bar{v}) = \sum_{k=1}^{T} F(k,\bar{v}(k)) = \sum_{k=1}^{T} F(k,d).$$

Moreover, for all $u \in W$ such that $\Phi(u) < r_i$, $i = 1, 2$, taking (3.1) and (3.2) into account, one has $\max_{k\in[1,T]} |u(k)| \le c_i$, $i = 1, 2$. Therefore,

$$\sup_{u\in\Phi^{-1}(-\infty,r_i)} \Psi(u) = \sup_{\Phi(u)<r_i}\sum_{k=1}^{T} F(k,u(k)) \le \sum_{k=1}^{T}\max_{|\xi|\le c_i} F(k,u(k)), \quad i = 1, 2.$$

By (A0), $\bar{v} \in \Phi^{-1}(r_1, r_2)$, hence,

$$\begin{aligned}0 \le \beta(r_1,r_2) &\le \frac{\sup_{u\in\Phi^{-1}(r_1,r_2)}\Psi(u) - \Psi(\bar{v})}{r_2 - \Phi(\bar{v})} \\ &\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u) - \Psi(\bar{v})}{r_2 - \Phi(\bar{v})} \\ &\le \frac{\sum_{k=1}^{T}\max_{|\xi|\le c_2} F(k,\xi) - \sum_{k=1}^{T} F(k,d)}{\frac{(c_2)^p}{p(T+1)^{\frac{p(p-2)}{4}}} - \frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 - \frac{Td^p}{p}} \\ &= a_d(c_2).\end{aligned}$$

On the other hand, one has

$$\begin{aligned}\rho(r_1,r_2) &\ge \frac{\Psi(\bar{v}) - \sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{\Phi(\bar{v}) - r_1} \\ &\ge \frac{\sum_{k=1}^{T} F(k,d) - \sum_{k=1}^{T}\max_{|\xi|\le c_1} F(k,\xi)}{\frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 + \frac{Td^p}{p} - \frac{(c_1)^p}{p(T+1)^{\frac{p(p-2)}{4}}}} \\ &\ge \frac{\sum_{k=1}^{T}\max_{|\xi|\le c_1} F(k,\xi) - \sum_{k=1}^{T} F(k,d)}{\frac{(c_1)^p}{p(T+1)^{\frac{p(p-2)}{4}}} - \frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 - \frac{Td^p}{p}} = a_d(c_1).\end{aligned}$$

Hence, from Assumption (A1), we get $\beta(r_1, r_2) < \rho(r_1, r_2)$.
Therefore, owing to Theorem 2.4, for each $\lambda \in]\frac{1}{a_d(c_1)}, \frac{1}{a_d(c_2)}[$, the functional I_λ admits one critical point $u_0 \in W$ such that $r_1 < \Phi(u_0) < r_2$. Hence, the proof is completed. □

We now present an example to illustrate the result of Theorem 4.1.

Example 4.2. Put $c_1 = 0.1$, $c_2 = 20$, $d = 0.2$ and by taking $\alpha = 0.5$, $T = 9$, $p = 4$ and calculating, $a_{0.2}(0.1) = 63.722$ and $a_{0.2}(20) = 11.980$. Indeed $\frac{1}{(\Gamma(0.5))^2} \sum_{k=1}^{10} \left((k)^{\overline{-0.5}}\right)^2 = 1.791343942$, hence $A(0)$ reduce to

$$\frac{(c_1)^4}{400} < \frac{d^2}{(\Gamma(0.5))^2} \sum_{k=1}^{10} \left((k)^{\overline{-0.5}}\right)^2 + \frac{9d^4}{4} < \frac{(c_2)^4}{400},$$

and then

$$\frac{(c_1)^4}{400} < 1.791343942 d^2 + 2.25 d^4 < \frac{(c_2)^4}{400},$$

where it holds, since $0.00000025 < 0.07525375768 < 400$. Therefore

$$\begin{aligned} a_d(c_1) &= \frac{\sum_{k=1}^{T} \max_{|\xi| \le c_1} F(k,\xi) - \sum_{k=1}^{T} F(k,d)}{\frac{(c_1)^p}{p(T+1)^{\frac{p(p-2)}{4}}} - \frac{d^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} \left((k)^{\overline{-\alpha}}\right)^2 - \frac{Td^p}{p}} \\ &= \frac{\sum_{k=1}^{10} \max_{|\xi| \le c_1} \frac{1}{4}\xi^3 (\ln \frac{k+1}{k}) - \sum_{k=1}^{10} \frac{1}{4} d^3 (\ln \frac{k+1}{k})}{\frac{(c_1)^4}{400} - \frac{d^2}{(\Gamma(0.5))^2} \sum_{k=1}^{10} \left((k)^{\overline{-0.5}}\right)^2 - \frac{9d^4}{4}} \\ &= \frac{\frac{1}{4}(c_1^3 - d^3) \sum_{k=1}^{10} (\ln \frac{k+1}{k})}{\frac{(c_1)^4}{400} - 1.791343942 d^2 - 2.25 d^4} \\ &= \frac{\frac{1}{4}(0.1^3 - 0.2^3) \ln(11)}{\frac{(0.1)^4}{400} - 1.791343942(0.2^2) - 2.25(0.2^4)} = 63.722, \end{aligned}$$

and

$$\begin{aligned} a_d(c_2) &= \frac{\sum_{k=1}^{T} \max_{|\xi| \le c_2} F(k,\xi) - \sum_{k=1}^{T} F(k,d)}{\frac{(c_2)^p}{p(T+1)^{\frac{p(p-2)}{4}}} - \frac{d^2}{(\Gamma(1-\alpha))^2} \sum_{k=1}^{T} \left((k)^{\overline{-\alpha}}\right)^2 - \frac{Td^p}{p}} \\ &= \frac{\sum_{k=1}^{10} \max_{|\xi| \le c_2} \frac{1}{4}\xi^3 (\ln \frac{k+1}{k}) - \sum_{k=1}^{10} \frac{1}{4} d^3 (\ln \frac{k+1}{k})}{\frac{(c_2)^4}{400} - \frac{d^2}{(\Gamma(0.5))^2} \sum_{k=1}^{10} \left((k)^{\overline{-0.5}}\right)^2 - \frac{9d^4}{4}} \\ &= \frac{\frac{1}{4}(c_2^3 - d^3) \sum_{k=1}^{10} (\ln \frac{k+1}{k})}{\frac{(c_2)^4}{400} - 1.791343942 d^2 - 2.25 d^4} \\ &= \frac{\frac{1}{4}(20^3 - 0.2^3) \ln(11)}{\frac{(20)^4}{400} - 1.791343942(0.2^2) - 2.25(0.2^4)} = 11.980. \end{aligned}$$

Then, for every $\lambda \in]0.016, 0.083[$ the problem

$$\begin{cases} {}_{10}\nabla_k^\alpha \left({}_k\nabla_0^\alpha (u(k))\right) + {}_k\nabla_0^\alpha \left({}_{10}\nabla_k^\alpha (u(k))\right) + \phi_4(u(k)) = \frac{3}{4}\lambda u(k)^2 (\ln \frac{k+1}{k}), \quad k \in [1,9], \\ u(0) = u(10) = 0, \end{cases}$$

has at least one non-trivial solution u_0.

Here we point out an another immediate consequence of Theorem 4.1 as follows.

Theorem 4.3. *Let $f : [1,T]_{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function and assume that there exist two positive constants c and d with*

(A0) $\frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 + \frac{Td^p}{p} < \frac{(c)^p}{p(T+1)^{\frac{p(p-2)}{4}}}$,

such that

(A1) $\frac{\sum_{k=1}^{T}F(k,d)}{\frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2+\frac{Td^p}{p}} > p(T+1)^{\frac{p(p-2)}{4}}\frac{\sum_{k=1}^{T}\max_{|\xi|\le c}F(k,\xi)}{c^p}$.

Then for any

$$\lambda \in \left]\frac{\frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 + \frac{Td^p}{p}}{\sum_{k=1}^{T}F(k,d)}, \frac{c^p}{p(T+1)^{\frac{p(p-2)}{4}}\sum_{k=1}^{T}\max_{|\xi|\le c}F(k,\xi)}\right[,$$

the problem (1.1) has at least one non-trivial solution in W.

Proof. Applying Theorem 4.1, we have the conclusion, by picking $c_1 = 0$ and $c_2 = c$. Indeed, owing to our assumptions, one has

$$\begin{aligned} a_d(0) &= \frac{\sum_{k=1}^{T}F(k,d)}{\frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 + \frac{Td^p}{p}} \\ &> \frac{\sum_{k=1}^{T}\max_{|\xi|\le c}F(k,\xi) - \sum_{k=1}^{T}F(k,d)}{\frac{(c)^p}{p(T+1)^{\frac{p(p-2)}{4}}} - \frac{d^2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 - \frac{Td^p}{p}} \\ &= a_d(c). \end{aligned}$$

Hence, the proof is completed. □

Theorem 4.4. *Let $h : [1,T] \to \mathbb{R}$ be a positive and essentially bounded function and $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function and assume that there exist two positive constants c and d with $d^2\left(\frac{2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 + T\right) < c^2$,*

such that

$$\frac{\int_0^d g(t)dt}{d^2} > \left(\frac{2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 + T\right)\frac{\int_0^c g(t)dt}{c^2}.$$

Then for any

$$\lambda \in \left]\frac{\frac{2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 + T}{2\sum_{k=1}^{T}h(k)}\frac{d^2}{\int_0^d g(t)dt}, \frac{1}{2\sum_{k=1}^{T}h(k)}\frac{c^2}{\int_0^c g(t)dt}\right[,$$

the problem (1.2) has at least one non-trivial solution in the space $\{u : [0,T+1] \to \mathbb{R} : u(0) = u(T+1) = 0\}$.

Remark 4.5. We point out Theorem 4.4 is an immediate consequence of Theorem 4.3, by selecting $p = 2$ and $f(k,t) = h(k)g(t)$ for all $(k,t) \in [1,T]_{\mathbb{N}_0} \times \mathbb{R}$ be separable variable which satisfies (A0) and (A1).

Proof of Theorem 1.1:
For fixed $\lambda > 0$ as in the conclusion, the condition $\lim_{d\to 0^+} \frac{g(d)}{d} = +\infty$ implies $\lim_{d\to 0^+} \frac{\int_0^d g(t)dt}{d^2} = +\infty$, therefor there exists positive fixed constant d such that

$$\frac{\frac{2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 + T}{2\sum_{k=1}^{T} h(k)} \frac{d^2}{\int_0^d g(t)dt} < \lambda.$$

On the other hand for fixed $\lambda < +\infty$ as in the conclusion, the condition $\lim_{c\to+\infty} \frac{g(c)}{c} = 0$ implies $\lim_{c\to+\infty} \frac{\int_0^c g(t)dt}{c^2} = 0$, so for fixed d a positive constant c satisfying

$$d^2\left(\frac{2}{(\Gamma(1-\alpha))^2}\sum_{k=1}^{T}\left((k)^{\overline{-\alpha}}\right)^2 + T\right) < c^2,$$

can be chosen such that

$$\lambda < \frac{1}{2\sum_{k=1}^{T} h(k)} \frac{c^2}{\int_0^c g(t)dt}.$$

Hence, the conclusion follows from Theorem 4.4. Finally we present an example of Theorem 1.1.

Example 4.6. The following discrete boundary-value problem

$$\begin{cases} {}_{T+1}\nabla_k^\alpha\left({}_k\nabla_0^\alpha(u(k))\right) + {}_k\nabla_0^\alpha\left({}_{T+1}\nabla_k^\alpha(u(k))\right) = \lambda e^{-u(k)} - u(k), & k \in [1,T]_{\mathbb{N}_0}, \\ u(0) = u(T+1) = 0, \end{cases} \tag{4.1}$$

for any $\lambda \in \left]0, +\infty\right[$, has at least one non-trivial solution u_0, since $\lim_{c\to+\infty} \frac{e^{-c}}{c} = 0$ and $\lim_{d\to 0^+} \frac{e^{-d}}{d} = \infty$.

Funding
This work was supported by Sari agricultural sciences and natural resources university [grant number 03-1401-18].

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The author was the only one contributing to the manuscript. The author declares that he has fully written and approved the final manuscript.

Author details
Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University,

Sari, Iran
Email: mohsen.khaleghi@rocketmail.com and m.khaleghi@sanru.ac.ir

Acknowledgments

The author is thankful to the reviewers and the editor for their comments which have improved the quality of the article.

References

[1] T. Abdeljawad, *On delta and nabla Caputo fractional differences and dual identities,* Discrete Dynamics in Nature and Society 2013 (2013).
[2] T. Abdeljawad and F. Atici, *On the Definitions of Nabla Fractional Operators,* Abstract and applied Analysis, 2012 (2012), Article ID 406757, 13 pages, doi:10.1155/2012/406757.
[3] FM. Atici and PW. Eloe, *Discrete fractional calculus with the nabla operator,* Electron. J. Qual. Theory Differ. Equ. Spec. Ed. I 2009, 3 (2009).
[4] F.M. Atici and P.W. Eloe, *Initial value problems in discrete fractional calculus,* Proc. Amer. Math. Soc. 137(3) (2009) 981–989.
[5] F.M. Atici and P.W. Eloe, *Discrete fractional calculus with the nabla operator,* Electron. J. Qual. Theory Differ. Equ. Spec. Ed. I (3) (2009) 1–12.
[6] F.M. Atici and P.W. Eloe, *Two-point boundary value problems for finite fractional difference equations,* J. Difference Equ. Appl. 17(04) (2011) 445–456.
[7] F.M. Atici and S. Şengül, *Modeling with fractional difference equations,* J. Math. Anal. Appl. 369 (2010) 1–9.
[8] R.P. Agarwal, K. Perera and D. O'Regan, *Multiple positive solutions of singular discrete p-Laplacian problems via variational methods,* Adv. Diff. Equ. 2 (2005) 93–99.
[9] R.P. Agarwal, K. Perera, and D. O'Regan, *Multiple positive solutions of singular and nonsingular discrete problems via variational methods,* Nonl. Anal. TMA 58(2004) 69–73.
[10] G. Bonanno, *A Critical point theorem via the Ekeland variational principle,* Nonl. Anal. TMA 75 (2012) 2992–3007.
[11] G. Bonanno, B. Di Bella and D. O'Regan, *Non-trivial solutions for nonlinear fourth-order elastic beam equations,* Compu. Math. Appl. 62 (2011) 1862–1869.
[12] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel,* Prog. Fract. Differ. Appl. 1(2) (2015) 73–85.
[13] S. Dhar and L. Kong, *A critical point approach to multiplicity results for a fractional boundary value problem,* Bulletin of the Malaysian Mathematical Sciences Society, (2020) 1–17.
[14] W. Dong, J. Xu and D. O Regan, *Solutions for a fractional difference boundary value problem,* Adv. Difference Equ. 2013, 2013:319
[15] C. Goodrich and A. C. Peterson, *Discrete Fractional Calculus,* Springer international publishing Switzerland, New York, 2015.
[16] J. Henderson and H.B. Thompson, *Existence of multiple solutions for second order discrete boundary value problems,* Comput. Math. Appl. 43 (2002), 1239–1248.
[17] M. Khaleghi Moghadam and M. Avci, *Existence results to a nonlinear $p(k)$-Laplacian difference equation,* J. Difference Equ. Appl. 23(10) (2017), 1055–1068.
[18] M. Khaleghi Moghadam and J. Henderson, *Triple solutions for a dirichlet boundary value problem involving a perturbed discrete $p(k)$- laplacian operator,* Open Math. J., 15 (2017), 1075–1089.
[19] P. Li, L. Xu, H. Wang and Y. Wang, *The existence of solutions for perturbed fractional differential equations with impulses via Morse theory,* Boundary Value Problems, 2020(1),(2020) 1–13.
[20] P. Lindqvist, *On the equation $div(|u|^{p-2}u) + |u|^{p-2}u = 0$,* Proc. Amer. Math. Soc. 109 (1990), 157–164.
[21] H. L. Gray and N. F. Zhang, *On a new definition of the fractional difference,* Math. Comp. 50 , no. 182, (1988) 513–529.
[22] B. Ricceri, *A general variational principle and some of its applications,* J. Comput. Appl. Math. 113(1-2) (2000) 401–415.
[23] ZS. Xie, YF. Jin and CM. Hou, *Multiple solutions for a fractional difference boundary value problem via variational approach.* Abstr. Appl. Anal. 2012, 143914 (2012)

[24] SG. Samko, AA. Kilbas and OI. Marichev, *Fractional integrals and derivatives: Theory and Applications,* Breach Science Publishers: London, UK, 1993.
[25] M.E. Samei, G.K. Ranjbar and V. Hedayati,*Existence of solutions for equations and inclusions of multi-term fractional q-integro-differential with non-separated and initial boundary conditions,* J. Inequal. Appl. 2019, 273 (2019).
[26] E. Shishkina and S. Sitnik, *Transmutations, singular and fractional differential equations with applications to mathematical physics,* Academic Press, 2020.

(Mohsen Khaleghi Moghadam) Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, Sari, Iran.
Email address: m.khaleghi@sanru.ac.ir